On Heavy-Tailed Rare Event Analysis

Chang-Han Rhee

Centrum Wiskunde & Informatica, Amsterdam

Rare Events

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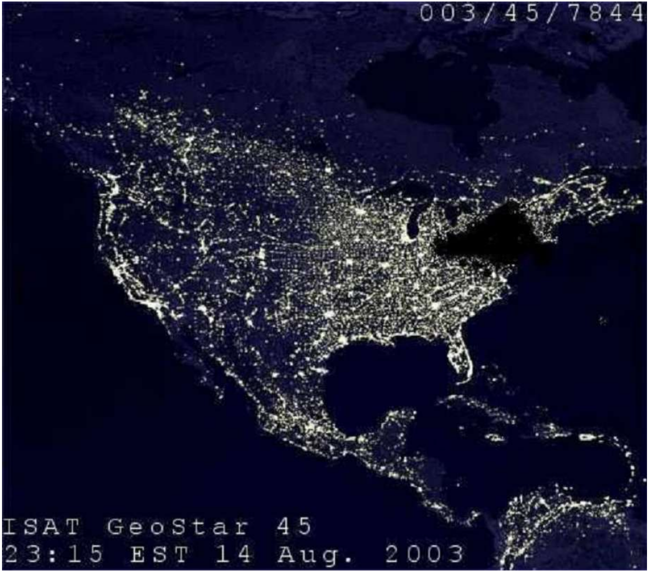

Rare Events

Although rare, rare events matter.

Need for understanding ‘how often?’ & ‘why?’

Rare Events depend on "Tail Behaviors"

Light-Tailed Distributions

- Extreme Values are Very Rare
- Normal, Exponential, etc

Heavy-Tailed Distributions

- Extreme Values are Frequent
- **Power Law**, **Weibull**, etc

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arise because

EVERYTHING goes wrong.

(Conspiracy Principle)

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Systemwide rare events

arise because of

A FEW Catastrophes.

(Catastrophe Principle)

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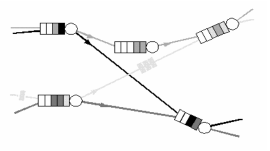
Heavy-tailed rare events are NOT understood well.

But, Heavy Tails are Everywhere:

Computer systems

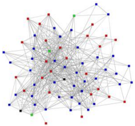


delays, files, ...

Finance

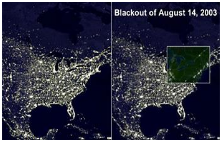
losses

Social networks



popularity, contagion

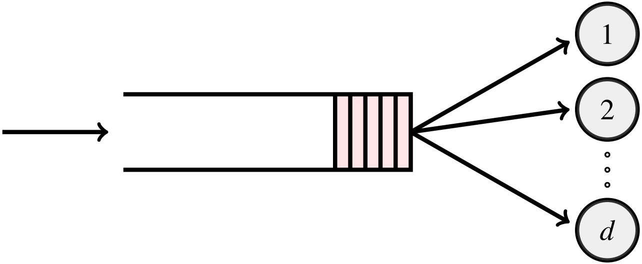
Energy Systems



blackouts

For Example: Open Problem Posed by Whitt (2000)

Congestion of Multiple Server Queue:



How many big jobs are needed to create large queue lengths?

In case of Weibull tails, NOT EVEN a conjecture!

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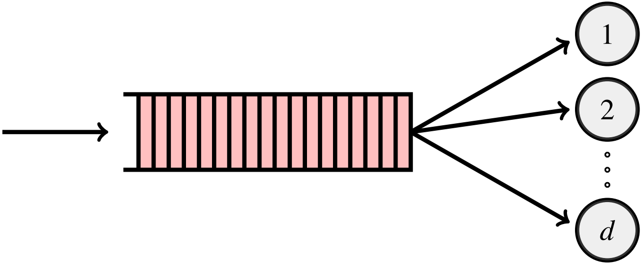

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In many applications, one can write the rare event of interest as

$$\{\bar{S}_n \in A\}$$

where $\bar{S}_n$ is the whole trajectory of a random walk.

We want to understand $\mathbf{P}(\bar{S}_n \in A)$ and $\mathbf{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A)$

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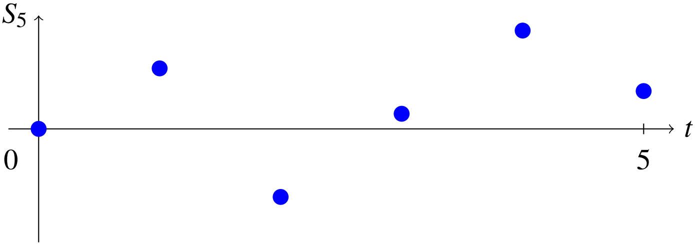

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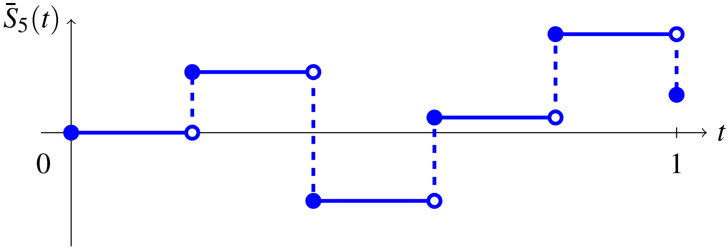

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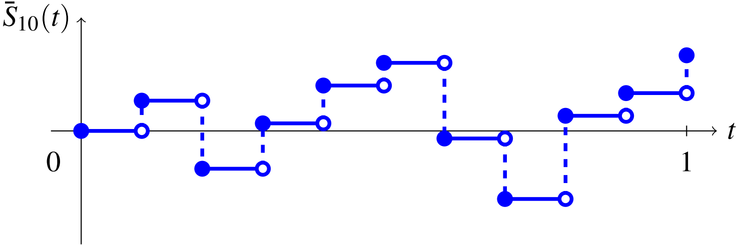

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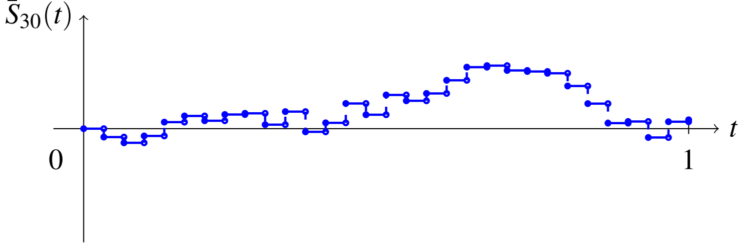

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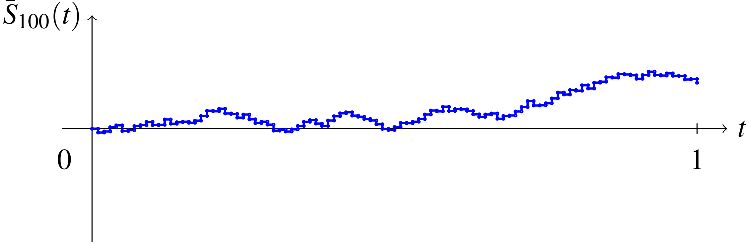

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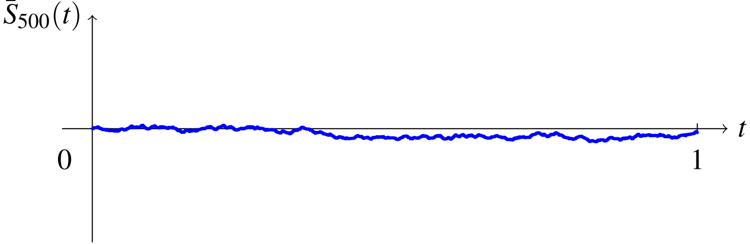

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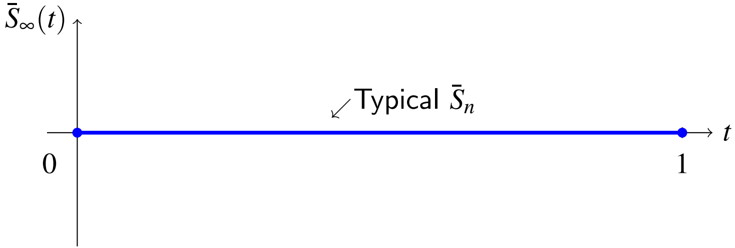

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Outline

Part 1. Large Deviations for Power Law Tails

R., Blanchet, Zwart (2016)

Under second round review at *Annals of Probability*

Part 2. Heavy-Tailed Rare Event Simulation

Chen, Blanchet, **R.**, Zwart (2017)

Mathematics of Operations Research

Part 3. Large Deviations for Weibull Tails

Bazbha, Blanchet, **R.**, Zwart (2017)

Submitted to *Annals of Applied Probability*

Part 1. Large Deviations for Power Law Tails

i.e., $\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}, \ \alpha > 0$

What's already known: principle of a single big jump

"In heavy-tailed systems, rare events arise due to one big anomaly."

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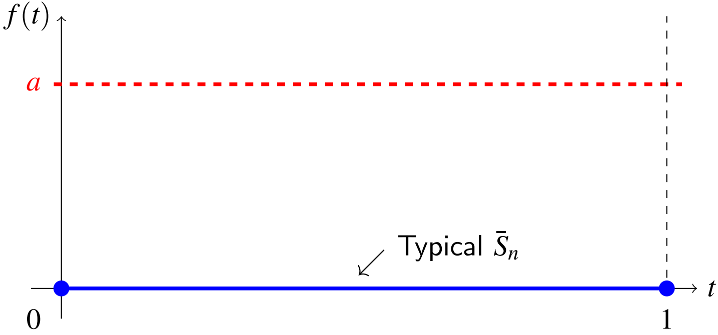

- $A = \{f \in \mathbb{D} : f$ crosses level a on $[0,1]\}$
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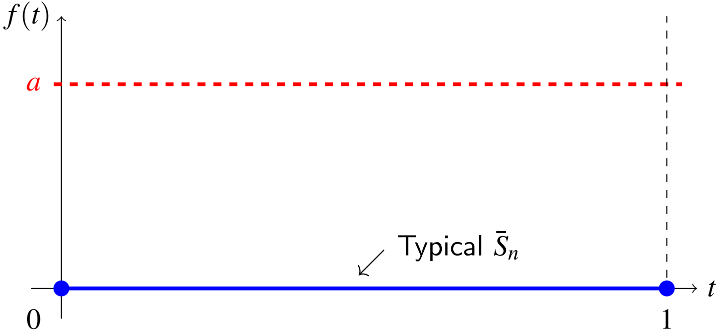

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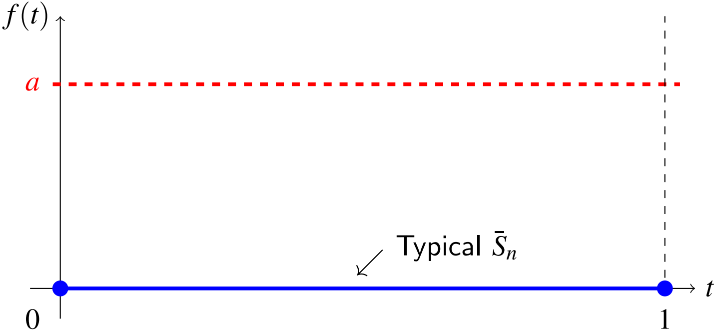

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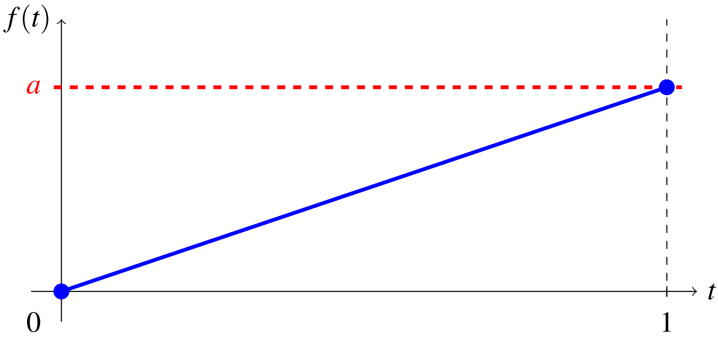

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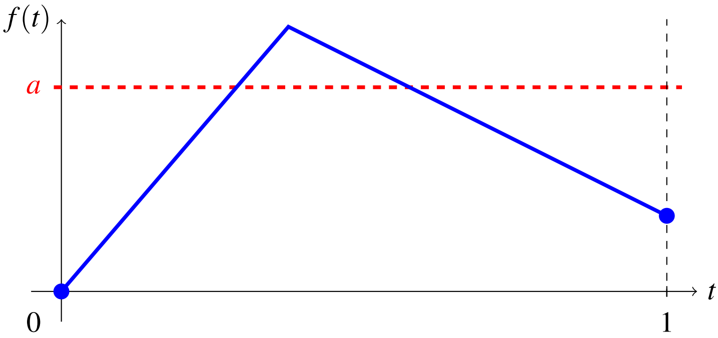

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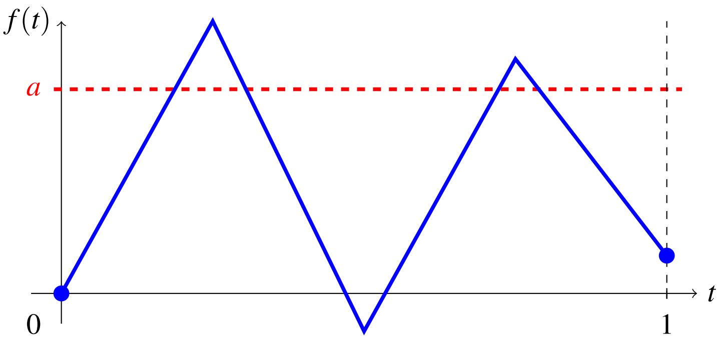

