On Heavy-Tailed Rare Event Analysis

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Rare Events
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Although rare, rare events matter.

Need for understanding ‘how often?’ & ‘why?’
Rare Events depend on “Tail Behaviors”

Light-Tailed Distributions
- Extreme Values are Very Rare
- Normal, Exponential, etc

Heavy-Tailed Distributions
- Extreme Values are Frequent
- Power Law, Weibull, etc
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Structural difference in the way systemwide rare events arise.
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Systemwide rare events arise because EVERYTHING goes wrong.

(Conspiracy Principle)

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Systemwide rare events arise because of A FEW Catastrophes.

(Catastrophe Principle)

Structural difference in the way systemwide rare events arise.
Light-tailed rare-event analysis has a long & successful history.
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- Large Deviations Theory answers ‘How rare?’, ’Why?’
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Light-tailed rare-event analysis has a long & successful history. In fact, ‘The only plausible scenario!’

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Varadhan won Abel Prize in 2007
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Heavy-tailed rare events are NOT understood well.
But, Heavy Tails are Everywhere:

- Computer systems
  - delays, files, ...
- Finance
  - losses
- Social networks
  - popularity, contagion
- Energy Systems
  - blackouts

$P_{\text{size}} > n \approx 1/n^\alpha$
For Example: Open Problem Posed by Whitt (2000)

Congestion of Multiple Server Queue:

How many big jobs are needed to create large queue lengths?

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In many applications, one can write the rare event of interest as

\[ \{\tilde{S}_n \in A\} \]

where \( \tilde{S}_n \) is the whole trajectory of a random walk.

We want to understand \( P(\tilde{S}_n \in A) \) and \( P(\tilde{S}_n \in \cdot | \tilde{S}_n \in A) \) for as general \( A \) as possible.
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How rare?

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Goal: Systematic Tools for Heavy-Tailed Systems

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- $S_k \triangleq X_1 + \cdots + X_k$.

- $X_i$: iid, $\mathbb{E}X_i = 0$, $\mathbb{P}(X_i \geq x) = x - (\alpha + 1)$, $\alpha > 0$ or $\mathbb{P}(X_i \geq x) = \exp(-x^\alpha)$, $\alpha \in (0, 1)$. 

$\mathbb{P}(\bar{S}_n \in A) \to 0$ - How fast?

$\mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \to ?$ - What is the most likely scenario?

$\mathbb{P}(\bar{S}_n \in \cdot) = ?$ - How to compute?
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- \( \bar{S}_n(t) \triangleq S_{[nt]}/n, \quad t \in [0, 1] \) (scaled process)
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- How fast?
- What is the most likely scenario?
- How to compute?

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\mathbb{P}(\tilde{S}_n(0) \in \cdot | \tilde{S}_n(t) \in A) \to ?
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Heavy-Tailed

- $\bar{S}_{30}(t)$
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\setlength{boxsep}{1pt}
\fbox{\begin{flushright}
\includegraphics[width=\columnwidth]{figure.png}
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- What is the most likely scenario? $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \to \ ?$

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• $\mathbb{P}(\bar{S}_n \in A) \to 0$ for “general” $A \subseteq \mathbb{D}$

Goal
- How fast? $\quad \mathbb{P}(\bar{S}_n \in A) \sim ?$
- What is the most likely scenario? $\quad \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \to ?$
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$\mathbb{P}(\bar{S}_n \in A) = \ ?$
Outline

Part 1. Large Deviations for Power Law Tails

R., Blanchet, Zwart (2016)
Under second round review at Annals of Probability

Part 2. Heavy-Tailed Rare Event Simulation

Chen, Blanchet, R., Zwart (2017)
Mathematics of Operations Research

Part 3. Large Deviations for Weibull Tails

Bazbha, Blanchet, R., Zwart (2017)
Submitted to Annals of Applied Probability
Part 1. Large Deviations for Power Law Tails

\[ i.e., \ P(X_i \geq x) = x^{-(\alpha+1)}, \ \alpha > 0 \]
What’s already known: principle of a single big jump

“In heavy-tailed systems, rare events arise due to one big anomaly.”
What’s already known: principle of a single big jump

\[ A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \} \]

\[ \mathbf{P}(\bar{S}_n \in A) \] (Ruin Probability of Insurance Firm)
What’s already known: principle of a single big jump

\[ f(t) \]

\[ a \]

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This is typical $\bar{S}_n | \bar{S}_n \in A$
What’s already known: principle of a single big jump

- $A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \}$
- $P(\bar{S}_n \in A) \sim c n^{-\alpha}$ (Ruin Probability of Insurance Firm)
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\[ \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow \mathbb{P}(Z \mathbb{1}_{[U \leq t]} \in \cdot) \text{ for some r.v.-s } Z \text{ and } U \]
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Hult, Lindskog, Mikosch, Samorodnitsky (2005)
Are all heavy-tailed rare events due to a single big jump?

No, by no means, absolutely not:

- Multiple server queues
- Queueing networks
- Re-insured insurance line
- Down-and-in barrier option
- Many more
Principle of a single big jump is just a tip of the iceberg!
Illustration: What if Large Claims are Reinsured?

\[ f(t) \]

- \( A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \text{ and jump sizes } \leq b \} \)

- \( \mathbb{P}(\bar{S}_n \in A) \sim cn^{-\alpha} \)

- \( \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \overset{?}{\to} \mathbb{P}(Z\mathbb{1}_{U \leq t} \in \cdot) \) for some r.v.-s \( Z \) and \( U \)
Illustration: What if Large Claims are Reinsured?

