

# Sample Path Large Deviations for Lévy Processes and Random Walks with Regularly Varying Increments

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## Abstract

Let  $X$  be a Lévy process with regularly varying Lévy measure  $\nu$ . We obtain sample-path large deviations for scaled processes  $\bar{X}_n(t) \triangleq X(nt)/n$  and obtain a similar result for random walks. Our results yield detailed asymptotic estimates in scenarios where multiple big jumps in the increment are required to make a rare event happen; we illustrate this through detailed conditional limit theorems. In addition, we investigate connections with the classical large deviations framework. In that setting, we show that a weak large deviation principle (with logarithmic speed) holds, but a full large deviation principle does not hold.

**Keywords** Sample Path Large Deviations · Regular Variation ·  $\mathbb{M}$ -convergence · Lévy Processes

**Mathematics Subject Classification** 60F10 · 60G17 · 60B10

## 1 Introduction

In this paper, we develop sample-path large deviations for one-dimensional Lévy processes and random walks, assuming the jump sizes are heavy-tailed. Specifically, let  $X(t), t \geq 0$ , be a centered Lévy process with regularly varying Lévy measure  $\nu$ . Assume that  $\mathbf{P}(X(1) > x)$  is regularly varying of index  $-\alpha$ , and that  $\mathbf{P}(X(1) < -x)$  is regularly varying of index  $-\beta$ ; i.e. there exist slowly varying functions  $L_+$  and  $L_-$  such that

$$\mathbf{P}(X(1) > x) = L_+(x)x^{-\alpha}, \quad \mathbf{P}(X(1) < -x) = L_-(x)x^{-\beta}. \quad (1.1)$$

Throughout the paper, we assume  $\alpha, \beta > 1$ . We also consider spectrally one-sided processes; in that case only  $\alpha$  plays a role. Define  $\bar{X}_n = \{\bar{X}_n(t), t \in [0, 1]\}$ , with  $\bar{X}_n(t) = X(nt)/n, t \geq 0$ . We are interested in large deviations of  $\bar{X}_n$ .

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This topic fits well in a branch of limit theory that has a long history, has intimate connections to point processes and extreme value theory, and is still a subject of intense activity. The investigation of tail estimates of the one-dimensional distributions of  $\bar{X}_n$  (or random walks with heavy-tailed step size distribution) was initiated in [Nagaev \(1969, 1977\)](#). The state of the art of such results is well summarized in [Borovkov and Borovkov \(2008\)](#); [Denisov et al. \(2008\)](#); [Embrechts et al. \(1997\)](#); [Foss et al. \(2011\)](#). In particular, [Denisov et al. \(2008\)](#) describe in detail how fast  $x$  needs to grow with  $n$  for the asymptotic relation

$$\mathbf{P}(X(n) > x) = n\mathbf{P}(X(1) > x)(1 + o(1)) \quad (1.2)$$

to hold, as  $n \rightarrow \infty$ , in settings that go beyond (1.1). If (1.2) is valid, the so-called *principle of one big jump* is said to hold. A functional version of this insight has been derived in [Hult et al. \(2005\)](#). A significant number of studies investigate the question of if and how the principle of a single big jump is affected by the impact of (various forms of) dependence, and cover stable processes, autoregressive processes, modulated processes, and stochastic differential equations; see [Buraczewski et al. \(2013\)](#); [Foss et al. \(2007\)](#); [Hult and Lindskog \(2007\)](#); [Konstantinides and Mikosch \(2005\)](#); [Mikosch and Wintenberger \(2013, 2016\)](#); [Mikosch and Samorodnitsky \(2000\)](#); [Samorodnitsky \(2004\)](#).

The problem we investigate in this paper is markedly different from all of these works. Our aim is to develop asymptotic estimates of  $\mathbf{P}(\bar{X}_n \in A)$  for a sufficiently general collection of sets  $A$ , so that it is possible to study continuous functionals of  $\bar{X}_n$  in a systematic manner. For many such functionals, and many sets  $A$ , the associated rare event will not be caused by a single big jump, but multiple jumps. The results in this domain (e.g. [Blanchet and Shi \(2012\)](#); [Foss and Korshunov \(2012\)](#); [Zwart et al. \(2004\)](#)) are few, each with an ad-hoc approach. As in large deviations theory for light tails, it is desirable to have more general tools available.

Another aspect of heavy-tailed large deviations we aim to clarify in this paper is the connection with the standard large-deviations approach, which has not been touched upon in any of the above-mentioned references. In our setting, the goal would be to obtain a function  $I$  such that

$$-\inf_{\xi \in A^\circ} I(\xi) \leq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{\log n} \leq \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{\log n} \leq -\inf_{\xi \in \bar{A}} I(\xi), \quad (1.3)$$

where  $A^\circ$  and  $\bar{A}$  are the interior and closure of  $A$ ; all our large deviations results are derived in the Skorokhod  $J_1$  topology. Equation (1.3) is a classical large deviations principle (LDP) with sub-linear speed (cf. [Dembo and Zeitouni \(2009\)](#)). Using existing results in the literature (e.g. [Denisov et al. \(2008\)](#)), it is not difficult to show that  $X(n)/n = \bar{X}_n(1)$  satisfies an LDP with rate function  $I_1 = I_1(x)$  which is 0 at 0, equal to  $(\alpha - 1)$  if  $x > 0$ , and  $(\beta - 1)$  if  $x < 0$ . This is a lower-semicontinuous function of which the level sets are not compact. Thus, in large-deviations terminology,  $I_1$  is a rate function, but is not a good one. This implies that techniques such as the projective limit approach cannot be applied. In fact, in [Section 4.4](#), we show that there does not exist an LDP of the

form (1.3) for general sets  $A$ , by giving a counterexample. A version of (1.3) for compact sets is derived in Section 4.3, as a corollary of our main results. A result similar to (1.3) for random walks with semi-exponential (Weibullian) tails has been derived in Gantert (1998) (see also Gantert (2000); Gantert et al. (2014) for related results). Though an LDP for finite-dimensional distributions can be derived, lack of exponential tightness also persists at the sample-path level. To make the rate function good (i.e., to have compact level sets), a topology chosen in Gantert (1998) is considerably weaker than any of the Skorokhod topologies (but sufficient for the application that is central in that work).