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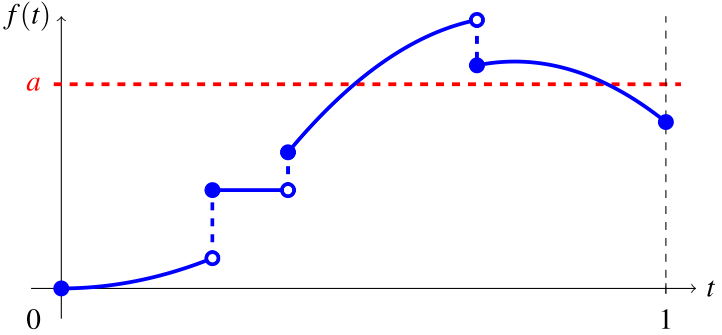

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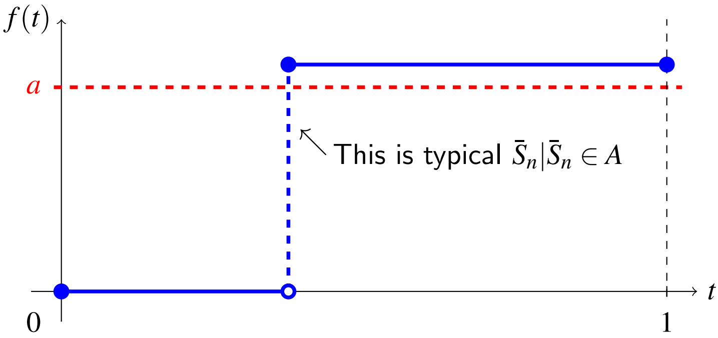

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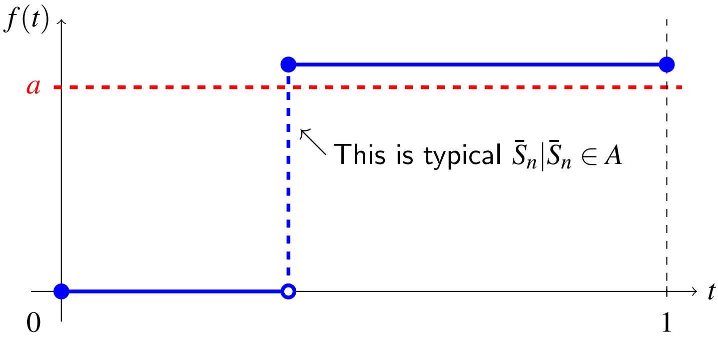
What's already known: principle of a single big jump



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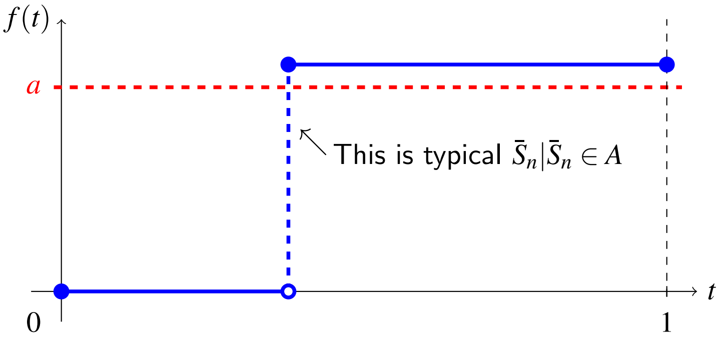

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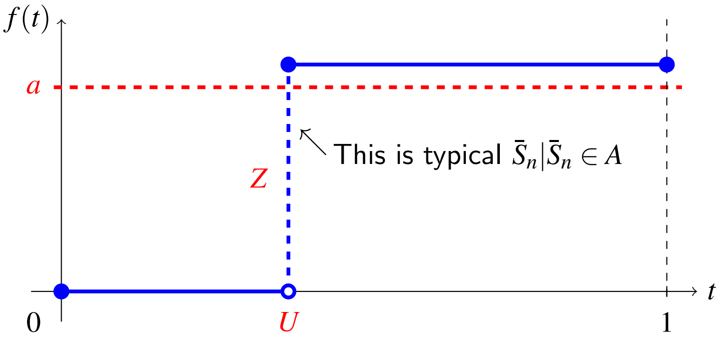

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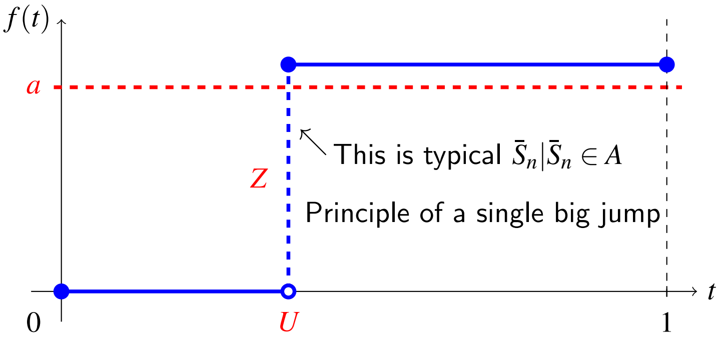

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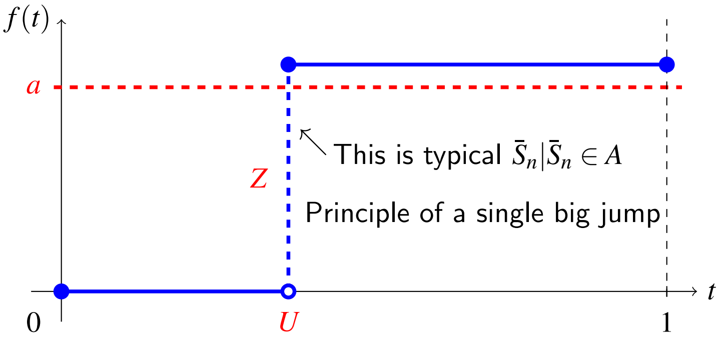

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Hult, Lindskog, Mikosch, Samorodnitsky (2005)

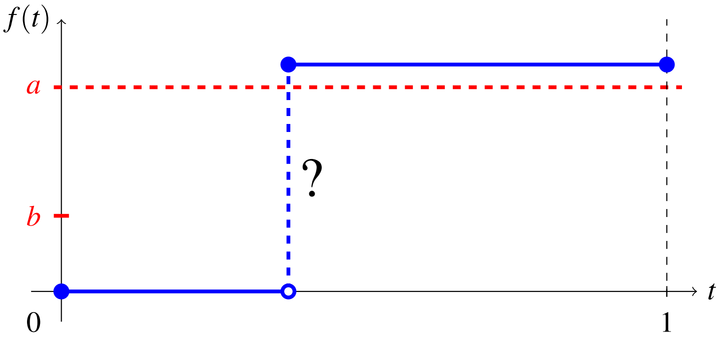
Are all heavy-tailed rare events due to a single big jump?

No, by no means, absolutely not:

- Multiple server queues
- Queueing networks
- Re-insured insurance line
- Down-and-in barrier option
- Many more

Principle of a single big jump is just a tip of the iceberg!

Illustration: What if Large Claims are Reinsured?



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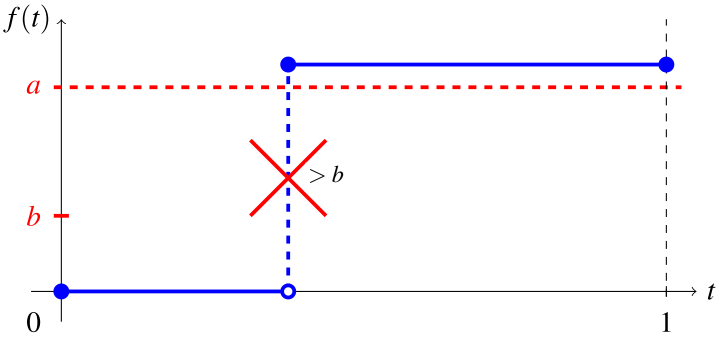

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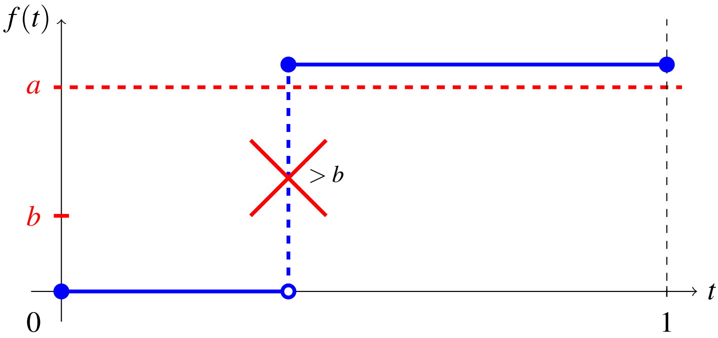

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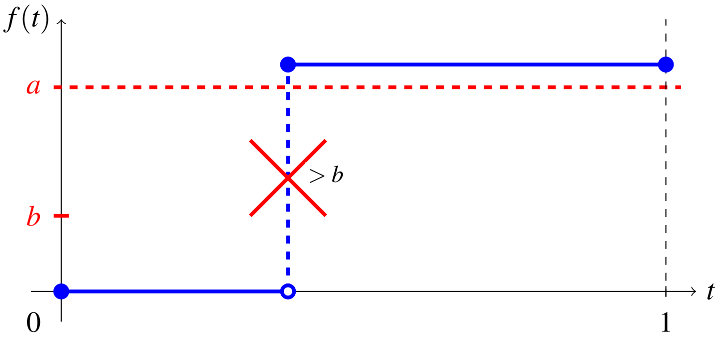

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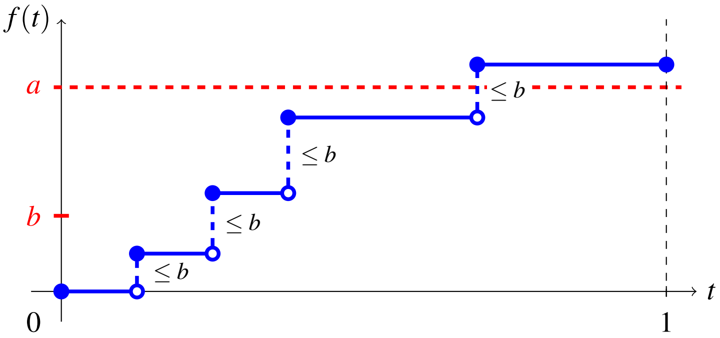

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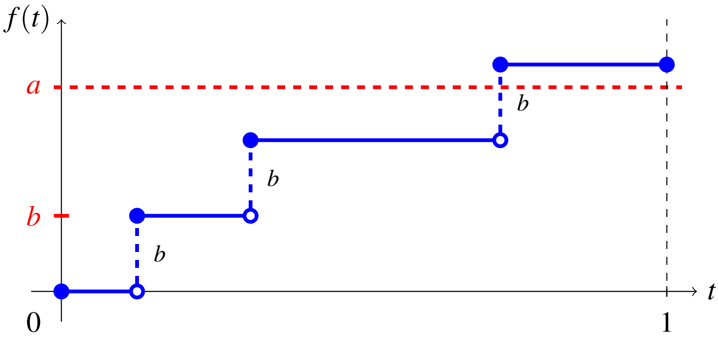

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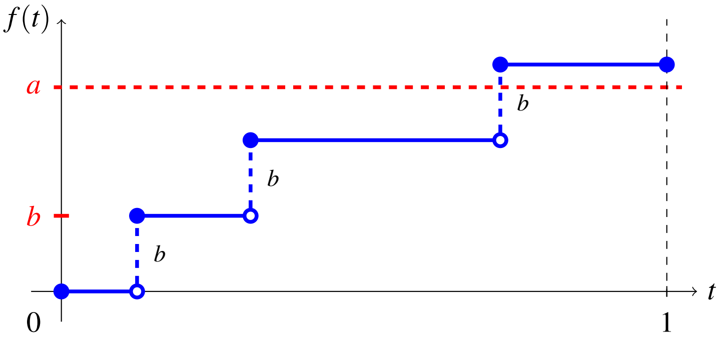

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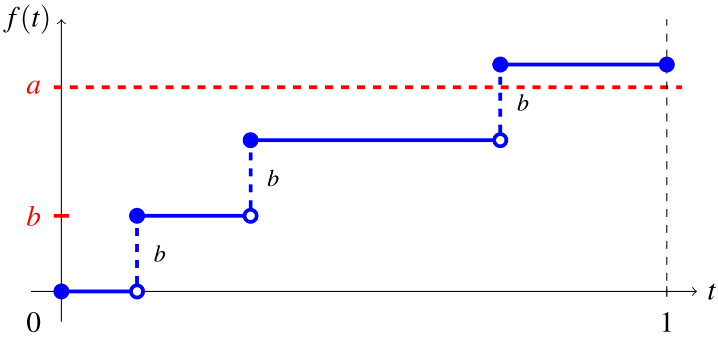

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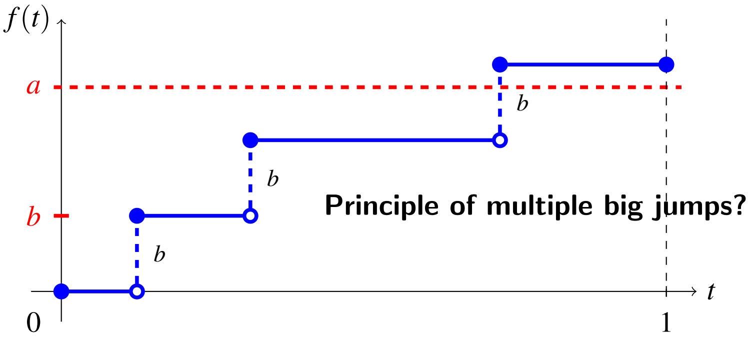

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Exact Asymptotics for Heavy-tailed Random Walks

$$\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$$

Theorem (R., Blanchet, Zwart, 2017)

For "general" $A \subseteq \mathbb{D}$

$$C(A^\circ) \leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq C(A^-).$$

- $\mathcal{J}(A)$: *min #jumps for step functions to be inside A*
- $C(\cdot)$: *a measure*

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LD power index

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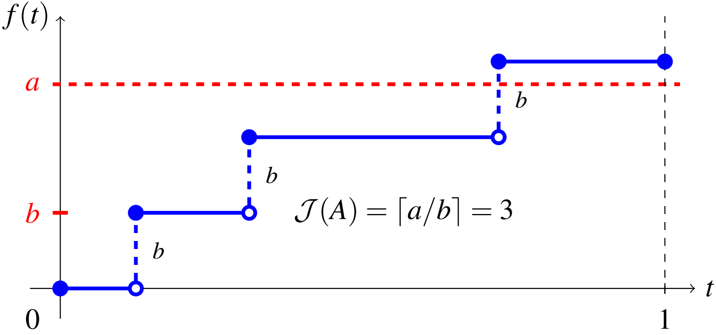
Back to Our Reinsurance Example: Conjecture Confirmed!