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• $A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b \}$

• $\mathbb{P}(\tilde{S}_n \in A) \not\sim cn^{-\alpha}$

• $\mathbb{P}(\tilde{S}_n \in \cdot | \tilde{S}_n \in A) \overset{\Delta}{=} \mathbb{P}(Z1_{[U \leq t]} \in \cdot)$ for some r.v.-s $Z$ and $U$
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- $\mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \not\rightarrow \mathbb{P}(Z \mathbb{1}_{[U \leq t]} \in \cdot)$ for some r.v.-s $Z$ and $U$
Illustration: What if Large Claims are Reinsured?

- $A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b \}$

- $P(\bar{S}_n \in A) \not\sim cn^{-\alpha}$

- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \not\sim P(Z \mathbb{1}_{[U \leq t]} \in \cdot)$ for some r.v.-s $Z$ and $U$
Illustration: What if Large Claims are Reinsured?

- \( A = \{ f \in \mathbb{D} : f \) crosses level \( a \) on \([0, 1]\) & jump sizes \( \leq b \}\)

- \( \mathbb{P}(\tilde{S}_n \in A) \not\sim cn^{-\alpha} \)

- \( \mathbb{P}(\tilde{S}_n \in \cdot | \tilde{S}_n \in A) \not\sim \mathbb{P}(Z \mathbb{1}_{[U \leq t]} \in \cdot) \) for some r.v.-s \( Z \) and \( U \)
Illustration: What if Large Claims are Reinsured?

- $A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b \}$

- $P(\bar{S}_n \in A) \not\sim cn^{-\alpha} \Rightarrow (cn^{-\alpha})^3$

- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \not\sim P(Z1_{[U \leq t]} \in \cdot)$ for some r.v.-s $Z$ and $U$
Illustration: What if Large Claims are Reinsured?

\[ A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \& \text{ jump sizes } \leq b \} \]

\[ P(\bar{S}_n \in A) \not\sim cn^{-\alpha} \Rightarrow (cn^{-\alpha})^3 \]

\[ P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \not\sim P(Z 1_{[U \leq t]} \in \cdot) \Rightarrow P(\sum_{i=1}^{3} Z_i 1_{[U_i \leq t]} \in \cdot) \]
Illustration: What if Large Claims are Reinsured?

- $A = \{f \in \mathbb{D} : f$ crosses level $a$ on $[0, 1]$ & jump sizes $\leq b\}$

- $P(\bar{S}_n \in A) \not\sim c n^{-\alpha} \Rightarrow (c n^{-\alpha})^3$

- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \not\sim P(Z \mathbb{1}_{[U \leq t]} \in \cdot) \Rightarrow P\left(\sum_{i=1}^{3} Z_i \mathbb{1}_{[U_i \leq t]} \in \cdot\right)$
Exact Asymptotics for Heavy-tailed Random Walks

\[ P(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0 \]

**Theorem (R., Blanchet, Zwart, 2017)**

For “general” \( A \subseteq \mathbb{D} \)

\[ C(A^\circ) \leq \liminf_{n \to \infty} \frac{P(\bar{S}_n \in A)}{n^{-\alpha J(A)}} \leq \limsup_{n \to \infty} \frac{P(\bar{S}_n \in A)}{n^{-\alpha J(A)}} \leq C(A^-). \]

- \( J(A) \): min \# jumps for step functions to be inside \( A \)
- \( C(\cdot) \): a measure
\( P(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0 \)

**Theorem (R., Blanchet, Zwart, 2017)**

For “general” \( A \subseteq \mathbb{D} \)

\[ P(\bar{S}_n \in A) \sim n^{-\alpha J(A)} \]

- \( J(A) \): min \# jumps for step functions to be inside \( A \)
Exact Asymptotics for Heavy-tailed Random Walks

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**Theorem (R., Blanchet, Zwart, 2017)**

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\[ P(\bar{S}_n \in A) \sim n^{-\alpha J(A)} \]

- \( J(A) \): min \# jumps for step functions to be inside \( A \)
Back to Our Reinsurance Example: Conjecture Confirmed!

\[ f(t) \]

\[ a \]

\[ b \]

\[ 0 \]

\[ 1 \]

\[ t \]

- \[ A = \{ f \in D : f \text{ crosses level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b \} \]

- \[ P(\bar{S}_n \in A) \sim n^{-\lfloor a/b \rfloor} \alpha \]

- \[ P(\bar{S}_n \in A) \rightarrow P\left( \sum_{i=1}^{\lfloor a/b \rfloor} Z_i 1_{[U_i \leq t]} \in \cdot \right) \]
Conspiracy vs Catastrophy

\[ A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b \} \]

Light-Tailed Claim Size

Reinsurance makes no difference.

Heavy-Tailed Claim Size

Reinsurance helps!
Conspiracy vs Catastrophy

\[ A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b \} \]

Light-Tailed Claim Size

Heavy-Tailed Claim Size

Reinsurance makes no difference.
Conspiracy vs Catastrophe

\[ A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b \} \]

Light-Tailed Claim Size

Reinsurance makes no difference.