The approach followed in the present paper is based on the recent developments in the theory of regular variation. In particular, in Lindskog et al. (2014), the classical notion of regular variation is re-defined through a new convergence concept called  $\mathbb{M}$ -convergence (this is in itself a refinement of other reformulations of regular variation in function spaces; see de Haan and Lin (2001); Hult and Lindskog (2005, 2006)). In Section 2, we further investigate the  $\mathbb{M}$ -convergence framework by deriving a number of general results that facilitate the development of our proofs.

This paves the way towards our main large deviations results, which are presented in Section 3. We actually obtain estimates that are sharper than (1.3), though we impose a condition on  $A$ . For one-sided Lévy processes, our result takes the form

$$C_{\mathcal{J}(A)}(A^\circ) \leq \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty))^{\mathcal{J}(A)}} \leq \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty))^{\mathcal{J}(A)}} \leq C_{\mathcal{J}(A)}(\bar{A}). \quad (1.4)$$

Precise definitions can be found in Section 3.1; for now we just mention that  $C_j$  is a measure on the Skorokhod space, and  $\mathcal{J}(\cdot)$  is an integer valued set function defined as  $\mathcal{J}(A) = \inf_{\xi \in A \cap \mathbb{D}_s^\uparrow} \mathcal{D}_+(\xi)$ , where  $\mathcal{D}_+(\xi)$  is the number of discontinuities of  $\xi$ , and  $\mathbb{D}_s^\uparrow$  is the set of all non-increasing step functions vanishing at the origin. Throughout the paper, we adopt the convention that the infimum over an empty set is  $\infty$ . Letting  $\mathbb{D}_j$  and  $\mathbb{D}_{<j}$  be the sets of step functions vanishing at the origin with precisely  $j$  and at most  $j-1$  steps respectively, we note that the measure  $C_j$ , defined on  $\mathbb{D} \setminus \mathbb{D}_{<j}$  has its support on  $\mathbb{D}_j$ . A crucial assumption for (1.4) to hold is that the Skorokhod  $J_1$  distance between the sets  $A$  and  $\mathbb{D}_{<\mathcal{J}(A)}$  is strictly positive. For  $A$  such that  $\mathcal{J}(A) = 1$  this result corresponds to the one shown in Hult et al. (2005). (Note that Hult et al. (2005) deals with multivariate regular variation whereas we focus on 1-dimensional regular variation in this paper.) The interpretation of the “rate function”  $\mathcal{J}(A)$  is that it provides the number of jumps in the Lévy process that are necessary to make the event  $A$  happen. This can be seen as an extension of the principle of a single big jump to multiple jumps. A rigorous statement on when (1.4) holds can be found in Theorem 3.2, which is the first main result of the paper.

The result that comes closest to (1.4) is Theorem 5.1 in Lindskog et al. (2014) which considers the  $\mathbb{M}$ -convergence of  $\nu[n, \infty)^{-j} \mathbf{P}(X/n \in A)$ . This result could be used as a starting point to investigate rare events that happen on a time-scale of  $O(1)$ . However, in the large-deviations scaling we consider, rare events that

happen on a time-scale of  $O(n)$ . Controlling the Lévy process on this larger time-scale requires more delicate estimates, eventually leading to an additional factor  $n^j$  in the asymptotic results. We further show that the choice  $j = \mathcal{J}(A)$  is the only choice that leads to a non-trivial limit. One useful notion that we develop and rely on in our setting is a form of asymptotic equivalence, which can best be compared with exponential equivalence in classical large deviations theory.

In Section 3.2 we present sample-path large deviations for two-sided Lévy processes. Our main results in this case are Theorems 3.3–3.5. In the two-sided case, determining the most likely path requires resolving significant combinatorial issues which do not appear in the one-sided case. The polynomial rate of decay for  $\mathbf{P}(\bar{X}_n \in A)$ , which was described by the function  $\mathcal{J}(A)$  in the one-sided case, has a more complicated description; the corresponding polynomial rate in the two-sided case is

$$\inf_{\xi, \zeta \in \mathbb{D}_s^\uparrow; \xi - \zeta \in A} (\alpha - 1)\mathcal{D}_+(\xi) + (\beta - 1)\mathcal{D}_+(\zeta). \quad (1.5)$$

Note that this is a result that one could expect from the result for one-sided Lévy processes and a heuristic application of the contraction principle. A rigorous treatment of the two-sided case requires a more delicate argument compared to the one-sided case: in the one-sided case, the argument simplifies since if one takes  $j$  largest jumps away from  $\bar{X}_n$ , then the probability that the residual process is of significant size is  $o((n\nu[n, \infty))^j)$  so that it does not contribute in (1.4), while in two-sided case, taking  $j$  largest upward jumps and  $k$  largest downward jumps from  $\bar{X}_n$  doesn't guarantee that the residual process remains small with high enough probability—i.e., the probability that the residual process is of significant size cannot be bounded by  $o((n\nu[n, \infty))^j (n\nu(-\infty, -n])^k)$ . In addition, it may be the case that multiple pairs  $(j, k)$  of jumps lead to optimal solutions of (1.5). To overcome such difficulties, we first develop general tools—Lemma 2.2 and 2.3—that establish a suitable notion of  $\mathbb{M}$ -convergence on product spaces. Using these results, we prove in Theorem 5.1 the suitable  $\mathbb{M}$ -convergence for multiple Lévy processes in the associated product space. Viewing the two-sided Lévy process as a superposition of one-sided Lévy processes, we then apply the continuous mapping principle for  $\mathbb{M}$ -convergence to Theorem 5.1 to establish our main results. Although no further implications are discussed in this paper, we believe that Theorem 5.1 itself is of independent interest as well because it can be applied to generate large deviations results for a general class of functionals of multiple Lévy processes.