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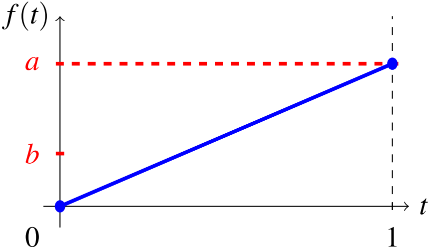
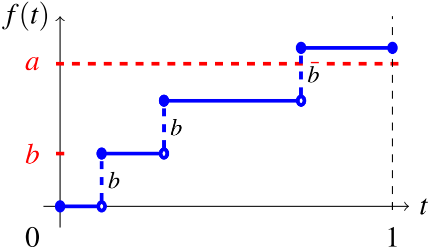
Conspiracy vs Catastrophy

$$A = \{f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0,1] \ \& \text{ jump sizes} \le b\}$$



Light-Tailed Claim Size



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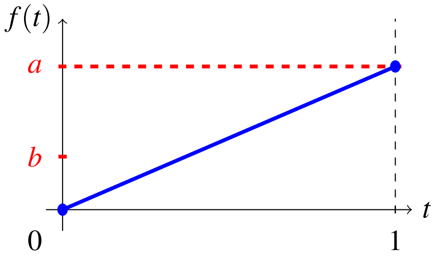

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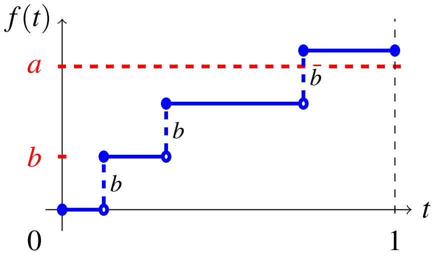

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Reinsurance makes no difference.

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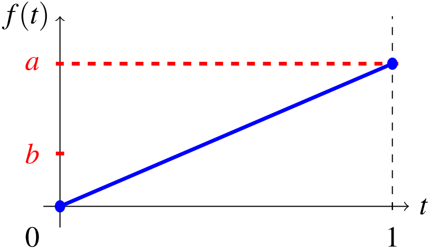

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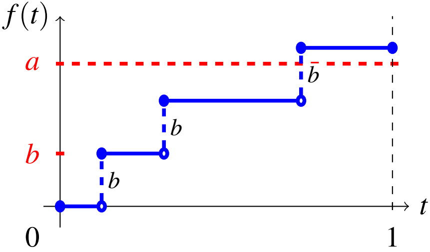


Heavy-Tailed Claim Size

Reinsurance helps!

Two-sided Random Walks

$$\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0 \qquad \text{and} \qquad \mathbf{P}(X_i \leq -x) = x^{-(\beta+1)}, \quad \beta > 0.$$

Theorem (R., Blanchet, Zwart, 2017)

For "general" $A \subseteq \mathbb{D}$,

$$\mathbf{P}(\bar{S}_n \in A) \sim n^{-\overbrace{\{\alpha\mathcal{J}(A)+\beta\mathcal{K}(A)\}}^{\text{Cost of Jumps}}}.$$

- $\mathcal{J}(A)$: *# of upward jumps*
- $\mathcal{K}(A)$: *# of downward jumps*

of step functions that minimize the cost of staying inside A

Connection to Impulse Control Problem

The “rate of decay” is determined by a discrete optimization problem:

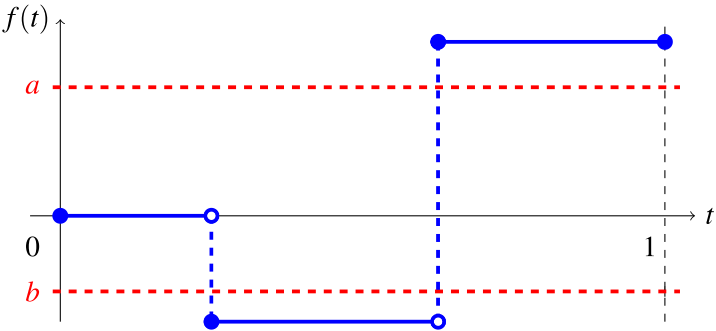
$$\begin{aligned} \alpha\mathcal{J}(A)+\beta\mathcal{K}(A) \quad = \quad & \min_{j,k}\ \alpha j+\beta k \\ & \text{subject to } (j,k)\in\mathbb{Z}_+^2 \\ & \mathbb{D}_{j,k}\cap A\neq\emptyset \end{aligned}$$

where

$$\mathbb{D}_{j,k}=\{\text{step functions w/ } j \text{ upward jumps and } k \text{ downward jumps}\}$$

Different from variational calculus that arises in light-tailed case!

Example: Barrier Option



- $A = \{f \in \mathbb{D} : f$ is below b at some point & end up above $a\}$
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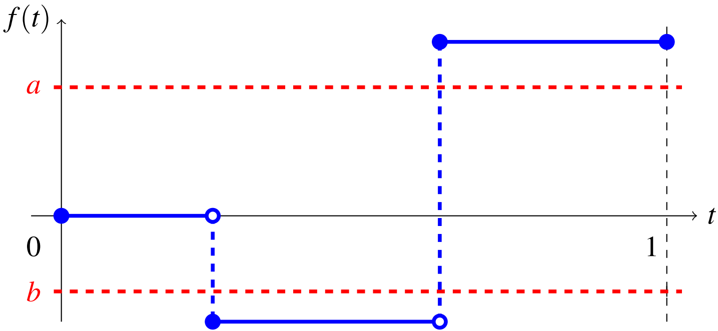

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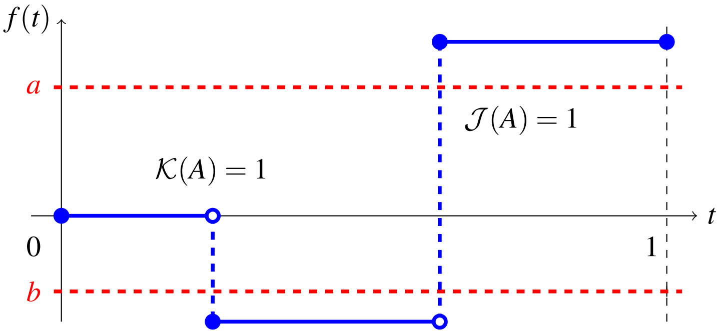

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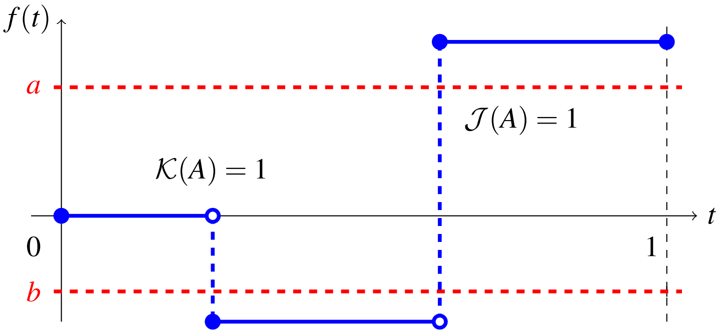

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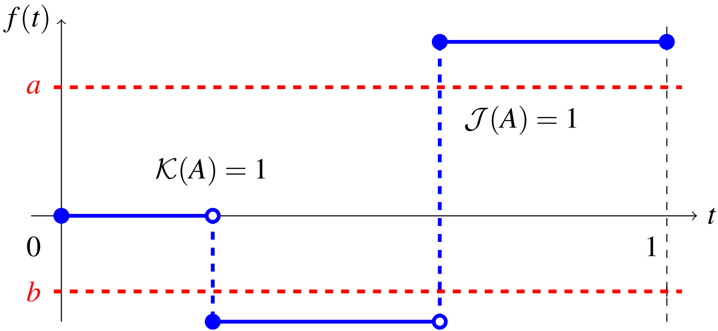

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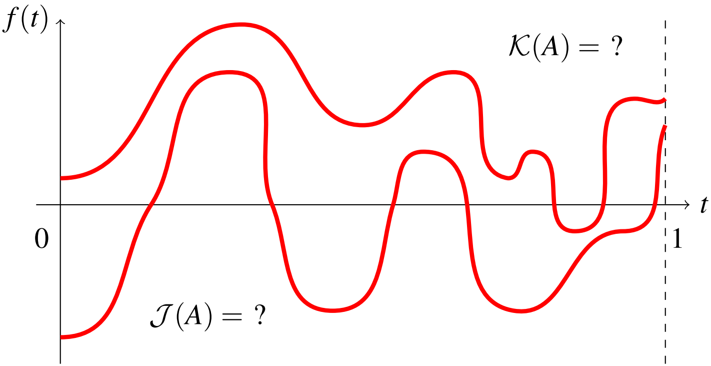

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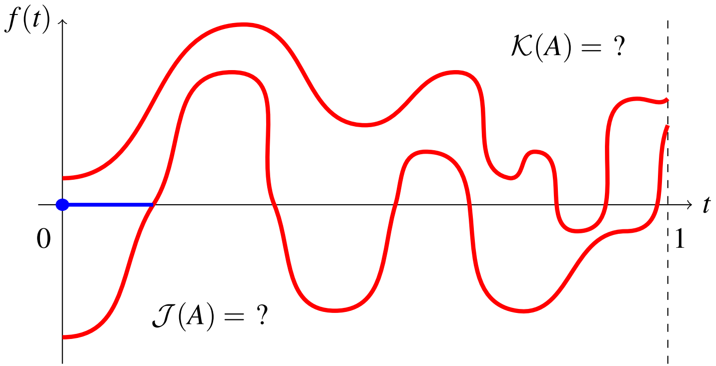

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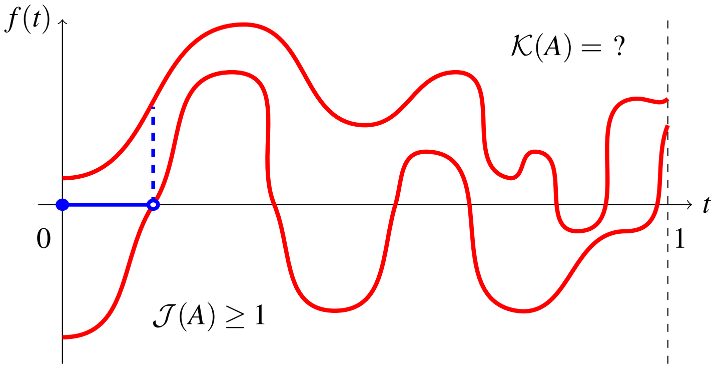

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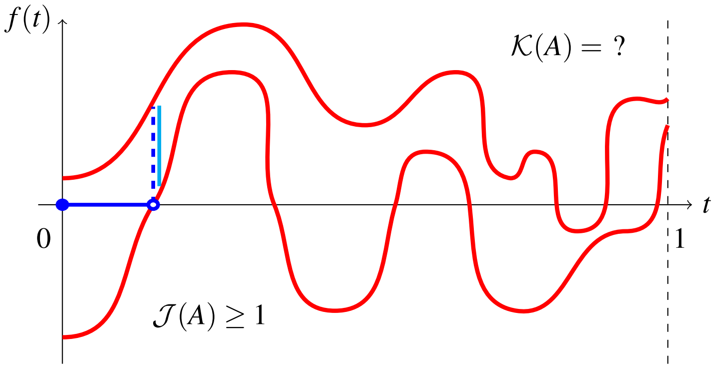

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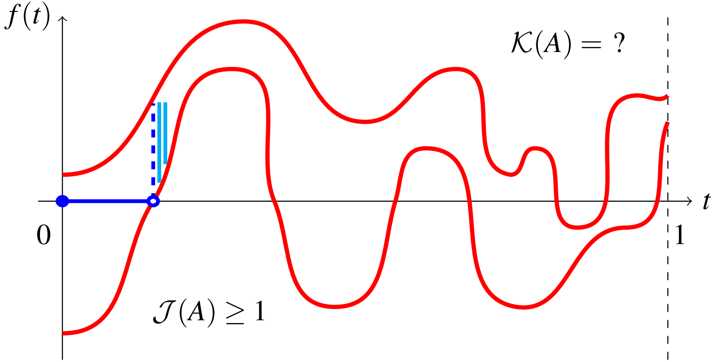

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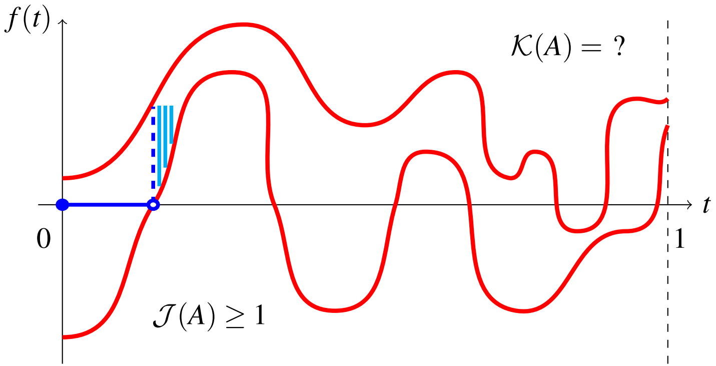