Heavy-Tailed Claim Size

Reinsurance helps!
Two-sided Random Walks

\[ P(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0 \quad \text{and} \quad P(X_i \leq -x) = x^{-(\beta+1)}, \quad \beta > 0. \]

**Theorem (R., Blanchet, Zwart, 2017)**

For “general” \( A \subseteq \mathbb{D} \),

\[ P(\bar{S}_n \in A) \sim n^{-\{\alpha J(A) + \beta K(A)\}}. \]

- \( J(A) \): \# of upward jumps
- \( K(A) \): \# of downward jumps

of step functions that minimize the cost of staying inside \( A \)
Connection to Impulse Control Problem

The “rate of decay” is determined by a discrete optimization problem:

\[
\alpha J(A) + \beta K(A) = \min_{j,k} \alpha j + \beta k
\]

subject to \((j, k) \in \mathbb{Z}_+^2\)

\[
\mathbb{D}_{j,k} \cap A \neq \emptyset
\]

where

\[
\mathbb{D}_{j,k} = \{\text{step functions w/ } j \text{ upward jumps and } k \text{ downward jumps}\}
\]

Different from variational calculus that arises in light-tailed case!
Example: Barrier Option

- $A = \{ f \in \mathbb{D} : f \text{ is below } b \text{ at some point & end up above } a \}$

- $\mathbb{P}(\bar{S}_n \in A) \sim$ ?

- $\mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow$ ?
Example: Barrier Option

- $A = \{ f \in \mathbb{D} : f \text{ is below } b \text{ at some point } \& \text{ end up above } a \}$

- $P(\bar{S}_n \in A) \sim ?$

- $P(\bar{S}_n \in | \bar{S}_n \in A) \rightarrow ?$
Example: Barrier Option

- $A = \{ f \in \mathbb{D} : f \text{ is below } b \text{ at some point } \& \text{ end up above } a \}$

- $P(\bar{S}_n \in A) \sim \ ?$

- $P(\bar{S}_n \in A | \bar{S}_n \in A) \rightarrow \ ?$
Example: Barrier Option

\[ f(t) \]

\[ a \]

\[ b \]

\[ \mathcal{K}(A) = 1 \]

\[ \mathcal{J}(A) = 1 \]

\[ 0 \]

\[ 1 \]

- \[ A = \{ f \in \mathbb{D} : f \text{ is below } b \text{ at some point \& end up above } a \} \]

- \[ \mathbb{P} (\bar{S}_n \in A) \sim n^{-(1\alpha + 1\beta)} \]

- \[ \mathbb{P} (\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \]
Example: Barrier Option

\[ A = \{ f \in \mathbb{D} : f \text{ is below } b \text{ at some point } \& \text{ end up above } a \} \]

\[ P(\bar{S}_n \in A) \sim n^{-(1\alpha + 1\beta)} \]

\[ P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow P(\mathbb{1}_{[U_1 \leq t]} - \mathbb{1}_{[U_2 \leq t]} \in \cdot) \]
Example: Sausage

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)
- \( \mathbb{P}(\tilde{S}_n \in A) \sim ? \)
- \( \mathbb{P}(\tilde{S}_n \in \cdot | \tilde{S}_n \in A) \rightarrow ? \)
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $\mathbb{P}(\bar{S}_n \in A) \sim ?$
- $\mathbb{P}(\bar{S}_n \in A | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)
- \( \mathbb{P}(\bar{S}_n \in A) \sim ? \)
- \( \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \)
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$

- $\mathbb{P}(\bar{S}_n \in A) \sim ?$

- $\mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{f \in \mathbb{D} : f \text{ stays between the two curves}\}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in A | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

\[ A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \]

\[ P(\bar{S}_n \in A) \sim ? \]

\[ P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \]
Example: Sausage

\[ f(t) \]

\[ \mathcal{J}(A) \geq 1 \]

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)

- \( \mathbb{P}(\bar{S}_n \in A) \sim ? \)

- \( \mathbb{P}(\bar{S}_n \in A) \rightarrow ? \)

\[ \mathcal{K}(A) = ? \]

\[ \sum_{j=1}^{3} Z_j + \sum_{k=1}^{2} Z_k \leq t \]
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in A | \bar{S}_n \in A) \to ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$

- $P(\tilde{S}_n \in A) \sim ?$

- $P(\tilde{S}_n \in \cdot | \tilde{S}_n \in A) \rightarrow ?$
Example: Sausage

\[ A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \]

\[ \mathbb{P}(\bar{S}_n \in A) \sim ? \]

\[ \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \]
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$

- $P(\bar{S}_n \in A) \sim ?$

- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim \ ?$
- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow \ ?$
Example: Sausage

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)
- \( \mathbb{P}(\bar{S}_n \in A) \sim ? \)
- \( \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \)

\[ f(t) \]

\[ K(A) = ? \]

\[ J(A) \geq 2 \]
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $\mathbb{P}(\bar{S}_n \in A) \sim ?$
- $\mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

\[ f(t) \]

\[ J(A) \geq 2 \]

\[ K(A) = ? \]

\[ 0 \]

\[ 1 \]

- \( A = \{ f \in D : f \text{ stays between the two curves} \} \)

- \( P(\bar{S}_n \in A) \sim ? \)

- \( P(\bar{S}_n \notin A) \rightarrow ? \)
Example: Sausage

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)
- \( P(\bar{S}_n \in A) \sim ? \)
- \( P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \)
Example: Sausage

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)
- \( P(\bar{S}_n \in A) \sim ? \)
- \( P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \to ? \)

\( f(t) \)

\( K(A) = ? \)

\( J(A) \geq 2 \)
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in A | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in A | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

\[\mathcal{K}(A) = ?\]

\[f(t)\]

\[\mathcal{J}(A) \geq 2\]

- \(A = \{f \in \mathbb{D} : f \text{ stays between the two curves}\}\)

- \(P(\bar{S}_n \in A) \sim ?\)

- \(P(\bar{S}_n \in A) \rightarrow ?\)
Example: Sausage

- $A = \{f \in \mathbb{D} : f \text{ stays between the two curves}\}$

- $P(\bar{S}_n \in A) \sim ?$

- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n = \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