We derive analogous results for random walks in Section 4.1. Random walks cannot be decomposed into independent components with small jumps and large jumps as easily as Lévy processes, making the analysis of random walks more technical if done directly. However, it is possible to follow an indirect approach. Given a random walk  $S_k, k \geq 0$ , one can study a subordinated version  $S_{N(t)}, t \geq 0$  with  $N(t), t \geq 0$  an independent unit rate Poisson process. The Skorokhod  $J_1$  distance between rescaled versions of  $S_k, k \geq 0$  and  $S_{N(t)}, t \geq 0$  can then be bounded in terms of the deviations of  $N(t)$  from  $t$ , which have been studied

thoroughly.

In Section 4.2, we provide conditional limit theorems which give a precise description of the limit behavior of  $\bar{X}_n$  given that  $\bar{X}_n \in A$  as  $n \rightarrow \infty$ . An early result of this type is given in Durrett (1980), which focuses on regularly varying random walks with finite variance conditioned on the event  $A = \{\bar{X}_n(1) > a\}$ . Using the recent results that we have discussed (e.g. Hult et al. (2005)) more general conditional limit theorems can be derived for single-jump events.

We prove an LDP of the form (1.3) in Section 4.3, where the upper bound requires a compactness assumption. We construct a counterexample showing that the compactness assumption cannot be totally removed, and thus, a full LDP does not hold. Essentially, if a rare event is caused by  $j$  big jumps, then the framework developed in this paper applies if each of these jumps is bounded away from below by a strictly positive constant. Our counterexample in Section 4.4 indicates that it is not trivial to remove this condition.

As one may expect, it is not possible to apply classical variational methods to derive an expression for the exponent  $\mathcal{J}(A)$ , as is often the case in large deviations for light tails. Nevertheless, there seems to be a generic connection with a class of control problems called impulse control problems. Equation (1.5) is a specific deterministic impulse-control problem, which is related to Barles (1985). We expect that techniques similar to those in Barles (1985) will be useful to characterize optimality of solutions for problems like (1.5). The latter challenge is not taken up in the present study and will be addressed elsewhere. Instead, in Section 6, we analyse (1.5) directly in several examples; see also Chen et al. (2017). In each case, a condition needs to be checked to see whether our framework is applicable. We provide a general result that essentially states that we only need to check this condition for step functions in  $A$ , which makes this check rather straightforward.

In summary, this paper is organized as follows. After developing some preliminary results in Section 2, we present our main results in Section 3. Applications to random walks and connections with classical large deviations theory are investigated in Section 4. Section 5 is devoted to proofs. We collect some useful bounds in Appendix A.

## 2 $\mathbb{M}$ -convergence

This section reviews and develops general concepts and tools that are useful in deriving our large deviations results. The proofs of the lemmas and corollaries stated throughout this section are provided in Section 5.1. We start with briefly reviewing the notion of  $\mathbb{M}$ -convergence, introduced in Lindskog et al. (2014).

Let  $(\mathbb{S}, d)$  be a complete separable metric space, and  $\mathcal{S}$  be the Borel  $\sigma$ -algebra on  $\mathbb{S}$ . Given a closed subset  $\mathbb{C}$  of  $\mathbb{S}$ , let  $\mathbb{S} \setminus \mathbb{C}$  be equipped with the relative topology as a subspace of  $\mathbb{S}$ , and consider the associated sub  $\sigma$ -algebra  $\mathcal{S}_{\mathbb{S} \setminus \mathbb{C}} \triangleq \{A : A \subseteq \mathbb{S} \setminus \mathbb{C}, A \in \mathcal{S}\}$  on it. Define  $\mathbb{C}^r \triangleq \{x \in \mathbb{S} : d(x, \mathbb{C}) < r\}$  for  $r > 0$ , and let  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  be the class of measures defined on  $\mathcal{S}_{\mathbb{S} \setminus \mathbb{C}}$  whose restrictions to  $\mathbb{S} \setminus \mathbb{C}^r$  are finite for all  $r > 0$ . Topologize  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  with a sub-

basis  $\{\nu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C}) : \nu(f) \in G\}$ :  $f \in \mathcal{C}_{\mathbb{S} \setminus \mathbb{C}}$ ,  $G$  open in  $\mathbb{R}_+$  where  $\mathcal{C}_{\mathbb{S} \setminus \mathbb{C}}$  is the set of real-valued, non-negative, bounded, continuous functions whose support is bounded away from  $\mathbb{C}$  (i.e.,  $f(\mathbb{C}^r) = \{0\}$  for some  $r > 0$ ). A sequence of measures  $\mu_n \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  converges to  $\mu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  if  $\mu_n(f) \rightarrow \mu(f)$  for each  $f \in \mathcal{C}_{\mathbb{S} \setminus \mathbb{C}}$ . Note that this notion of convergence in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  coincides with the classical notion of weak convergence of measures (Billingsley, 2013) if  $\mathbb{C}$  is an empty set. We say that a set  $A \subseteq \mathbb{S}$  is bounded away from another set  $B \subseteq \mathbb{S}$  if  $\inf_{x \in A, y \in B} d(x, y) > 0$ . An important characterization of  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ -convergence is as follows:

**Theorem 2.1** (Theorem 2.1 of Lindskog et al., 2014). *Let  $\mu, \mu_n \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ . Then  $\mu_n \rightarrow \mu$  in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  as  $n \rightarrow \infty$  if and only if*

$$\limsup_{n \rightarrow \infty} \mu_n(F) \leq \mu(F) \quad (2.1)$$

for all closed  $F \in \mathcal{S}_{\mathbb{S} \setminus \mathbb{C}}$  bounded away from  $\mathbb{C}$  and

$$\liminf_{n \rightarrow \infty} \mu_n(G) \geq \mu(G) \quad (2.2)$$

for all open  $G \in \mathcal{S}_{\mathbb{S} \setminus \mathbb{C}}$  bounded away from  $\mathbb{C}$ .