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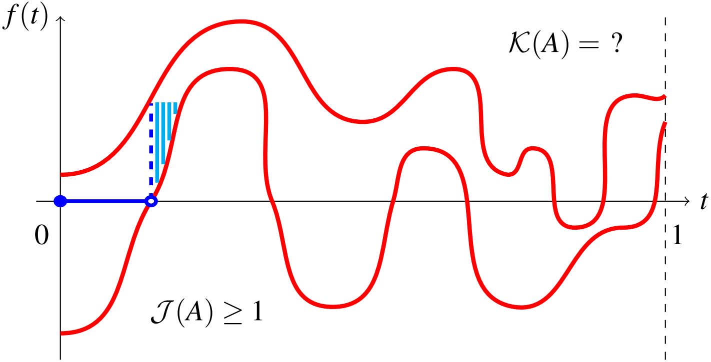

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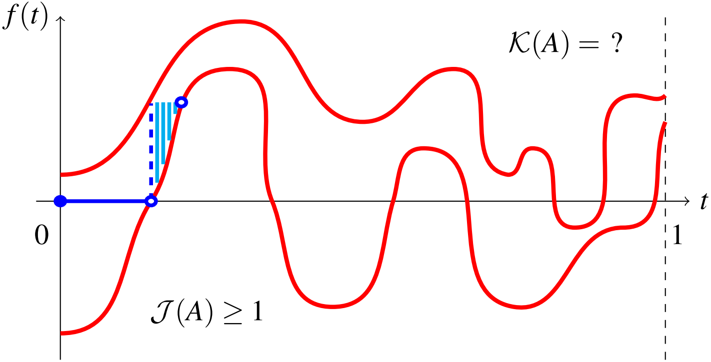

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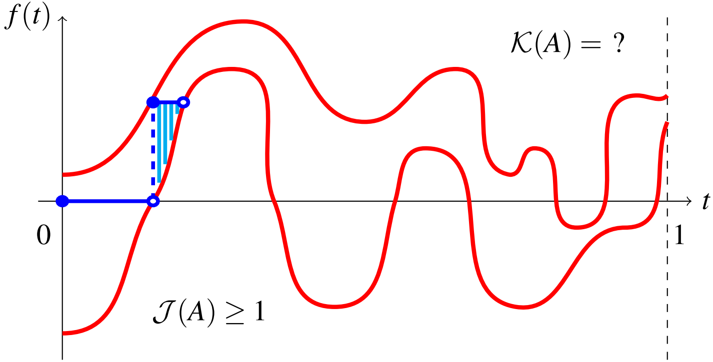

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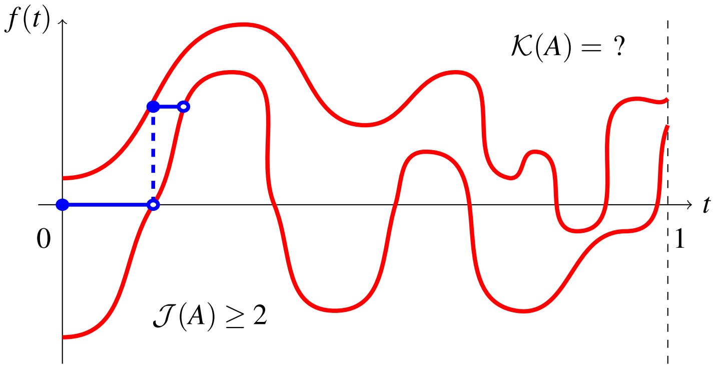

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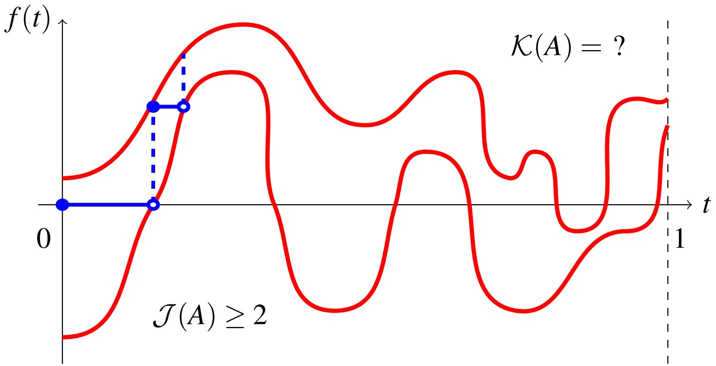

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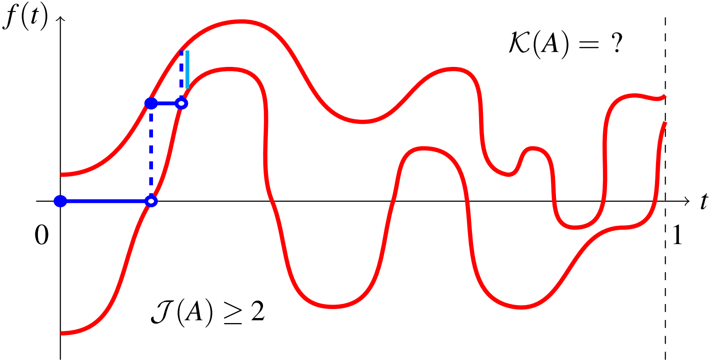

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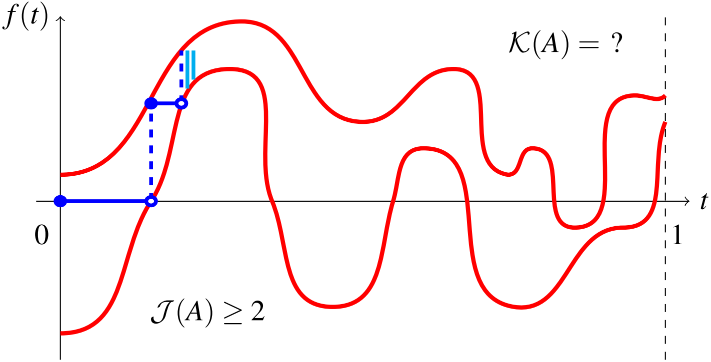

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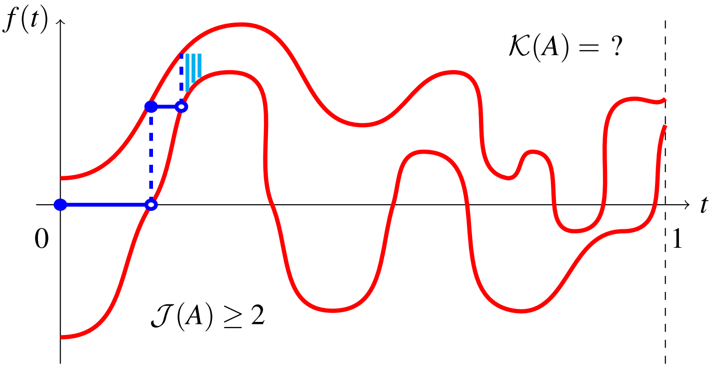

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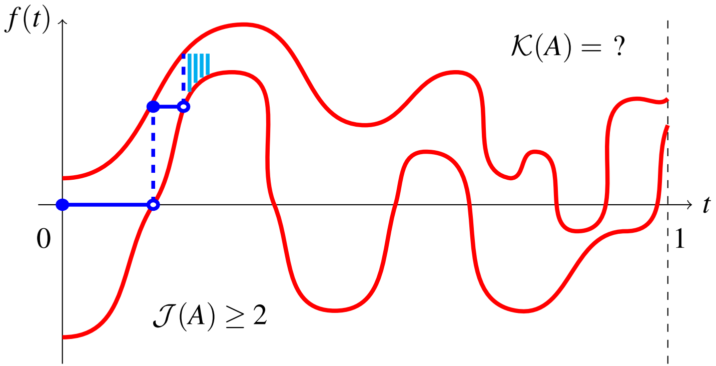

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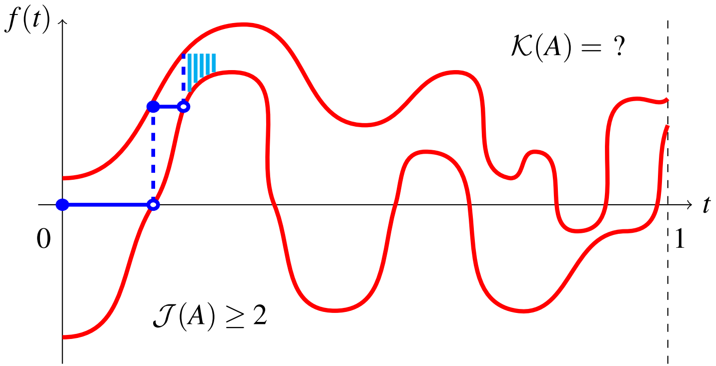

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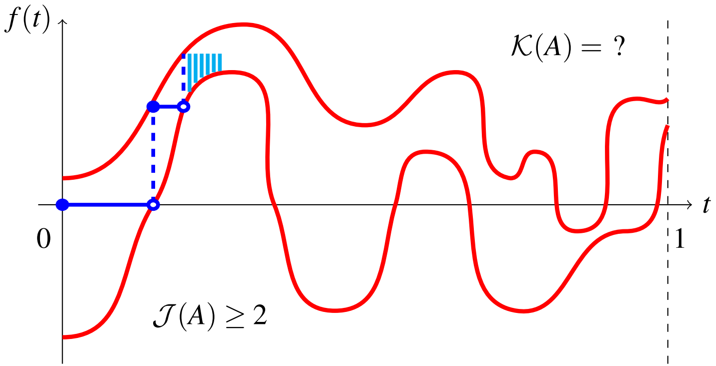

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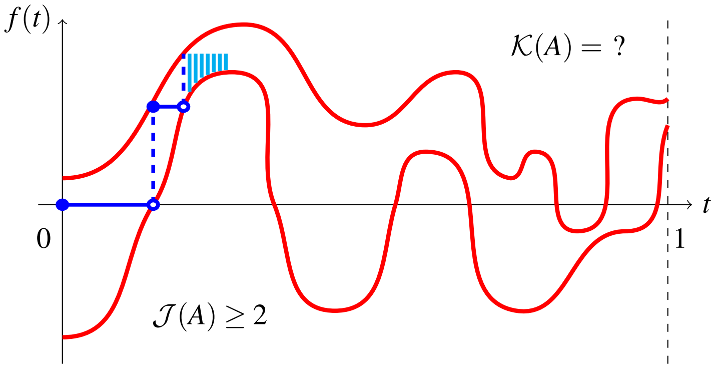

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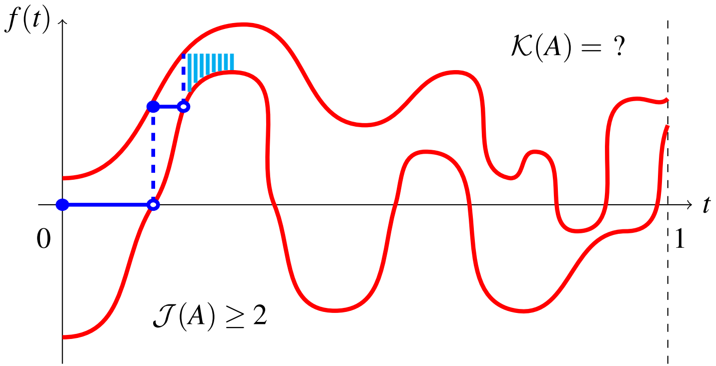

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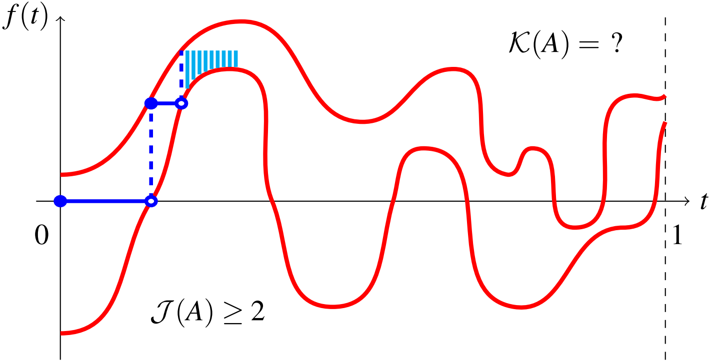

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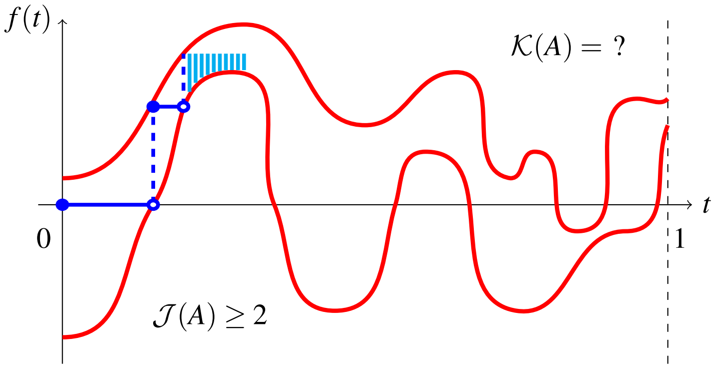

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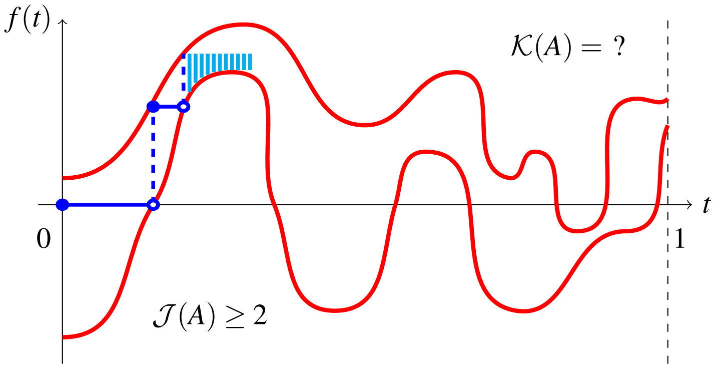