\( f(t) \)

\( K(A) = ? \)

\( J(A) \geq 2 \)

- \( A = \{ f \in \mathbb{D} : f \) stays between the two curves\}

- \( P(\tilde{S}_n \in A) \sim ? \)

- \( P(\tilde{S}_n \in A) \rightarrow ? \)
Example: Sausage

\[ A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \]

\[ \mathbb{P}(\bar{S}_n \in A) \sim ? \]

\[ \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \]
Example: Sausage

- $A = \{f \in \mathbb{D} : f \text{ stays between the two curves}\}$
- $\mathbb{P}(\bar{S}_n \in A) \sim ?$
- $\mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

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- \( P(\bar{S}_n \in A) \sim ? \)
- \( P(\bar{S}_n \in A | \bar{S}_n \in A) \rightarrow ? \)
Example: Sausage

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- $P(\bar{S}_n \in A) \sim ?$

- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
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- \( \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \)
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in A | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\tilde{S}_n \in A) \sim ?$
- $P(\tilde{S}_n \in A | \tilde{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $\mathbb{P}(\bar{S}_n \in A) \sim ?$
- $\mathbb{P}(\bar{S}_n \in A | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim ?$
- $P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $\mathbb{P}(\bar{S}_n \in A) \sim ?$
- $\mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$

- $\mathbb{P}(\bar{S}_n \in A) \sim \ ?$

- $\mathbb{P}(\bar{S}_n \in A) \rightarrow \ ?$
Example: Sausage

- $A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \}$
- $\mathbb{P}(\bar{S}_n \in A) \sim \cdots$
- $\mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow \cdots$
Example: Sausage

\[ f(t) \]

\[ \mathcal{J}(A) = 3 \]

\[ \mathcal{K}(A) = 2 \]

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)

- \( \mathbb{P}(\bar{S}_n \in A) \sim ? \)

- \( \mathbb{P}(\bar{S}_n \in A) \rightarrow ? \)
Example: Sausage

\[ f(t) \]

\[ \mathcal{J}(A) = 3 \]

\[ \mathcal{K}(A) = 2 \]

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)

- \( \mathbb{P}(\tilde{S}_n \in A) \sim ? \)

- \( \mathbb{P}(\tilde{S}_n \in \cdot | \tilde{S}_n \in A) \rightarrow ? \)
Example: Sausage

- \( A = \{ f \in \mathbb{D} : f \text{ stays between the two curves} \} \)
- \( \mathbb{P}(\bar{S}_n \in A) \sim n^{-(3\alpha + 2\beta)} \)
- \( \mathbb{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow ? \)
Example: Sausage

- \( A = \{ f \in \mathbb{D} : f \) stays between the two curves\} 

- \( P(\bar{S}_n \in A) \sim n^{-(3\alpha + 2\beta)} \)

- \( P(\bar{S}_n \in \cdot | \bar{S}_n \in A) \rightarrow P(\sum_{j=1}^{3} Z_j^+ 1_{U_j^+ \leq t} - \sum_{k=1}^{2} Z_k^- 1_{U_k^- \leq t} \in \cdot) \)
Example: Sausage

- $A = \{ f \in D : f \text{ stays between the two curves} \}$
- $P(\bar{S}_n \in A) \sim \exp(-nI^*)$

$I^*$: sol’n of optimization problem over absolutely continuous functions.
Example: Sausage

Conspiracy
Connection to variational problems
(continuous optimization)

Catastrophy
Connection with impulse control
(discrete optimization)
Lévy Processes and Multidimensional Processes

- Same results for Lévy processes.

- Similar results for vector valued processes with independent components as well (for both Lévy processes and random walks).

Analysis of Many Server Queues and even Queueing Networks!
Example - Stochastic Fluid Network

- Job size distribution: Pareto with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$

- Queue 3 experiences congestion because of what?

- Optimization problem reduces to a knapsack type problem
Example - Stochastic Fluid Network

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Example - Stochastic Fluid Network

Job size distribution: Pareto with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$

Queue 3 experiences congestion because of what?

Optimization problem reduces to a knapsack type problem

$\rho_1 = 0.8$

$\rho_2 = 0.8$

$\rho_3 = 1$

Cost = $\alpha_1 = 1$

Congestion!

(if $\mu_3 < 1 + 0.8 + 0.72 = 2.52$)
Example - Stochastic Fluid Network

- Job size distribution: Pareto with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$

- Queue 3 experiences congestion because of what?

- Optimization problem reduces to a knapsack type problem
Example - Stochastic Fluid Network

- **Queue 1**
  - \( \mu_1 = 1 \)
  - 80% output
  - 10% input

- **Queue 2**
  - \( \mu_2 = 1 \)
  - 80% output
  - 10% input

- **Queue 3**
  - \( \mu_3 \)
  - 10% input

\( \rho_1 = 0.8 \)
\( \rho_2 = 0.8 \)
\( \rho_3 = 1 \)

- **Job size distribution**: Pareto with \( \alpha_1 = 1 \), \( \alpha_2 = 1 \), \( \alpha_3 = 3 \)

- **Queue 3 experiences congestion** because of what?

- **Optimization problem reduces to a knapsack type problem**
Example - Stochastic Fluid Network

- Job size distribution: Pareto with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$

- Queue 3 experiences congestion because of what?

- Optimization problem reduces to a knapsack type problem
Example - Stochastic Fluid Network

- Job size distribution: Pareto with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$

- Queue 3 experiences congestion because of what?