We now introduce a new notion of equivalence between two families of random objects, which will prove to be useful in Section 3.1, and Section 4.1. Let  $F_\delta \triangleq \{x \in \mathbb{S} : d(x, F) \leq \delta\}$  and  $G^{-\delta} \triangleq ((G^c)_\delta)^c$ . (Compare these notations to  $\mathbb{C}^r$ ; note that we are using the convention that superscript implies open sets and subscript implies closed sets.)

**Definition 1.** *Suppose that  $X_n$  and  $Y_n$  are random elements taking values in a complete separable metric space  $(\mathbb{S}, d)$ , and  $\epsilon_n$  is a sequence of positive real numbers.  $Y_n$  is said to be asymptotically equivalent to  $X_n$  with respect to  $\epsilon_n$  if for each  $\delta > 0$ ,*

$$\limsup_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(d(X_n, Y_n) \geq \delta) = 0.$$

The usefulness of this notion of equivalence comes from the following lemma, which states that if  $Y_n$  is asymptotically equivalent to  $X_n$ , and  $X_n$  satisfies a limit theorem, then  $Y_n$  satisfies the same limit theorem. Moreover, it also allows one to extend the lower and upper bounds to more general sets in case there are asymptotically equivalent distributions that are supported on a subspace  $\mathbb{S}_0$  of  $\mathbb{S}$ :

**Lemma 2.1.** *Suppose that  $\epsilon_n^{-1} \mathbf{P}(X_n \in \cdot) \rightarrow \mu(\cdot)$  in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  for some sequence  $\epsilon_n$  and a closed set  $\mathbb{C}$ . In addition, suppose that  $\mu(\mathbb{S} \setminus \mathbb{S}_0) = 0$  and  $\mathbf{P}(X_n \in \mathbb{S}_0) = 1$  for each  $n$ . If  $Y_n$  is asymptotically equivalent to  $X_n$  with respect to  $\epsilon_n$ , then*

$$\liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in G) \geq \mu(G)$$

if  $G$  is open and  $G \cap \mathbb{S}_0$  is bounded away from  $\mathbb{C}$ ;

$$\limsup_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in F) \leq \mu(F)$$

if  $F$  is closed and there is a  $\delta > 0$  such that  $F_\delta \cap \mathbb{S}_0$  is bounded away from  $\mathbb{C}$ .

This lemma is particularly useful when we work in Skorokhod space, and  $\mathbb{S}_0$  is the class of step functions. Taking  $\mathbb{S}_0 = \mathbb{S}$ , a simpler version of Lemma 2.1 follows immediately:

**Corollary 2.1.** *Suppose that  $\epsilon_n^{-1}\mathbf{P}(X_n \in \cdot) \rightarrow \mu(\cdot)$  in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  for some sequence  $\epsilon_n$ . If  $Y_n$  is asymptotically equivalent to  $X_n$  with respect to  $\epsilon_n$ , then the law of  $Y_n$  has the same (normalized) limit, i.e.,  $\epsilon_n^{-1}\mathbf{P}(Y_n \in \cdot) \rightarrow \mu(\cdot)$  in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ .*

Next, we discuss the  $\mathbb{M}$ -convergence in a product space as a result of the  $\mathbb{M}$ -convergences on each space.

**Lemma 2.2.** *Suppose that  $\mathbb{S}_1, \dots, \mathbb{S}_d$  are separable metric spaces,  $\mathbb{C}_1, \dots, \mathbb{C}_d$  are closed subsets of  $\mathbb{S}_1, \dots, \mathbb{S}_d$ , respectively. If  $\mu_n^{(i)}(\cdot) \rightarrow \mu^{(i)}(\cdot)$  in  $\mathbb{M}(\mathbb{S}_i \setminus \mathbb{C}_i)$  for each  $i = 1, \dots, d$  then,*

$$\mu_n^{(1)} \times \dots \times \mu_n^{(d)}(\cdot) \rightarrow \mu^{(1)} \times \dots \times \mu^{(d)}(\cdot) \quad (2.3)$$

in  $\mathbb{M}\left(\left(\prod_{i=1}^d \mathbb{S}_i\right) \setminus \bigcup_{i=1}^d \left(\left(\prod_{j=1}^{i-1} \mathbb{S}_j\right) \times \mathbb{C}_i \times \left(\prod_{j=i+1}^d \mathbb{S}_j\right)\right)\right)$ .

It should be noted that Lemma 2.2 itself is not exactly “right” in the sense that the set we take away is unnecessarily large, and hence, has limited applicability. More specifically, the  $\mathbb{M}$ -convergence in (2.3) applies only to the sets that are contained in a “rectangular” domain  $\prod_{i=1}^d (\mathbb{S}_i \setminus \mathbb{C}_i)$ . Our next observation allows one to combine multiple instances of  $\mathbb{M}$ -convergences to establish a more refined one so that (2.3) applies to a class of sets that are not confined to a rectangular domain. In particular, we will see later in Theorem 3.3 and Theorem 5.1 that in combination with Lemma 2.2, the following lemma produces the “right”  $\mathbb{M}$ -convergence for two-sided Lévy processes and random walks.