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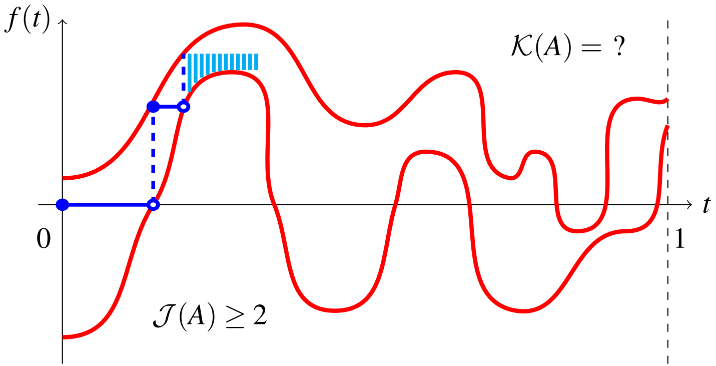

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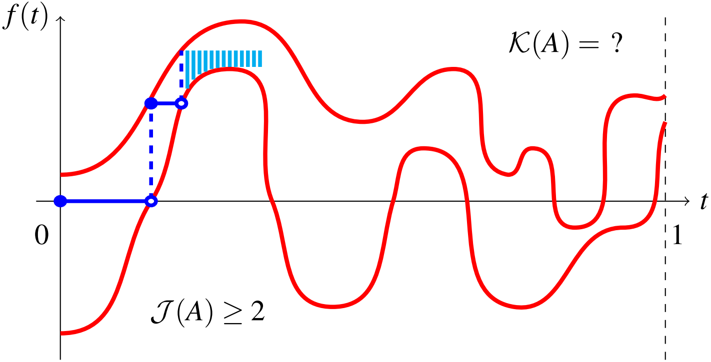

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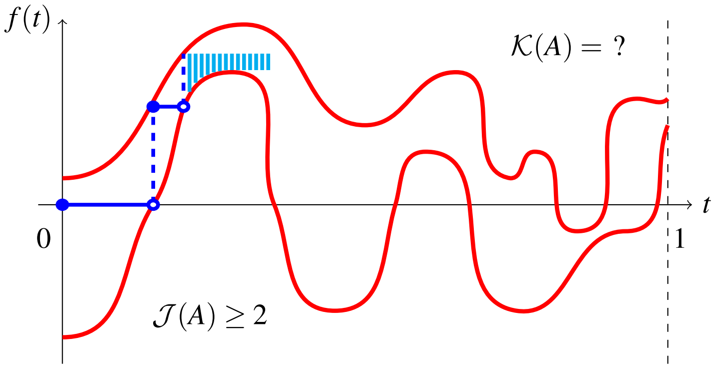

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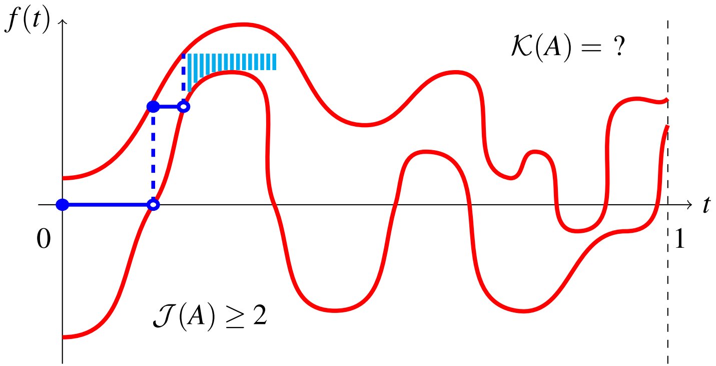

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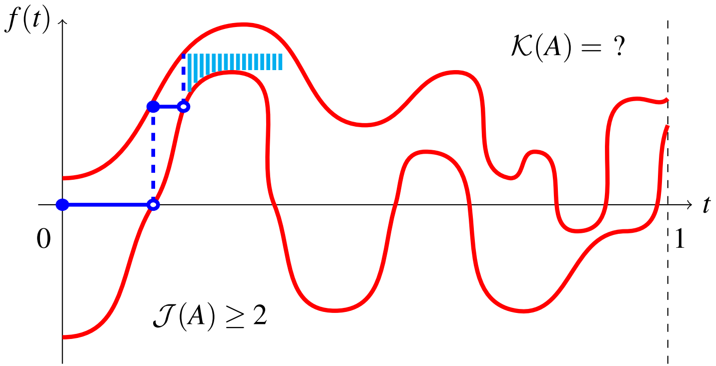

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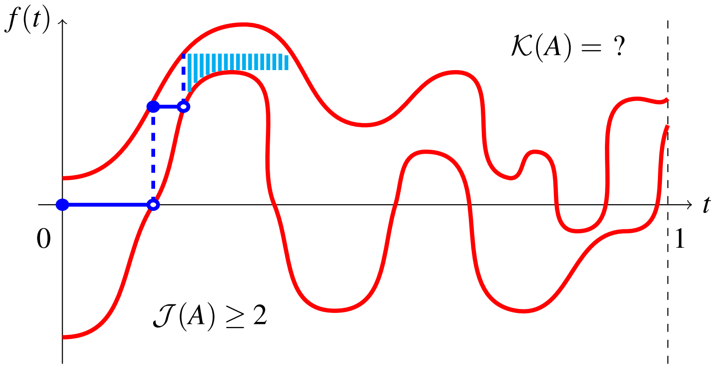

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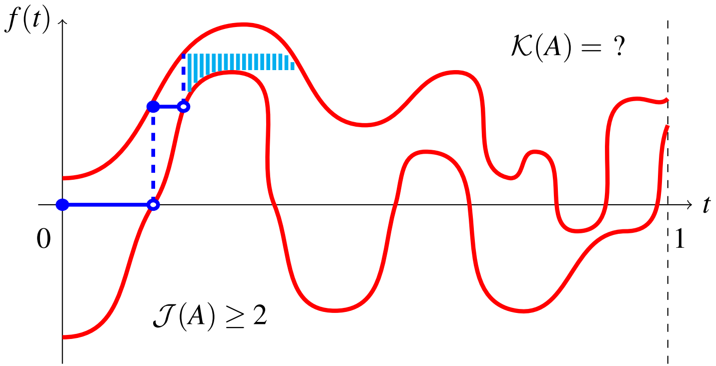

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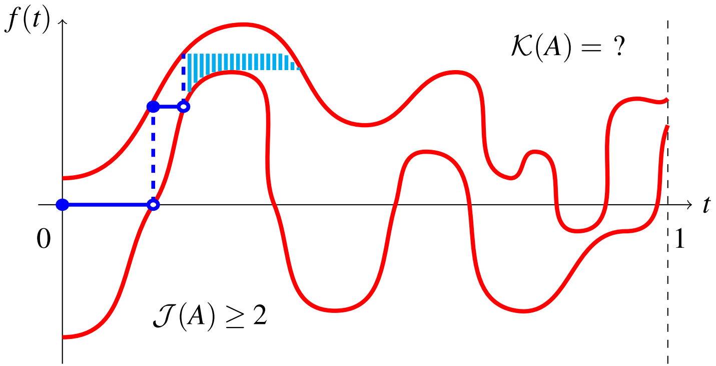

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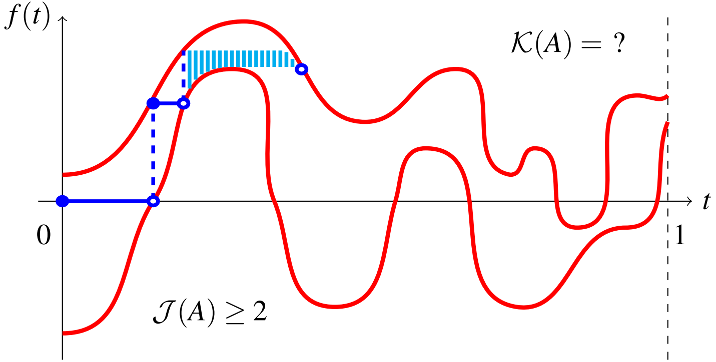

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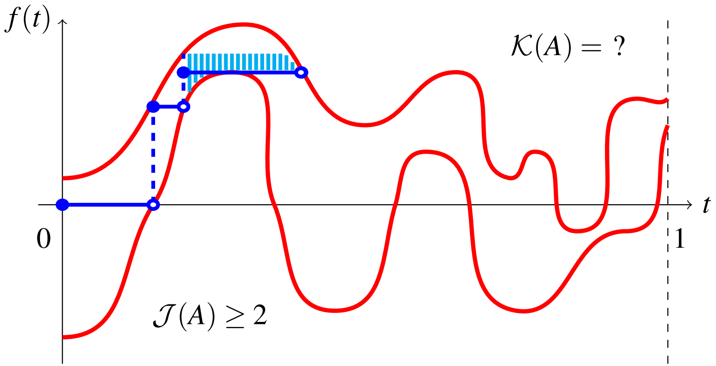

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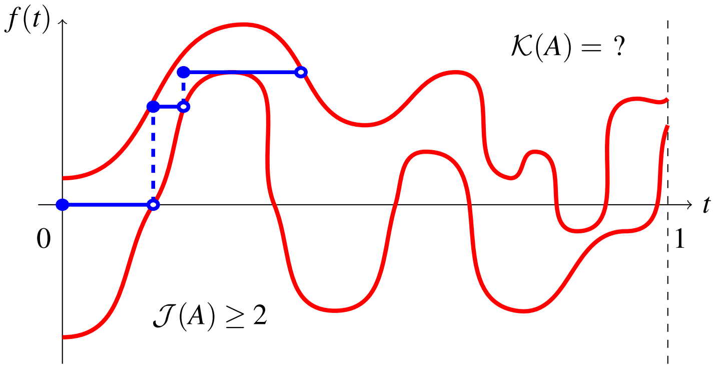

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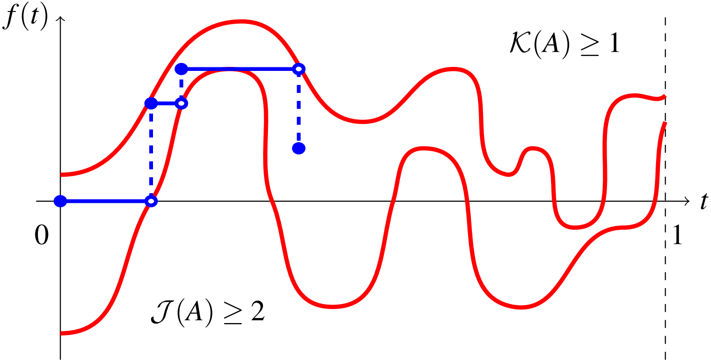

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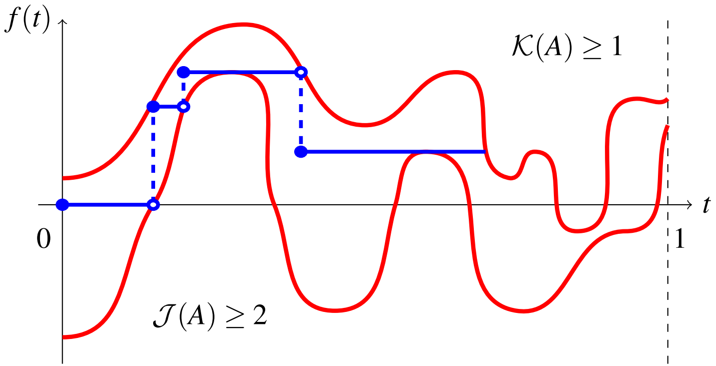

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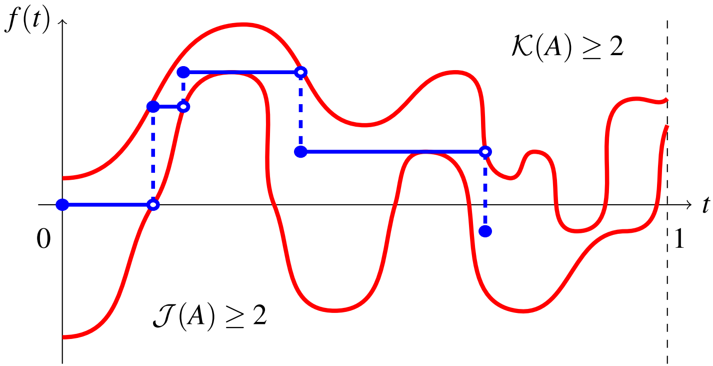

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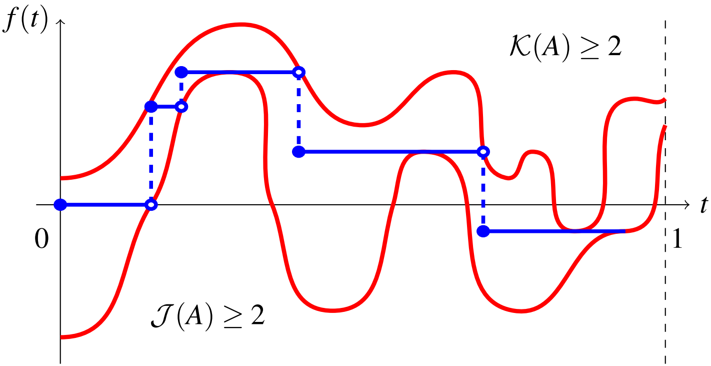

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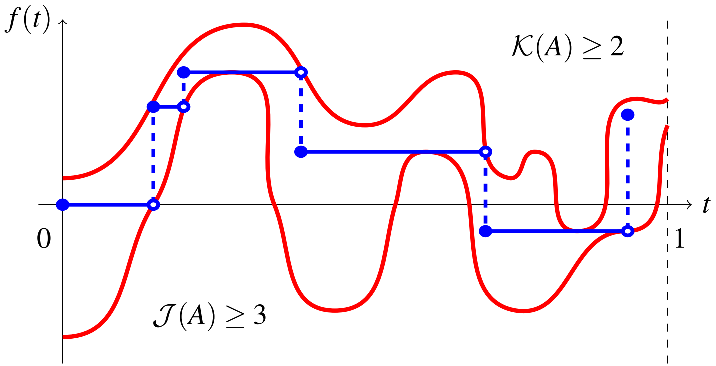