- Optimization problem reduces to a knapsack type problem

Congestion! (if $\mu_3 < 1 + 0.8 + 0.8 = 2.6$)

Cost = $\alpha_1 + \alpha_2 = 2$
Example - Stochastic Fluid Network

- Job size distribution: Pareto with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$
- Queue 3 experiences congestion because of what?
- Optimization problem reduces to a knapsack type problem
Example - Stochastic Fluid Network

- Job size distribution: Pareto with $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 3$

- Queue 3 experiences congestion because of what?

- Optimization problem reduces to a knapsack type problem
Part 2. Heavy-Tailed Rare Event Simulation

Computing $\mathbf{P}(\bar{S}_n \in A)$ for finite $n$
Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $P(\text{Head})$

- Flip the coin 100 times
- Count the number of head
- Divide by 100 and report the number

Should be reasonably close to 1/2
Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $P(\text{Edge})$

- Flip the coin 100 times
- Count the number of $\text{Edge}$
- Divide by 100 and report the number
Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $P(Edge)$

- Flip the coin 100 times

- Count the number of Edge: most likely to be 0

- Divide by 100 and report the number
Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\Pr(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge: most likely to be 0
- Divide by 100 and report the number: and hence, likely to be 0
Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $P(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge: most likely to be 0
- Divide by 100 and report the number: and hence, likely to be 0

Is 0 a useful answer?
Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $P(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number : and hence, likely to be 0

Is 0 a useful answer? No.

e.g., Nuclear Meltdown, Large-Scale Blackout, Large Financial Loss
Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

- Suppose \( \approx 10^{-6} \)
- Flip the coin 100 times
- Count the number of Edge
- Divide by 100 and report the number
Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $P(\text{Edge}) \approx 10^{-6}$

- Flip the coin 100 a few million times
- Count the number of Edge
- Divide by the total number of flips and report the number
Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $P(\text{Edge}) \approx 10^{-6}$

- Flip the coin 100 a few million times
- Count the number of Edge
- Divide by the total number of flips and report the number

Much harder than $P(\text{Head})$
Importance Sampling

- Construct an alternative universe
Importance Sampling

- Construct an alternative universe
- Perform experiments there

Finding a good alternative universe is crucial.
Importance Sampling

- Construct an alternative universe

- Perform experiments there

- Recover the true probability using the relationship between the two parallel universes
Importance Sampling

- Construct an alternative universe
  - i.e., Consider an ”importance distribution” $Q$

- Perform experiments there

- Recover the true probability using the relationship between the two parallel universes
Importance Sampling

- Construct an alternative universe
  - i.e., Consider an "importance distribution" $Q$

- Perform experiments there
  - i.e., Sample $\Pi^{(1)}_{\text{Edge}}, \ldots, \Pi^{(m)}_{\text{Edge}}$ from $Q$

- Recover the true probability using the relationship between the two parallel universes
Importance Sampling

- Construct an alternative universe
  - i.e., Consider an "importance distribution" $Q$

- Perform experiments there
  - i.e., Sample $P_{\text{Edge}}^{(1)}, \ldots, P_{\text{Edge}}^{(m)}$ from $Q$
    as well as the likelihood ratio $\left(\frac{dP}{dQ}\right)^{(1)}, \ldots, \left(\frac{dP}{dQ}\right)^{(m)}$

- Recover the true probability using the relationship between the two parallel universes
Importance Sampling

- Construct an alternative universe
  - i.e., Consider an "importance distribution" $Q$

- Perform experiments there
  - i.e., Sample $I_{\text{Edge}}^{(1)}, \ldots, I_{\text{Edge}}^{(m)}$ from $Q$
    
    as well as the likelihood ratio $(\frac{dP}{dQ})^{(1)}, \ldots, (\frac{dP}{dQ})^{(m)}$

- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m} \sum_{i=1}^{m} I_{\text{Edge}}^{(i)} (\frac{dP}{dQ})^{(i)}$ as an estimate of $P(\text{Edge})$

Finding a good alternative universe $Q$ is crucial.
Importance Sampling for $\mathbf{P}(\tilde{S}_n \in A)$

- Construct an alternative universe
  - i.e., Consider an ”importance distribution” $Q$

- Perform experiments there
  - i.e., Sample $\mathbb{I}_{\text{Edge}}^{(1)}, \ldots, \mathbb{I}_{\text{Edge}}^{(m)}$ from $Q$
    as well as the likelihood ratio $\left( \frac{d\mathbf{P}}{dQ} \right)^{(1)}, \ldots, \left( \frac{d\mathbf{P}}{dQ} \right)^{(m)}$

- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m} \sum_{i=1}^{m} \mathbb{I}_{\text{Edge}}^{(i)} \left( \frac{d\mathbf{P}}{dQ} \right)^{(i)}$ as an estimate of $\mathbf{P}(\text{Edge})$
Importance Sampling for $P(\bar{S}_n \in A)$

- Construct an alternative universe
  - i.e., Consider an “importance distribution” $Q_n$

- Perform experiments there
  - i.e., Sample $\Pi^{(1)}_{\{\bar{S}_n \in A\}}, \ldots, \Pi^{(m)}_{\{\bar{S}_n \in A\}}$ from $Q_n$
    as well as the likelihood ratio $\left(\frac{dP}{dQ_n}\right)^{(1)}, \ldots, \left(\frac{dP}{dQ_n}\right)^{(m)}$

- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m} \sum_{i=1}^{m} \Pi^{(i)}_{\{\bar{S}_n \in A\}} \left(\frac{dP}{dQ_n}\right)^{(i)}$ as an estimate of $P(\bar{S}_n \in A)$
Importance Sampling for $P(\bar{S}_n \in A)$