**Lemma 2.3.** *Consider a family of measures  $\{\mu^{(i)}\}_{i=0,1,\dots,m}$  and a family of closed subsets  $\{\mathbb{C}(i)\}_{i=0,1,\dots,m}$  of  $\mathbb{S}$  such that  $\frac{1}{\epsilon_n^{(i)}}\mathbf{P}(X_n \in \cdot) \rightarrow \mu^{(i)}(\cdot)$  in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C}(i))$  for  $i = 0, \dots, m$  where  $\{\{\epsilon_n(i) : n \geq 1\}\}_{i=0,1,\dots,m}$  is the family of associated normalizing sequences. Suppose that  $\mu^{(0)} \in \mathbb{M}(\mathbb{S} \setminus \bigcap_{i=0}^m \mathbb{C}(i))$ ;  $\limsup_{n \rightarrow \infty} \frac{\epsilon_n(i)}{\epsilon_n(0)} = 0$  for  $i = 1, \dots, m$ ; and for each  $r > 0$ , there exist positive numbers  $r_0, \dots, r_m$  such that  $\bigcap_{i=0}^m \mathbb{C}(i)^{r_i} \subseteq \left(\bigcap_{i=0}^m \mathbb{C}(i)\right)^r$ . Then*

$$\frac{1}{\epsilon_n(0)}\mathbf{P}(X_n \in \cdot) \rightarrow \mu^{(0)}$$

in  $\mathbb{M}(\mathbb{S} \setminus \bigcap_{i=0}^m \mathbb{C}(i))$ .

A version of the continuous mapping principle is satisfied by  $\mathbb{M}$ -convergence. Let  $(\mathbb{S}', d')$  be a complete separable metric space, and let  $\mathbb{C}'$  be a closed subset of  $\mathbb{S}'$ .

**Theorem 2.2** (Mapping theorem; Theorem 2.3 of Lindskog et al. (2014)). *Let  $h : (\mathbb{S} \setminus \mathbb{C}, \mathcal{S}_{\mathbb{S} \setminus \mathbb{C}}) \rightarrow (\mathbb{S}' \setminus \mathbb{C}', \mathcal{S}_{\mathbb{S}' \setminus \mathbb{C}'})$  be a measurable mapping such that  $h^{-1}(A')$*

is bounded away from  $\mathbb{C}$  for any  $A' \in \mathcal{S}'_{\mathbb{S}' \setminus \mathbb{C}'}$  bounded away from  $\mathbb{C}'$ . Then  $\hat{h} : \mathbb{M}(\mathbb{S} \setminus \mathbb{C}) \rightarrow \mathbb{M}(\mathbb{S}' \setminus \mathbb{C}')$  defined by  $\hat{h}(\nu) = \nu \circ h^{-1}$  is continuous at  $\mu$  provided  $\mu(D_h) = 0$ , where  $D_h$  is the set of discontinuity points of  $h$ .

For our purpose, the following slight extension will prove to be useful in developing rigorous arguments.

**Lemma 2.4.** *Let  $\mathbb{S}_0$  be a measurable subset of  $\mathbb{S}$ , and  $h : (\mathbb{S}_0, \mathcal{S}_{\mathbb{S}_0}) \rightarrow (\mathbb{S}' \setminus \mathbb{C}', \mathcal{S}'_{\mathbb{S}' \setminus \mathbb{C}'})$  be a measurable mapping such that  $h^{-1}(A')$  is bounded away from  $\mathbb{C}$  for any  $A' \in \mathcal{S}'_{\mathbb{S}' \setminus \mathbb{C}'}$  bounded away from  $\mathbb{C}'$ . Then  $\hat{h} : \mathbb{M}(\mathbb{S} \setminus \mathbb{C}) \rightarrow \mathbb{M}(\mathbb{S}' \setminus \mathbb{C}')$  defined by  $\hat{h}(\nu) = \nu \circ h^{-1}$  is continuous at  $\mu$  provided that  $\mu(\partial \mathbb{S}_0 \setminus \mathbb{C}^r) = 0$  and  $\mu(D_h \setminus \mathbb{C}^r) = 0$  for all  $r > 0$ , where  $D_h$  is the set of discontinuity points of  $h$ .*

When we focus on Lévy processes, we are specifically interested in the case where  $\mathbb{S}$  is  $\mathbb{R}_+^{\infty \downarrow} \times [0, 1]^\infty$ , where  $\mathbb{R}_+^{\infty \downarrow} \triangleq \{x \in \mathbb{R}_+^\infty : x_1 \geq x_2 \geq \dots\}$ , and  $\mathbb{S}'$  is the Skorokhod space  $\mathbb{D} = \mathbb{D}([0, 1], \mathbb{R})$  — the space of real-valued RCLL functions on  $[0, 1]$ . We use the usual product metrics  $d_{\mathbb{R}_+^{\infty \downarrow}}(x, y) = \sum_{i=1}^\infty \frac{|x_i - y_i| \wedge 1}{2^i}$  and  $d_{[0, 1]^\infty}(x, y) = \sum_{i=1}^\infty \frac{|x_i - y_i|}{2^i}$  for  $\mathbb{R}_+^{\infty \downarrow}$  and  $[0, 1]^\infty$ , respectively. For the finite product of metric spaces, we use the maximum metric; i.e., we use  $d_{\mathbb{S}_1 \times \dots \times \mathbb{S}_d}((x_1, \dots, x_d), (y_1, \dots, y_d)) \triangleq \max_{i=1, \dots, d} d_{\mathbb{S}_i}(x_i, y_i)$  for the product  $\mathbb{S}_1 \times \dots \times \mathbb{S}_d$  of metric spaces  $(\mathbb{S}_i, d_{\mathbb{S}_i})$ . For  $\mathbb{D}$ , we use the usual Skorokhod  $J_1$  metric  $d(x, y) \triangleq \inf_{\lambda \in \Lambda} \|\lambda - e\| \vee \|x \circ \lambda - y\|$ , where  $\Lambda$  denotes the set of all non-decreasing homeomorphisms from  $[0, 1]$  onto itself,  $e$  denotes the identity, and  $\|\cdot\|$  denotes the supremum norm. Let

$$S_j \triangleq \{(x, u) \in \mathbb{R}_+^{\infty \downarrow} \times [0, 1]^\infty : 0, 1, u_1, \dots, u_j \text{ are all distinct}\}.$$