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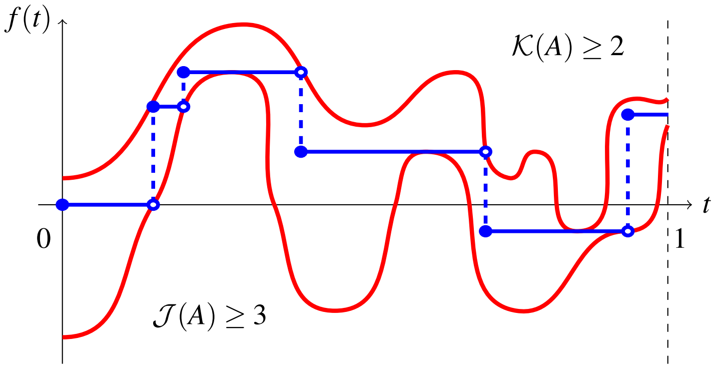

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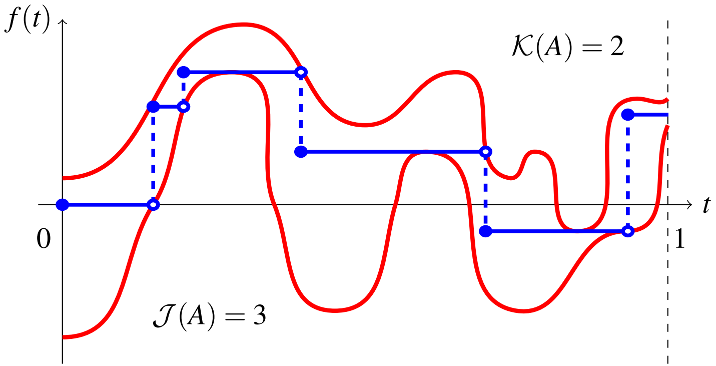

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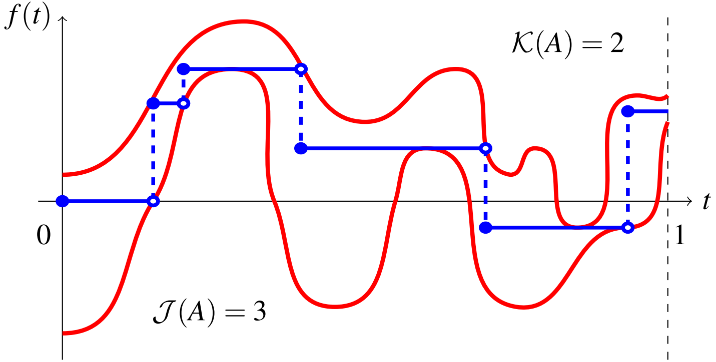

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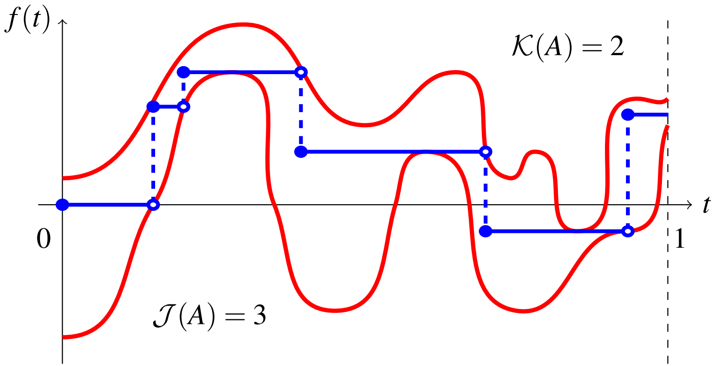

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- $\mathbf{P}(\bar{S}_n \in A) \sim n^{-(3\alpha + 2\beta)}$
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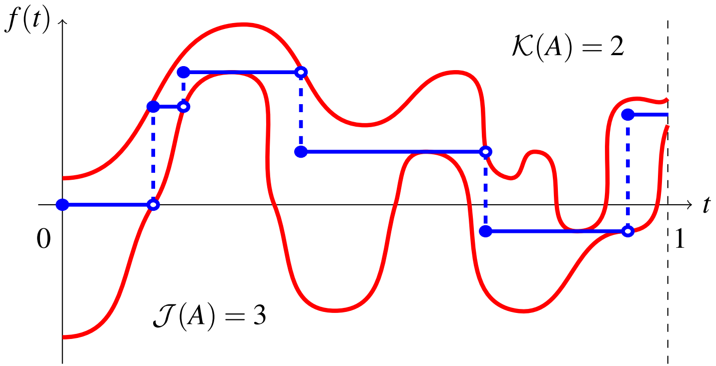

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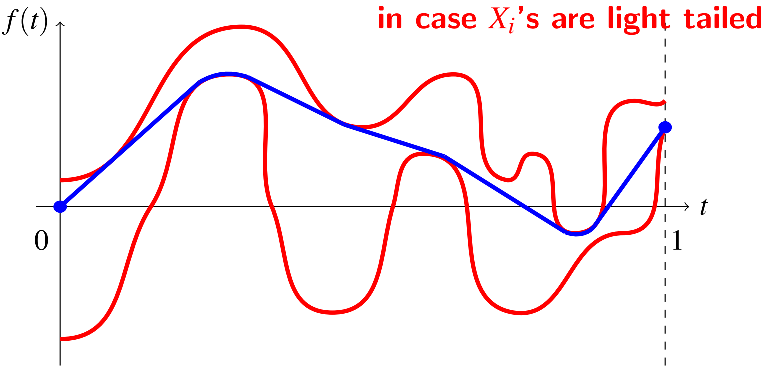

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- $\mathbf{P}(\bar{S}_n \in A) \sim \exp(-nI^*)$

I^*: sol'n of optimization problem over absolutely continuous functions.

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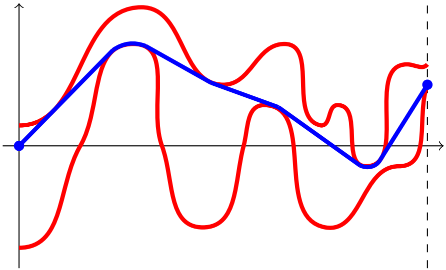
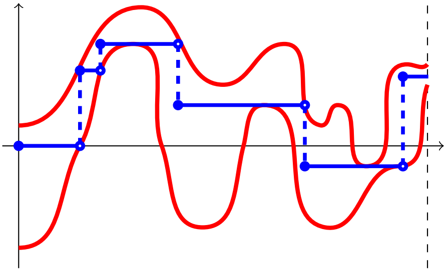




Conspiracy

Connection to variational problems

(continuous optimization)

Catastrophy

Connection with impulse control

(discrete optimization)

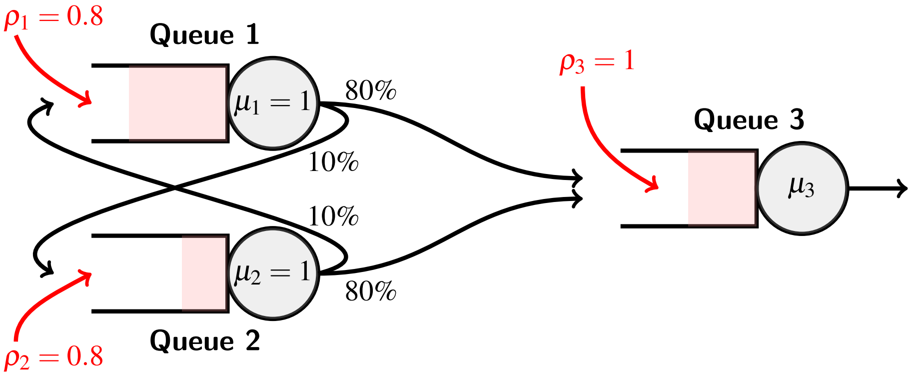
Lévy Processes and Multidimensional Processes

- Same results for Lévy processes.

- Similar results for vector valued processes with independent components as well (for both Lévy processes and random walks).

Analysis of Many Server Queues and even Queueing Networks!

Example - Stochastic Fluid Network



- Job size distribution: Pareto with $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 3$
- Queue 3 experiences congestion because of what?
- **Optimization problem reduces to a knapsack type problem**

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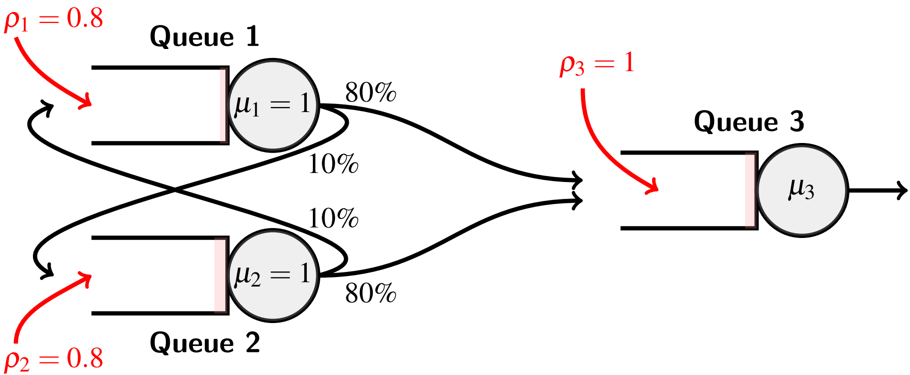

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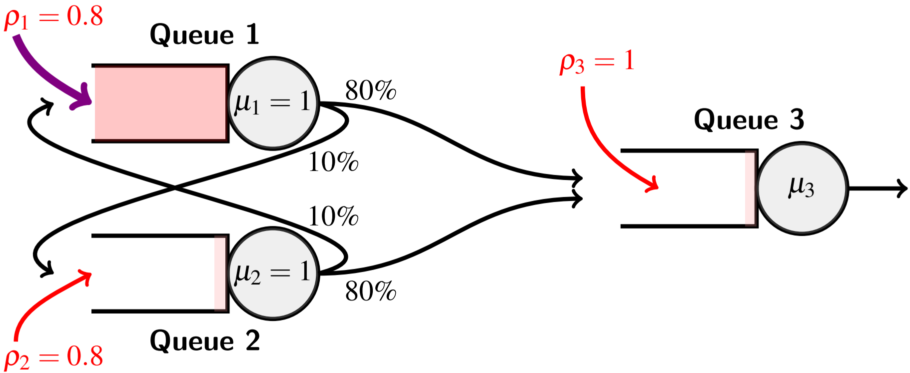

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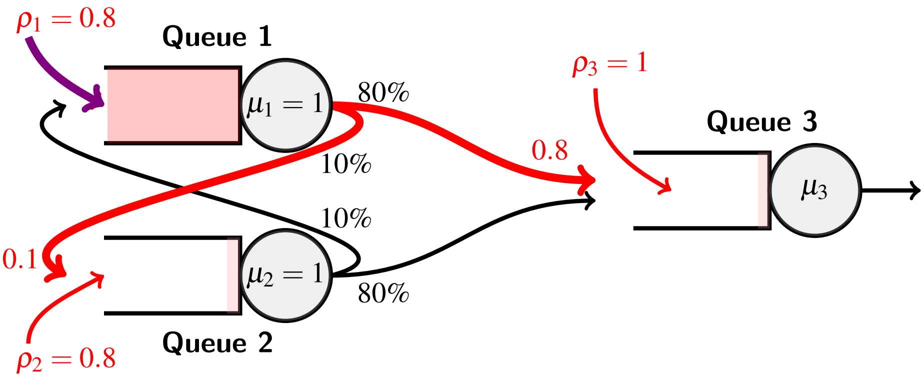

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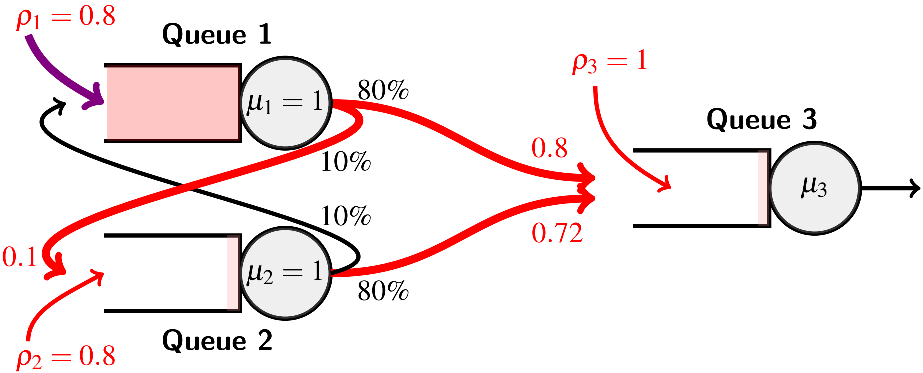

- Job size distribution: Pareto with $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 3$
- Queue 3 experiences congestion because of what?
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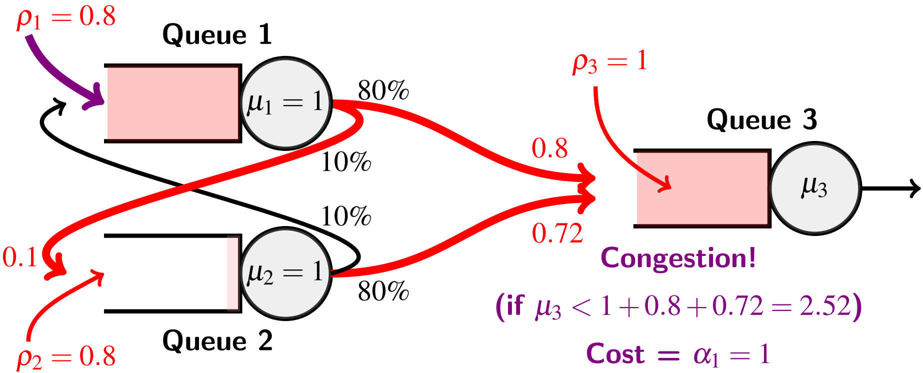