- Construct an alternative universe
  - i.e., Consider an ”importance distribution” $Q_n$

- Perform experiments there
  - i.e., Sample $\mathbb{I}_{\{\bar{S}_n \in A\}}^{(1)}, \ldots, \mathbb{I}_{\{\bar{S}_n \in A\}}^{(m)}$ from $Q_n$
    as well as the likelihood ratio $(\frac{dP}{dQ_n})^{(1)}, \ldots, (\frac{dP}{dQ_n})^{(m)}$

- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m} \sum_{i=1}^{m} \mathbb{I}_{\{\bar{S}_n \in A\}}^{(i)} (\frac{dP}{dQ_n})^{(i)}$ as an estimate of $P(\bar{S}_n \in A)$
    $\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n}$ : IS estimator
Importance Sampling for $P(\overline{S}_n \in A)$

- Construct an alternative universe
  - i.e., Consider an ”importance distribution” $Q_n$

- Perform experiments there
  - i.e., Sample $\mathbb{I}_{\{\overline{S}_n \in A\}}^{(1)}, \ldots, \mathbb{I}_{\{\overline{S}_n \in A\}}^{(m)}$ from $Q_n$

    as well as the likelihood ratio $(\frac{dP}{dQ_n})^{(1)}, \ldots, (\frac{dP}{dQ_n})^{(m)}$

- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m} \sum_{i=1}^{m} \mathbb{I}_{\{\overline{S}_n \in A\}}^{(i)} \left(\frac{dP}{dQ_n}\right)^{(i)}$ as an estimate of $P(\overline{S}_n \in A)$

  $\mathbb{I}_{\{\overline{S}_n \in A\}} \frac{dP}{dQ_n}$ : IS estimator

Finding a good alternative universe $Q_n$ is crucial.
Goal: Strongly Efficient IS Estimator

\[ \mathbb{I}\{\bar{S}_n \in A\} \frac{dP}{dQ_n} \] is a strongly efficient estimator for \( P(\bar{S}_n \in A) \), if

\[ \mathbb{E}_{Q_n} \left( \mathbb{I}\{\bar{S}_n \in A\} \frac{dP}{dQ_n} \right)^2 \sim P(\bar{S}_n \in A)^2 \]
Goal: Strongly Efficient IS Estimator

\[ \mathbb{I}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \] is a strongly efficient estimator for \( P(\bar{S}_n \in A) \), if

\[ \text{Error}^2 \]

\[ E_{Q_n} \left( \mathbb{I}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \sim P(\bar{S}_n \in A)^2 \]
Goal: Strongly Efficient IS Estimator

\[ \mathbb{I}\{\bar{S}_n \in A\} \frac{dP}{dQ_n} \] is a **strongly efficient** estimator for \( P(\bar{S}_n \in A) \), if

\[ \mathbb{E}_{Q_n} \left( \mathbb{I}\{\bar{S}_n \in A\} \frac{dP}{dQ_n} \right)^2 \sim P(\bar{S}_n \in A)^2 \]
Goal: Strongly Efficient IS Estimator

\[ \mathbb{I}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \] is a **strongly efficient** estimator for \( P(\bar{S}_n \in A) \), if

\[
E_{Q_n} \left( \mathbb{I}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \sim \ P(\bar{S}_n \in A)^2
\]

\[ \Rightarrow \] Number of simulation runs required remains bounded.
Goal: Strongly Efficient IS Estimator

\[ \mathbb{I}\{\bar{S}_n \in A\} \frac{dP}{dQ_n} \text{ is a strongly efficient estimator for } P(\bar{S}_n \in A), \text{ if} \]

\[ E^{Q_n} \left( \mathbb{I}\{\bar{S}_n \in A\} \frac{dP}{dQ_n} \right)^2 \sim P(\bar{S}_n \in A)^2 \]

⇒ Number of simulation runs required remains bounded.

Notoriously Hard for Heavy-Tailed Processes.
What is a good alternate universe $Q_n$ for $P(\bar{S}_n \in A)$?

General principle for making $\mathbb{I}\{\bar{S}_n \in A\} \frac{dP}{dQ_n}$ an efficient estimator:

- Choose $Q_n(\cdot)$ as close to $P(\cdot | \bar{S}_n \in A)$ as possible.

- Make sure that $\frac{dP}{dQ_n}$ does not blow up.
First All-Purpose Simulation Scheme for Heavy-Tails

- Fix $w \in (0, 1)$ and define

$$Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)$$
First All-Purpose Simulation Scheme for Heavy-Tails

- Fix $w \in (0, 1)$ and define $\frac{dP}{dQ_n}$ not too big

$$Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)$$
• Fix $w \in (0, 1)$ and define $Q_n(\cdot) \equiv wP(\cdot) + (1-w)P(\cdot | \bar{S}_n \in B^\gamma)$
First All-Purpose Simulation Scheme for Heavy-Tails

- \( B^\gamma \triangleq \{ \text{paths w/ at least } J(A) \text{ jumps of size } > \gamma \} \Rightarrow J(A) = J(B^\gamma) \)

- Fix \( w \in (0, 1) \) and define \( Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma) \) not too big

\[ \text{Q}_n(\cdot) \text{ close to } P(\cdot | \bar{S}_n \in A) \]

\[ Z_n \triangleq 1\{ \bar{S}_n \in A \} \]

\[ Z_n \] is strongly efficient for \( P(\cdot | \bar{S}_n \in A) \)