This set will play the role of  $\mathbb{S}_0$  of Lemma 2.4. Define  $T_j : S_j \rightarrow \mathbb{D}$  to be  $T_j(x, u) = \sum_{i=1}^j x_i 1_{[u_i, 1]}$ . Let  $\mathbb{D}_j$  be the subspaces of the Skorokhod space consisting of nondecreasing step functions, vanishing at the origin, with exactly  $j$  jumps, and  $\mathbb{D}_{\leq j} \triangleq \bigcup_{0 \leq i \leq j} \mathbb{D}_i$ —i.e., nondecreasing step functions vanishing at the origin with at most  $j$  jumps. Similarly, let  $\mathbb{D}_{< j} \triangleq \bigcup_{0 \leq i < j} \mathbb{D}_i$ . Define  $\mathbb{H}_j \triangleq \{x \in \mathbb{R}_+^{\infty \downarrow} : x_j > 0, x_{j+1} = 0\}$ , and  $\mathbb{H}_{< j} \triangleq \{x \in \mathbb{R}_+^{\infty \downarrow} : x_j = 0\}$ . The continuous mapping principle applies to  $T_j$ , as we can see in the following result.

**Lemma 2.5** (Lemma 5.3 and Lemma 5.4 of [Lindskog et al., 2014](#)). *Suppose  $A \subset \mathbb{D}$  is bounded away from  $\mathbb{D}_{< j}$ . Then,  $T_j^{-1}(A)$  is bounded away from  $\mathbb{H}_{< j} \times [0, 1]^\infty$ . Moreover,  $T_j : S_j \rightarrow \mathbb{D}$  is continuous.*

A consequence of Result 2.5 and Lemma 2.4 along with the observation that  $S_j$  is open is that one can derive a limit theorem in a path space from a limit theorem for jump sizes.

**Corollary 2.2.** *If  $\mu_n \rightarrow \mu$  in  $\mathbb{M}((\mathbb{R}_+^{\infty \downarrow} \times [0, 1]^\infty) \setminus (\mathbb{H}_{< j} \times [0, 1]^\infty))$ , and  $\mu(S_j^c \setminus (\mathbb{H}_{< j} \times [0, 1]^\infty)^r) = 0$  for all  $r > 0$ , then  $\mu_n \circ T_j^{-1} \rightarrow \mu \circ T_j^{-1}$  in  $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{< j})$ .*



To obtain the large deviations for two-sided Lévy measures, we will first establish the large deviations for independent spectrally positive Lévy processes, and then apply Lemma 2.4 with  $h(\xi, \zeta) = \xi - \zeta$ . The next lemma verifies two important conditions of Lemma 2.4 for such  $h$ . Let  $\mathbb{D}_{l,m}$  denote the subspace of the Skorokhod space consisting of step functions vanishing at the origin with exactly  $l$  upward jumps and  $m$  downward jumps. Given  $\alpha, \beta > 1$ , let  $\mathbb{D}_{< j,k} \triangleq \bigcup_{(l,m) \in \mathbb{I}_{< j,k}} \mathbb{D}_{l,m}$  and  $\mathbb{D}_{< (j,k)} \triangleq \bigcup_{(l,m) \in \mathbb{I}_{< j,k}} \mathbb{D}_l \times \mathbb{D}_m$ , where  $\mathbb{I}_{< j,k} \triangleq \{(l, m) \in \mathbb{Z}_+^2 \setminus \{(j, k)\} : (\alpha - 1)l + (\beta - 1)m \leq (\alpha - 1)j + (\beta - 1)k\}$  and  $\mathbb{Z}_+$  denotes the set of non-negative integers. Note that in the definition of  $\mathbb{I}_{< j,k}$ , the inequality is not strict; however, we choose to use the strict inequality in our notation to emphasize that  $(j, k)$  is not included in  $\mathbb{I}_{< j,k}$ .

**Lemma 2.6.** *Let  $h : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$  be defined as  $h(\xi, \zeta) \triangleq \xi - \zeta$ . Then,  $h$  is continuous at  $(\xi, \zeta) \in \mathbb{D} \times \mathbb{D}$  such that  $(\xi(t) - \xi(t-))(\zeta(t) - \zeta(t-)) = 0$  for all  $t \in (0, 1]$ . Moreover,  $h^{-1}(A) \subseteq \mathbb{D} \times \mathbb{D}$  is bounded away from  $\mathbb{D}_{< (j,k)}$  for any  $A \subseteq \mathbb{D}$  bounded away from  $\mathbb{D}_{< j,k}$ .*

We next characterize convergence-determining classes for the convergence in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ .

**Lemma 2.7.** *Suppose that (i)  $\mathcal{A}_p$  is a  $\pi$ -system; (ii) each open set  $G \subseteq \mathbb{S}$  bounded away from  $\mathbb{C}$  is a countable union of sets in  $\mathcal{A}_p$ ; and (iii) for each closed set  $F \subseteq \mathbb{S}$  bounded away from  $\mathbb{C}$ , there is a set  $A \in \mathcal{A}_p$  bounded away from  $\mathbb{C}$  such that  $F \subseteq A^\circ$  and  $\mu(A \setminus A^\circ) = 0$ . If, in addition,  $\mu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  and  $\mu_n(A) \rightarrow \mu(A)$  for every  $A \in \mathcal{A}_p$  such that  $A$  is bounded away from  $\mathbb{C}$ , then  $\mu_n \rightarrow \mu$  in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ .*

**Remark 1.** *Since  $\mathbb{S}$  is a separable metric space, the Lindelöf property holds. Therefore, a sufficient condition for assumption (ii) of Lemma 2.7 is that for every  $x \in \mathbb{S} \setminus \mathbb{C}$  and  $\epsilon > 0$ , there is  $A \in \mathcal{A}_p$  such that  $x \in A^\circ \subseteq B(x, \epsilon)$ . To see that this implies assumption (ii), note that for any given open set  $G$ , one can construct a cover  $\{(A_x)^\circ : x \in G\}$  of  $G$  by choosing  $A_x$  so that  $x \in (A_x)^\circ \subseteq G$  and then extract a countable subcover (due to the Lindelöf property) whose union is equal to  $G$ . Note also that if  $A$  in assumption (iii) is open, then  $\mu(A \setminus A^\circ) = \mu(\emptyset) = 0$  automatically.*