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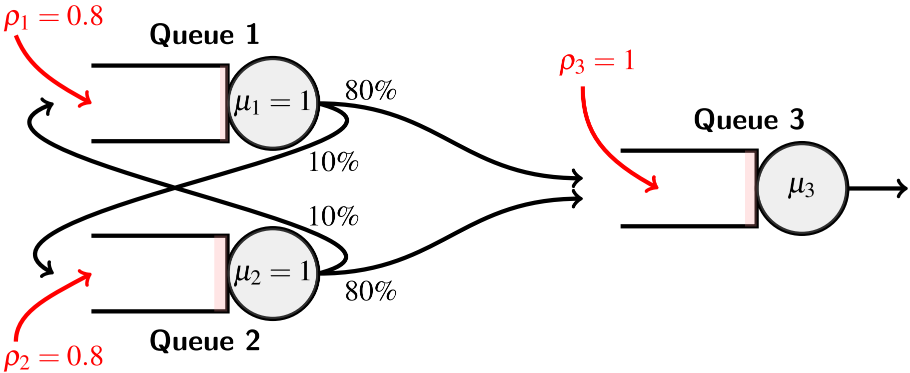

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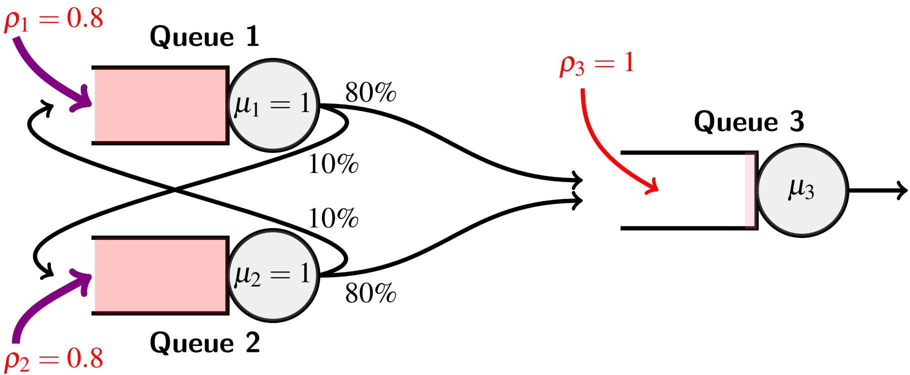

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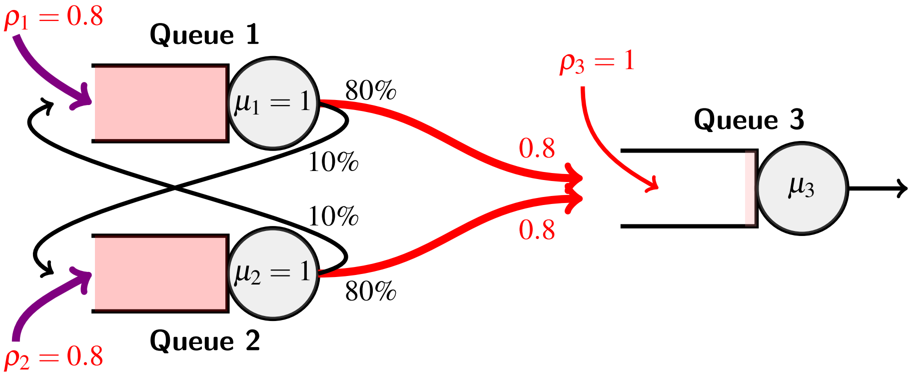

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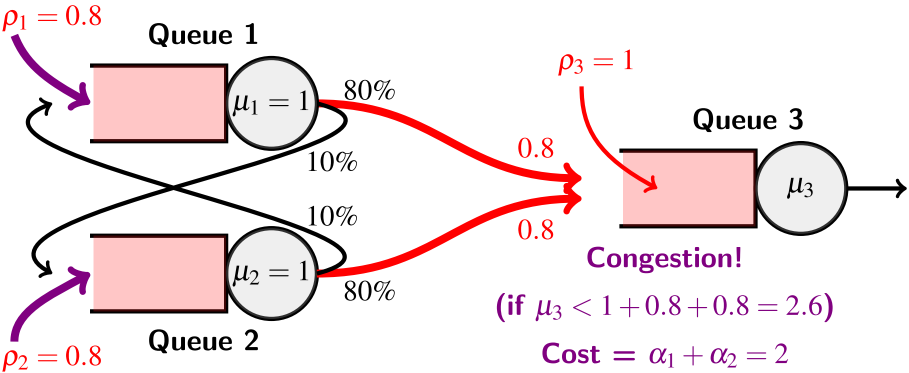

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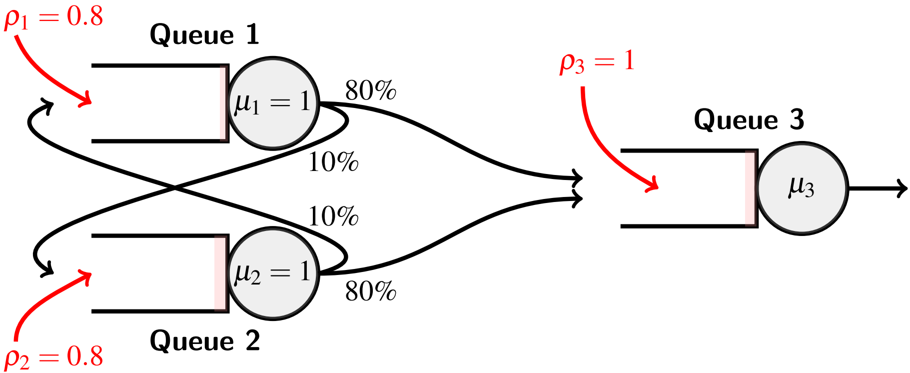

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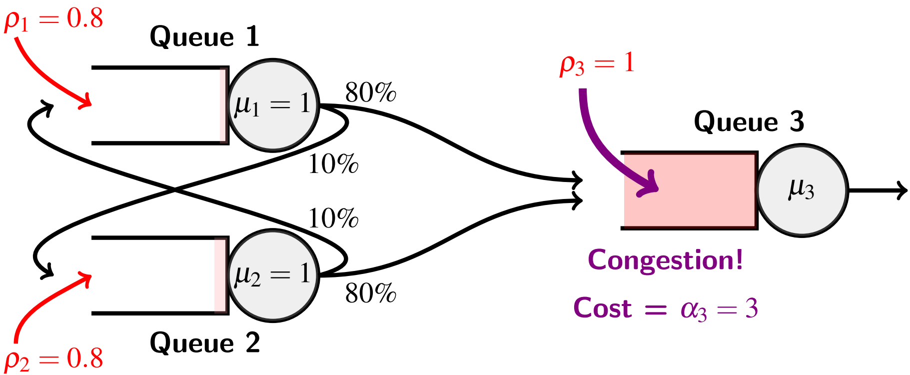

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Part 2. Heavy-Tailed Rare Event Simulation

Computing $\mathbf{P}(\bar{S}_n \in A)$ for finite n

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Head})$

- Flip the coin 100 times
- Count the number of head
- Divide by 100 and report the number

Should be reasonably close to $1/2$

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge
- Divide by 100 and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number : and hence, likely to be 0

Challenge of Rare Event Simulation

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Is 0 a useful answer?

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number : and hence, likely to be 0

Is 0 a useful answer? No.

e.g., Nuclear Meltdown, Large-Scale Blackout, Large Financial Loss

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{\textit{Suppose}}{\approx} 10^{-6}$

- Flip the coin 100 times
- Count the number of Edge
- Divide by 100 and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{\textit{Suppose}}{\approx} 10^{-6}$

- Flip the coin ~~100~~ a few million times
- Count the number of Edge
- Divide by the total number of flips and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{\textit{Suppose}}{\approx} 10^{-6}$

- Flip the coin ~~100~~ a few million times
- Count the number of Edge
- Divide by the total number of flips and report the number

Much harder than $\mathbf{P}$(Head)

Importance Sampling

- Construct an alternative universe

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- Perform experiments there

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- Perform experiments there
- Recover the true probability using the relationship between the two parallel universes

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- Construct an alternative universe
  - i.e., Consider an "importance distribution" $\mathbf{Q}$
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  - i.e., Sample $\mathbb{I}_{\text{Edge}}^{(1)}, \ldots, \mathbb{I}_{\text{Edge}}^{(m)}$ from $\mathbf{Q}$

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    as well as the likelihood ratio $\left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right)^{(1)}, \ldots, \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right)^{(m)}$
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- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m}\sum_{i=1}^{m} \mathbb{I}_{\text{Edge}}^{(i)} \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right)^{(i)}$ as an estimate of $\mathbf{P}(\text{Edge})$

Importance Sampling for $\mathbf{P}(\bar{S}_n \in A)$

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- Construct an alternative universe
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  - i.e., Sample $\mathbb{I}^{(1)}_{\{\bar{S}_n \in A\}}, \ldots, \mathbb{I}^{(m)}_{\{\bar{S}_n \in A\}}$ from $\mathbf{Q}_n$

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Finding a good alternative universe $\mathbf{Q}_n$ is crucial.

Goal: Strongly Efficient IS Estimator

$\mathbb{I}_{\{\bar{S}_n \in A\}} \dfrac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}$ is a **strongly efficient** estimator for $\mathbf{P}(\bar{S}_n \in A)$, if

$$\mathbf{E}^{\mathbf{Q}_n}\left(\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}\right)^2 \quad \sim \quad \mathbf{P}(\bar{S}_n \in A)^2$$

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Error ^{2} Target Quantity^2

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$\Rightarrow$ Number of simulation runs required remains bounded.

Goal: Strongly Efficient IS Estimator

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$\Rightarrow$ Number of simulation runs required remains bounded.

Notoriously Hard for Heavy-Tailed Processes.

What is a good alternate universe $\mathbf{Q}_n$ for $\mathbf{P}(\bar{S}_n \in A)$?

General principle for making $\mathbb{I}_{\{\bar{S}_n \in A\}} \dfrac{d\mathbf{P}}{d\mathbf{Q}_n}$ an efficient estimator:

- **Choose $\mathbf{Q}_n(\cdot)$ as close to $\mathbf{P}(\cdot|\bar{S}_n \in A)$ as possible.**

- **Make sure that $\frac{d\mathbf{P}}{d\mathbf{Q}_n}$ does not blow up.**

First All-Purpose Simulation Scheme for Heavy-Tails

- Fix $w \in (0, 1)$ and define

$$\mathbf{Q}_n(\,\cdot\,) \triangleq w\mathbf{P}(\,\cdot\,) + (1 - w)\mathbf{P}(\,\cdot\,|\,\bar{S}_n \in B^\gamma)$$

First All-Purpose Simulation Scheme for Heavy-Tails

- Fix $w \in (0,1)$ and define $\frac{d\mathbf{P}}{d\mathbf{Q}_n}$ not too big

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First All-Purpose Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{\text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma\} \;\Rightarrow\; \mathcal{J}(A) = \mathcal{J}(B^\gamma)$
- Fix $w \in (0,1)$ and define $\frac{d\mathbf{P}}{d\mathbf{Q}_n}$ not too big

$\mathbf{Q}_n(\cdot)$ close to $\mathbf{P}(\cdot|\bar{S}_n \in A)$

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- $B^\gamma \triangleq \{$paths w/ at least $\mathcal{J}(A)$ jumps of size $> \gamma\}$ $\Rightarrow$ $\mathcal{J}(A) = \mathcal{J}(B^\gamma)$

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- If we choose γ so that $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$,

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- $B^\gamma \triangleq \{$paths w/ at least $\mathcal{J}(A)$ jumps of size $>\gamma\}$

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First All-Purpose Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{\text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma\} \Rightarrow \mathcal{J}(A) = \mathcal{J}(B^\gamma)$

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First All-Purpose Simulation Scheme for Heavy-Tails

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$$\sim \left(n^{-\alpha\mathcal{J}(A)}\right)^2$$

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First All-Purpose Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{$paths w/ at least $\mathcal{J}(A)$ jumps of size $>\gamma\}$ $\Rightarrow$ $\mathcal{J}(A) = \mathcal{J}(B^\gamma)$

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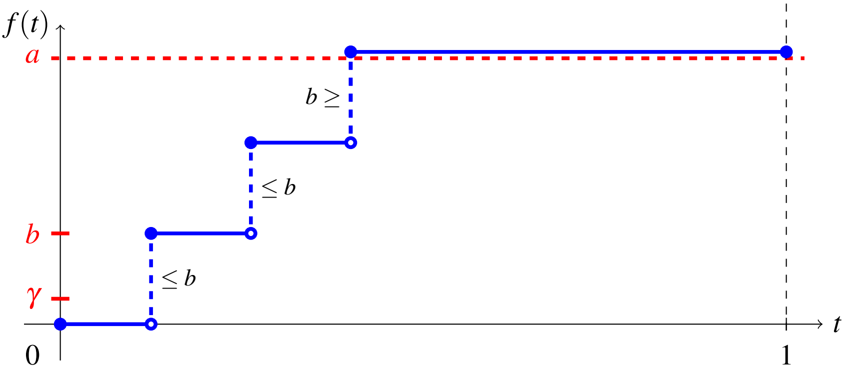
How to ensure $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$: Reinsurance Example



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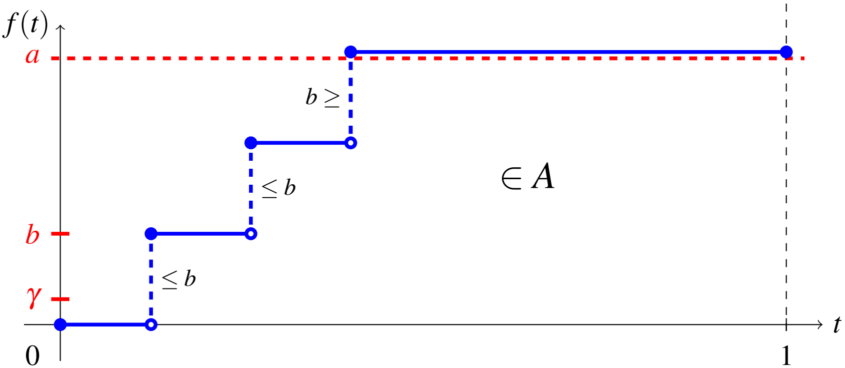