Chen, Blanchet, R., and Zwart (2017)
First All-Purpose Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{\text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma\} \Rightarrow \mathcal{J}(A) = \mathcal{J}(B^\gamma)$

- Fix $w \in (0, 1)$ and define $\frac{dP}{dQ_n}$ not too big

  \[Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)\]

- If we choose $\gamma$ so that $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$,
First All-Purpose Simulation Scheme for Heavy-Tails

- \( B^\gamma \triangleq \{ \text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma \} \implies \mathcal{J}(A) = \mathcal{J}(B^\gamma) \)

- Fix \( w \in (0, 1) \) and define \( \frac{dP}{dQ_n} \) not too big

  \[ Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma) \]

- If we choose \( \gamma \) so that \( \mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A) \),

  \[
  \mathbb{E}^{Q_n} \left( \mathbb{1}_{\{ S_n \in A \}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \setminus B^\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B^\gamma)
  \]
First All-Purpose Simulation Scheme for Heavy-Tails

- \(B^\gamma \triangleq \{\text{paths w/ at least } J(A) \text{ jumps of size } > \gamma\}\)

- Fix \(w \in (0, 1)\) and define

\[Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)\]

- If we choose \(\gamma\) so that \(J(A \setminus B^\gamma) \geq 2J(A)\),

\[\mathbb{E}^{Q_n} \left( \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \setminus B^\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B^\gamma)\]
• \( B^{\gamma} \triangleq \{ \text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma \} \)

• Fix \( w \in (0, 1) \) and define

\[
Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \tilde{S}_n \in B^{\gamma})
\]

• If we choose \( \gamma \) so that \( \mathcal{J}(A \setminus B^{\gamma}) \geq 2\mathcal{J}(A) \),

\[
\mathbb{E}^{Q_n} \left( \mathbb{1}_{\{\tilde{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} \mathbb{P}(\tilde{S}_n \in A \setminus B^{\gamma}) + \mathbb{P}(\tilde{S}_n \in A) \cdot \mathbb{P}(\tilde{S}_n \in B^{\gamma})
\]
• $B^\gamma \triangleq \{\text{paths w/ at least } J(A) \text{ jumps of size } > \gamma\}$

• Fix $w \in (0, 1)$ and define

$$Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)$$

• If we choose $\gamma$ so that $J(A \setminus B^\gamma) \geq 2J(A)$,

$$E^{Q_n} \left( \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \setminus B^\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B^\gamma)$$
First All-Purpose Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{\text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma\} \Rightarrow \mathcal{J}(A) = \mathcal{J}(B^\gamma)$

- Fix $w \in (0,1)$ and define

  $$Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)$$

- If we choose $\gamma$ so that $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$,

  $$\mathbb{E}_{Q_n} \left( \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \setminus B^\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B^\gamma)$$

\[\sim \begin{align*}
n^{-\alpha \mathcal{J}(A \setminus B^\gamma)} &+ n^{-\alpha \mathcal{J}(B^\gamma)} \\
&+ n^{-\alpha \mathcal{J}(A)} \end{align*}\]
First All-Purpose Simulation Scheme for Heavy-Tails

- \( B^\gamma \triangleq \{ \text{paths w/ at least } J(A) \text{ jumps of size } > \gamma \} \Rightarrow J(A) = J(B^\gamma) \)

- Fix \( w \in (0, 1) \) and define
  \[
  Q_n(\cdot) \triangleq wP(\cdot) + (1-w)P(\cdot | \bar{S}_n \in B^\gamma)
  \]

- If we choose \( \gamma \) so that \( J(A \setminus B^\gamma) \geq 2J(A) \),
  \[
  \mathbb{E}^{Q_n}\left( \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \setminus B^\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B^\gamma)
  \]

\( Z_n \equiv \mathbb{1}_{\{\bar{S}_n \in A\}} \) is strongly efficient for \( P(\bar{S}_n \in A) \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!

Chen, Blanchet, R., and Zwart (2017)
First All-Purpose Simulation Scheme for Heavy-Tails

- \( B^\gamma \triangleq \{ \text{paths w/ at least } J(A) \text{ jumps of size } > \gamma \} \quad \Rightarrow \quad J(A) = J(B^\gamma) \)

- Fix \( w \in (0, 1) \) and define

\[
Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)
\]

- If we choose \( \gamma \) so that \( J(A \setminus B^\gamma) \geq 2J(A) \),

\[
E^{Q_n} \left( \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \setminus B^\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B^\gamma)
\]

\[
\sim (n^{-\alpha J(A)})^2
\]
First All-Purpose Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{\text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma\} \Rightarrow \mathcal{J}(A) = \mathcal{J}(B^\gamma)$

- Fix $w \in (0, 1)$ and define

  $$Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)$$

- If we choose $\gamma$ so that $\mathcal{J}(A \backslash B^\gamma) \geq 2\mathcal{J}(A)$,

  $$E^{Q_n} \left( \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \backslash B^\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B^\gamma)$$

  $$\approx (n^{-\alpha}\mathcal{J}(A))^2 \sim (P(\bar{S}_n \in A))^2$$

\[ n^{-\alpha J(A \backslash B^\gamma)} \quad n^{-\alpha J(A)} \quad n^{-\alpha J(B^\gamma)} \]
• $B_\gamma \triangleq \{\text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma\} \Rightarrow \mathcal{J}(A) = \mathcal{J}(B_\gamma)$

• Fix $w \in (0, 1)$ and define

$$Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B_\gamma)$$

• If we choose $\gamma$ so that $\mathcal{J}(A \setminus B_\gamma) \geq 2\mathcal{J}(A)$,

$$E^{Q_n} \left( \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \setminus B_\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B_\gamma)$$

$$\sim (n^{-\alpha \mathcal{J}(A)})^2 \sim (P(\bar{S}_n \in A))^2$$

$$Z_n \triangleq \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \text{ is strongly efficient for } P(\bar{S}_n \in A)!$$