### 3 Sample-Path Large Deviations

In this section, we present large-deviations results for scaled Lévy processes with heavy-tailed Lévy measures. Section 3.1 studies a special case, where the Lévy measure is concentrated on the positive part of the real line, and Section 3.2 extends this result to Lévy processes with two-sided Lévy measures. In both cases, let  $X_n(t) \triangleq X(nt)$  be a scaled process of  $X$ , where  $X$  is a Lévy process with a Lévy measure  $\nu$ . Recall that  $X_n$  has Itô representation (see, for example,

Section 2 of [Kyprianou, 2014](#)):

$$X_n(s) = nsa + B(ns) + \int_{|x| \leq 1} x[N([0, ns] \times dx) - ns\nu(dx)] + \int_{|x| > 1} xN([0, ns] \times dx),$$

with  $a$  a drift parameter,  $B$  a Brownian motion, and  $N$  a Poisson random measure with mean measure  $\text{Leb} \times \nu$  on  $[0, n] \times (0, \infty)$ ;  $\text{Leb}$  denotes the Lebesgue measure.

### 3.1 One-sided Large Deviations

Let  $X$  be a Lévy process with Lévy measure  $\nu$ . In this section, we assume that  $\nu$  is a regularly varying (at infinity, with index  $-\alpha < -1$ ) Lévy measure concentrated on  $(0, \infty)$ . Consider a centered and scaled process

$$\bar{X}_n(s) \triangleq \frac{1}{n} X_n(s) - sa - \mu_1^+ \nu_1^+ s, \quad (3.1)$$

where  $\mu_1^+ \triangleq \frac{1}{\nu_1^+} \int_{[1, \infty)} x\nu(dx)$ , and  $\nu_1^+ \triangleq \nu[1, \infty)$ . For each constant  $\gamma > 1$ , let  $\nu_\gamma(x, \infty) \triangleq x^{-\gamma}$ , and let  $\nu_\gamma^j$  denote the restriction (to  $\mathbb{R}_+^{j\downarrow}$ ) of the  $j$ -fold product measure of  $\nu_\gamma$ . Let  $C_0(\cdot) \triangleq \delta_0(\cdot)$  be the Dirac measure concentrated on the zero function. Additionally, for each  $j \geq 1$ , define a measure  $C_j \in \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{< j})$  concentrated on  $\mathbb{D}_j$  as  $C_j(\cdot) \triangleq \mathbf{E} \left[ \nu_\alpha^j \{y \in (0, \infty)^j : \sum_{i=1}^j y_i 1_{[U_i, 1]} \in \cdot\} \right]$ , where the random variables  $U_i, i \geq 1$  are i.i.d. uniform on  $[0, 1]$ .

The proof of the main result of this section hinges critically on the following limit theorem.

**Theorem 3.1.** *For each  $j \geq 0$ ,*

$$(n\nu[n, \infty))^{-j} \mathbf{P}(\bar{X}_n \in \cdot) \rightarrow C_j(\cdot), \quad (3.2)$$

*in  $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{< j})$ , as  $n \rightarrow \infty$ . Moreover,  $\bar{X}_n$  is asymptotically equivalent to a process that assumes values in  $\mathbb{D}_{\leq \mathcal{J}(A)}$  almost surely.*

*Proof Sketch.* The proof of Theorem 3.1 is based on establishing the asymptotic equivalence of  $\bar{X}_n$  and the process obtained by just keeping its  $j$  biggest jumps, which we will denote by  $\tilde{J}_n^{\leq j}$  in Section 5. Such an equivalence is established via Proposition 5.1, and Proposition 5.2. Then, Proposition 5.3 identifies the limit of  $\tilde{J}_n^{\leq j}$ , which coincides with the limit in (3.2). The full proof of Theorem 3.1 is provided in Section 5.2.  $\square$

Recall that  $\mathbb{D}_s^\uparrow$  denotes the subset of  $\mathbb{D}$  consisting of non-decreasing step functions vanishing at the origin, and  $\mathcal{D}_+(\xi)$  denotes the number of upward jumps of an element  $\xi$  in  $\mathbb{D}$ . Finally, set

$$\mathcal{J}(A) \triangleq \inf_{\xi \in \mathbb{D}_s^\uparrow \cap A} \mathcal{D}_+(\xi). \quad (3.3)$$

Now we are ready to present the main result of this section, which is the following large-deviations theorem for  $\bar{X}_n$ .

**Theorem 3.2.** *Suppose that  $A$  is a measurable set. If  $\mathcal{J}(A) < \infty$ , and if  $A_\delta \cap \mathbb{D}_{\leq \mathcal{J}(A)}$  is bounded away from  $\mathbb{D}_{< \mathcal{J}(A)}$  for some  $\delta > 0$ , then*

$$\begin{aligned} C_{\mathcal{J}(A)}(A^\circ) &\leq \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty))^{\mathcal{J}(A)}} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty))^{\mathcal{J}(A)}} \leq C_{\mathcal{J}(A)}(\bar{A}). \end{aligned} \quad (3.4)$$

If  $\mathcal{J}(A) = \infty$ , and  $A_\delta \cap \mathbb{D}_{\leq i+1}$  is bounded away from  $\mathbb{D}_{\leq i}$  for some  $\delta > 0$  and  $i \geq 0$ , then

$$\lim_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty))^i} = 0. \quad (3.5)$$

In particular, in case  $\mathcal{J}(A) < \infty$ , (3.4) holds if  $A$  is bounded away from  $\mathbb{D}_{< \mathcal{J}(A)}$ ; in case  $\mathcal{J}(A) = \infty$ , (3.5) holds if  $A$  is bounded away from  $\mathbb{D}_{\leq i}$ .