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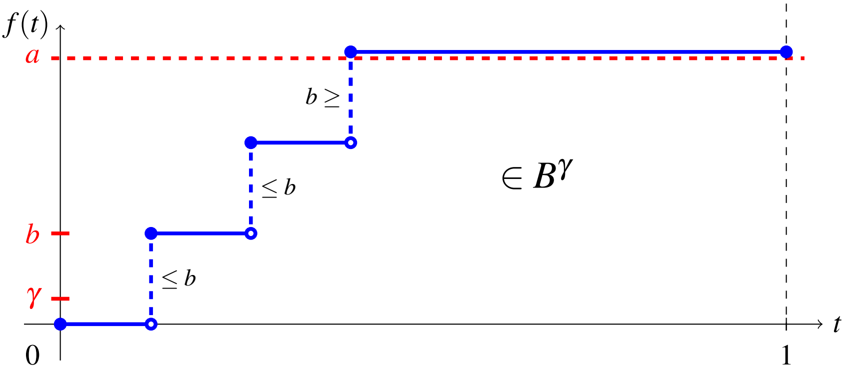

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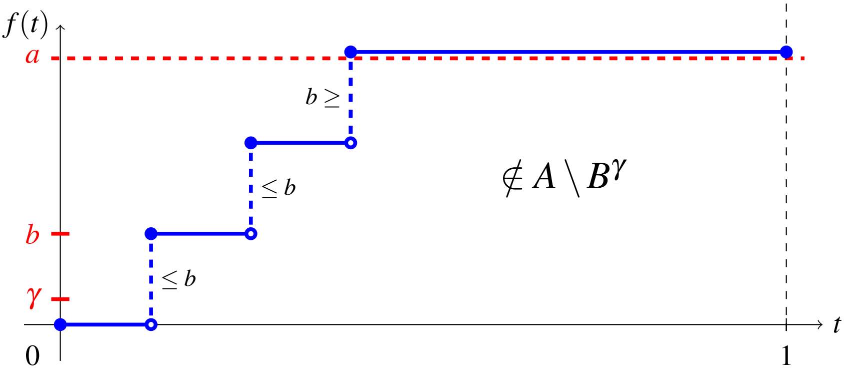

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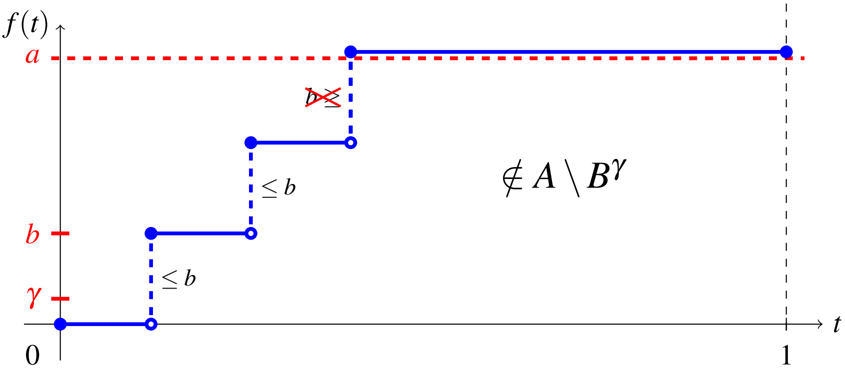

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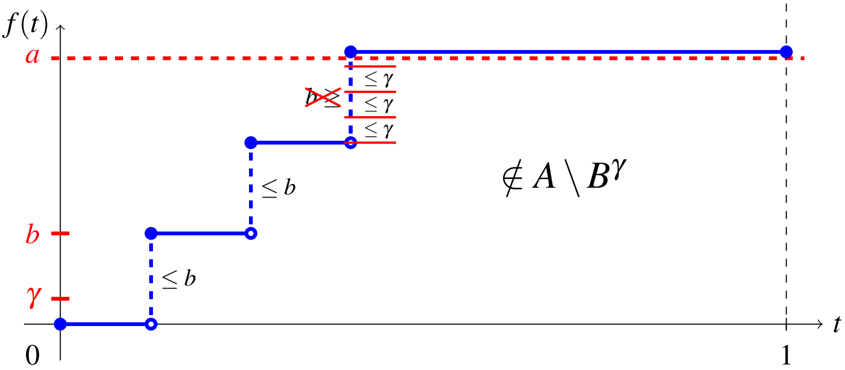

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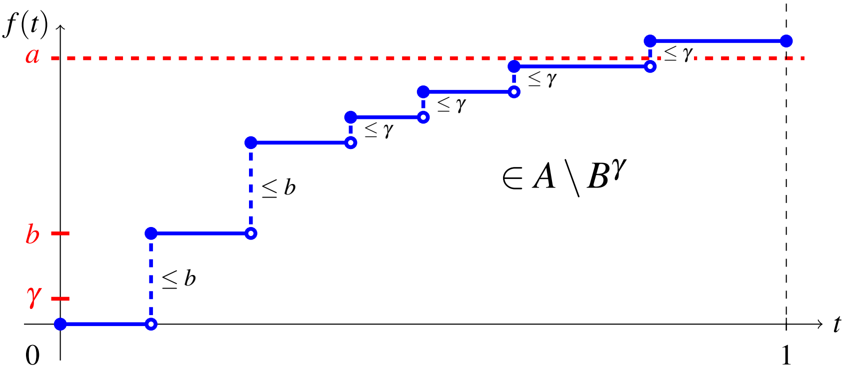

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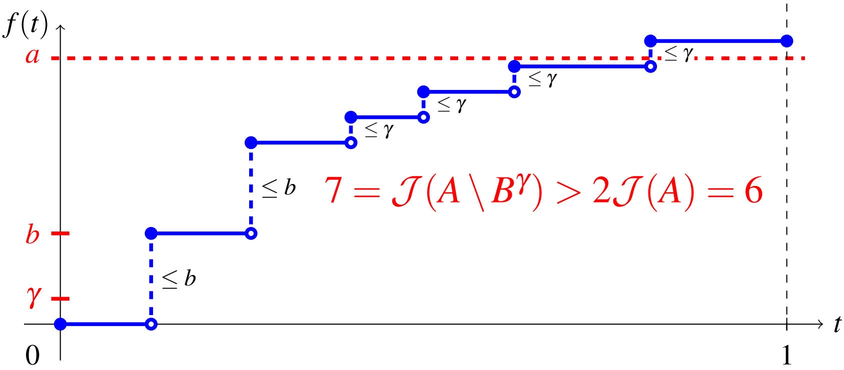

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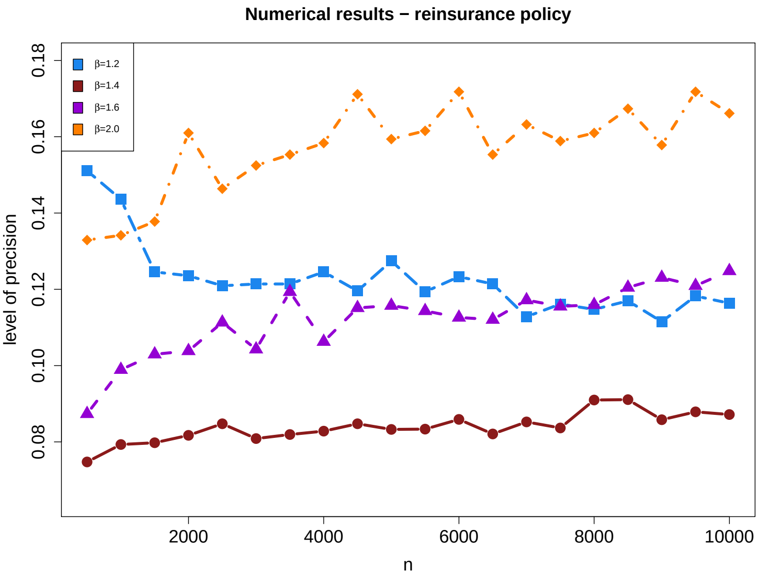
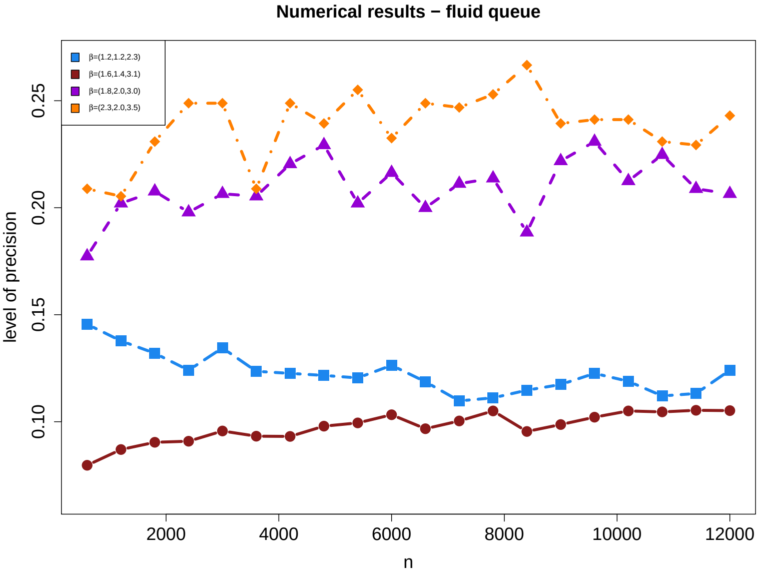
Numerical Experiments for Reinsurance Example



Numerical Results for Queueing Network



Part 3. Large Deviations for Weibull Tails

$$\mathbf{P}(X_i \geq x) = \exp(-x^{\alpha}),\ \ \alpha \in (0,1)$$

Rate function depends on the size of the jump

Theorem (Bazhba, Blanchet, R., Zwart 2017)

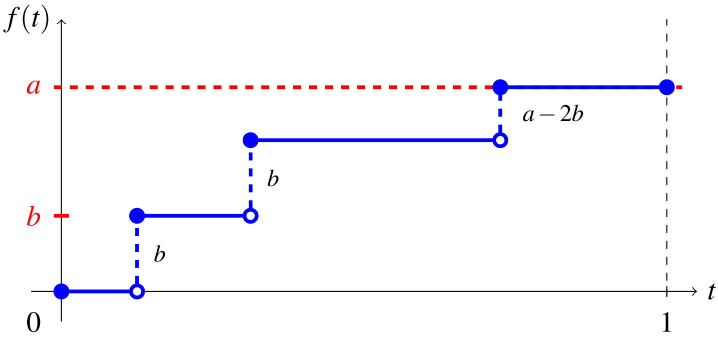
Suppose that $\mathbf{P}(X_i \geq x) = \exp(-x^{\alpha})$, $\alpha \in (0,1)$. *Then*

$$\mathbf{P}(\bar{S}_n \in A) \sim \exp(-n^{\alpha} \inf_{f \in A} I(f))$$

where

$$I(f) = \begin{cases} \sum_t \big(f(t) - f(t-)\big)^{\alpha}, & \text{if } f \text{ is a nondecreasing step function} \\ \infty, & \text{otherwise.} \end{cases}$$

Back to our old example



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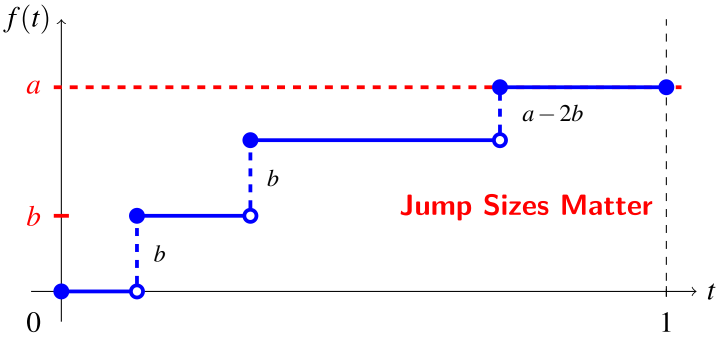

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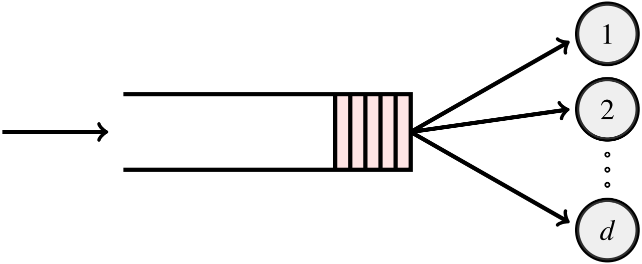
In case service time distribution is Weibull, i.e., $\mathbf{P}(S > x) = \exp(-x^{\alpha})$,



Tail Asymptotics? Most likely scenario?

So far, NOT EVEN a reasonable conjecture!

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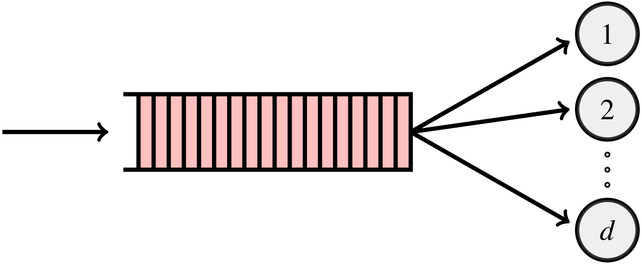

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Solution to Open Problem Posed by Whitt (2000)

Summary

- **Systematic tools** for rare-event analysis of heavy-tailed systems

- The **principle of multiple (least expensive) big jumps**

- Solved **longstanding open problems** in simulation and queueing theory

- **New connections** btwn rare event analysis and (discrete) optimization

  e.g., impulse control, knapsack problem, L^α-norm minimization $\alpha < 1$