Chen, Blanchet, R., and Zwart (2017)
First All-Purpose Simulation Scheme for Heavy-Tails

- \( B^\gamma \triangleq \{ \text{paths w/ at least } J(A) \text{ jumps of size } > \gamma \} \Rightarrow J(A) = J(B^\gamma) \)

- Fix \( w \in (0, 1) \) and define

\[
Q_n(\cdot) \triangleq wP(\cdot) + (1 - w)P(\cdot | \bar{S}_n \in B^\gamma)
\]

- If we choose \( \gamma \) so that \( J(A \setminus B^\gamma) \geq 2J(A) \),

\[
E_{Q_n} \left( \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \right)^2 \leq \frac{1}{w} P(\bar{S}_n \in A \setminus B^\gamma) + P(\bar{S}_n \in A) \cdot P(\bar{S}_n \in B^\gamma)
\]

\[
\sim (n^{-\alpha J(A)})^2 \sim (P(\bar{S}_n \in A))^2
\]

\( Z_n \triangleq \mathbb{1}_{\{\bar{S}_n \in A\}} \frac{dP}{dQ_n} \text{ is strongly efficient for } P(\bar{S}_n \in A) \)!

Chen, Blanchet, R., and Zwart (2017)
How to ensure $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$: Reinsurance Example

- $A = \{\text{paths that cross level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b\}$
- $B^\gamma = \{\text{paths with at least 3 jumps of size } > \gamma\}$
- $A \setminus B^\gamma = \{\text{paths that cross level } a \text{ on } [0, 1] \text{ & all jump sizes } \leq b \text{ & at most 2 jumps of size } > \gamma\}$
How to ensure $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$: Reinsurance Example

$$f(t)$$

- $A = \{\text{paths that cross level } a \text{ on } [0, 1] \& \text{jump sizes } \leq b\}$
- $B^\gamma = \{\text{paths with at least 3 jumps of size } > \gamma\}$
- $A \setminus B^\gamma = \{\text{paths that cross level } a \text{ on } [0, 1] \& \text{all jump sizes } \leq b \& \text{at most 2 jumps of size } > \gamma\}$
How to ensure $\mathcal{J}(A \setminus B^{\gamma}) \geq 2\mathcal{J}(A)$: Reinsurance Example

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- $A \setminus B^{\gamma} = \{\text{paths that cross level } a \text{ on } [0, 1] \\
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How to ensure $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$: Reinsurance Example

- $A = \{\text{paths that cross level } a \text{ on } [0, 1] \& \text{jump sizes } \leq b\}$
- $B^\gamma = \{\text{paths with at least 3 jumps of size } > \gamma\}$
- $A \setminus B^\gamma = \{\text{paths that cross level } a \text{ on } [0, 1] \& \text{all jump sizes } \leq b \& \text{at most 2 jumps of size } > \gamma\}$
How to ensure $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$: Reinsurance Example

$7 = \mathcal{J}(A \setminus B^\gamma) > 2\mathcal{J}(A) = 6$

- $A = \{\text{paths that cross level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b\}$
- $B^\gamma = \{\text{paths with at least 3 jumps of size } > \gamma\}$
- $A \setminus B^\gamma = \{\text{paths that cross level } a \text{ on } [0, 1]$
  & \text{ all jump sizes } \leq b \text{ & at most 2 jumps of size } > \gamma\}$
Numerical Experiments for Reinsurance Example

Numerical results – reinsurance policy

- $\beta = 1.2$
- $\beta = 1.4$
- $\beta = 1.6$
- $\beta = 2.0$

![Graph showing numerical results for different levels of precision.](image)
Numerical Results for Queueing Network

Numerical results – fluid queue

- $\beta = (1.2, 1.2, 2.3)$
- $\beta = (1.6, 1.4, 3.1)$
- $\beta = (1.8, 2.0, 3.0)$
- $\beta = (2.3, 2.0, 3.5)$

Graph showing the level of precision for different values of $\beta$. The graph includes markers and lines indicating the numerical results for each level of precision.
Part 3. Large Deviations for Weibull Tails

\[ P(X_i \geq x) = \exp(-x^\alpha), \quad \alpha \in (0, 1) \]
Rate function depends on the size of the jump

Theorem (Bazhba, Blanchet, R., Zwart 2017)

Suppose that \( P(X_i \geq x) = \exp(-x^\alpha), \alpha \in (0, 1) \). Then

\[
P(\bar{S}_n \in A) \sim \exp(-n^\alpha \inf_{f \in A} I(f))
\]

where

\[
I(f) = \begin{cases} 
\sum_t (f(t) - f(t^-))^\alpha, & \text{if } f \text{ is a nondecreasing step function} \\
\infty, & \text{otherwise}
\end{cases}
\]
Back to our old example

\[ A = \{ f \in \mathbb{D} : f \text{ crosses level } a \text{ on } [0, 1] \text{ & jump sizes } \leq b \} \]

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**Solution to Open Problem Posed by Whitt (2000)**
Summary

- **Systematic tools** for rare-event analysis of heavy-tailed systems

- The **principle of multiple (least expensive) big jumps**

- Solved **longstanding open problems** in simulation and queueing theory

- **New connections** btwn rare event analysis and (discrete) optimization
  - e.g., impulse control, knapsack problem, $L^\alpha$-norm minimization $\alpha < 1$