*Proof.* We first consider the case  $\mathcal{J}(A) < \infty$ . Note that  $\mathcal{J}(A^\circ) > \mathcal{J}(A)$  implies that  $A^\circ$  doesn't contain any element of  $\mathbb{D}_{\leq \mathcal{J}(A)}$ . Since  $C_{\mathcal{J}(A)}$  is supported on  $\mathbb{D}_{\leq \mathcal{J}(A)}$ ,  $A^\circ$  is a  $C_{\mathcal{J}(A)}$ -null set. Therefore, the lower bound holds trivially if  $\mathcal{J}(A^\circ) > \mathcal{J}(A)$ . On the other hand,  $\mathcal{J}(A) = \mathcal{J}(\bar{A})$ . To see this, suppose not—i.e.,  $\mathcal{J}(\bar{A}) < \mathcal{J}(A)$ . Then, there exists  $\zeta \in \mathbb{D}_s^\uparrow \cap \bar{A}$  such that  $\zeta \in \mathbb{D}_{< \mathcal{J}(A)}$ . This implies that  $\zeta \in A_\delta \cap \mathbb{D}_{\leq \mathcal{J}(A)}$  for any  $\delta > 0$ , which is contradictory to the assumption that  $A_\delta \cap \mathbb{D}_{\leq \mathcal{J}(A)}$  is bounded away from  $\mathbb{D}_{< \mathcal{J}(A)}$  for some  $\delta > 0$ . In view of these observations, we can assume w.l.o.g. that  $\mathcal{J}(A^\circ) = \mathcal{J}(A) = \mathcal{J}(\bar{A})$ . Now, from Theorem 3.1 with  $j = \mathcal{J}(A^\circ)$  along with the lower bound of Lemma 2.1,

$$\begin{aligned} C_{\mathcal{J}(A)}(A^\circ) &= C_{\mathcal{J}(A^\circ)}(A^\circ) \leq \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A^\circ)}{(n\nu[n, \infty))^{\mathcal{J}(A^\circ)}} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty))^{\mathcal{J}(A)}}. \end{aligned}$$

Similarly, from Theorem 3.1 with  $j = \mathcal{J}(\bar{A})$  along with the upper bound of Lemma 2.1,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty))^{\mathcal{J}(A)}} &\leq \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in \bar{A})}{(n\nu[n, \infty))^{\mathcal{J}(\bar{A})}} \\ &\leq C_{\mathcal{J}(\bar{A})}(\bar{A}) = C_{\mathcal{J}(A)}(\bar{A}). \end{aligned}$$

In case  $\mathcal{J}(A) = \infty$ , we reach the conclusion by applying Theorem 3.1 with  $j = i$  along with noting that  $C_i(\bar{A}) = 0$ .  $\square$

Theorem 3.2 dictates the “right” choice of  $j$  in Theorem 3.1 for which (3.2) can lead to a limit in  $(0, \infty)$ . We conclude this section with an investigation of

a sufficient condition for  $C_j$ -continuity; i.e., we provide a sufficient condition on  $A$  which guarantees  $C_j(\partial A) = 0$ . The latter property implies

$$C_j(A^\circ) = C_j(A) = C_j(\bar{A}), \quad (3.6)$$

implying that the liminf and limsup in our asymptotic estimates yield the same result. Assume that  $A$  is a subset of  $\mathbb{D}_j$  bounded away from  $\mathbb{D}_{<j}$ ; i.e.,  $d(A, \mathbb{D}_{<j}) > \gamma$  for some  $\gamma > 0$ . Consider a path  $\xi \in A$ . Note that every  $\xi \in \mathbb{D}_j$  is determined by the pair of jump sizes and jump times  $(x, u) \in (0, \infty)^j \times [0, 1]^j$ ; i.e.,  $\xi(t) = \sum_{i=1}^j x_i 1_{[u_i, 1]}(t)$ . Formally, we define a mapping  $\hat{T}_j : \hat{S}_j \rightarrow \mathbb{D}_j$  by  $\hat{T}_j(x, u) = \sum_{i=1}^j x_i 1_{[u_i, 1]}$ , where  $\hat{S}_j \triangleq \{(x, u) \in \mathbb{R}_+^{j+} \times [0, 1]^j : 0, 1, u_1, \dots, u_j \text{ are all distinct}\}$ . Since  $d(A, \mathbb{D}_{<j}) > \gamma$ , we know that  $\hat{T}_j(x, u) \in A$  implies  $x \in (\gamma, \infty)^j$ ; see Lemma 5.4 (b). In view of this, we can see that (3.6) holds if the Lebesgue measure of  $\hat{T}_j^{-1}(\partial A)$  is 0 since  $C_j(A) = \int_{(x, u) \in \hat{T}_j^{-1}(A)} dud\nu_\alpha^j(x)$ . One of the typical settings that arises in applications is that the set  $A$  can be written as a finite combination of unions and intersections of  $\phi_1^{-1}(A_1), \dots, \phi_m^{-1}(A_m)$ , where each  $\phi_i : \mathbb{D} \rightarrow \mathbb{S}_i$  is a continuous function, and all sets  $A_i$  are subsets of general topological space  $\mathbb{S}_i$ . If we denote this operation of taking unions and intersections by  $\Psi$  (i.e.,  $A = \Psi(\phi_1^{-1}(A_1), \dots, \phi_m^{-1}(A_m))$ ), then

$$\Psi(\phi_1^{-1}(A_1^\circ), \dots, \phi_m^{-1}(A_m^\circ)) \subseteq A^\circ \subseteq A \subseteq \bar{A} \subseteq \Psi(\phi_1^{-1}(\bar{A}_1), \dots, \phi_m^{-1}(\bar{A}_m)).$$

Therefore, (3.6) holds if  $\hat{T}_j^{-1}(\Psi(\phi_1^{-1}(\bar{A}_1), \dots, \phi_m^{-1}(\bar{A}_m))) \setminus \hat{T}_j^{-1}(\Psi(\phi_1^{-1}(A_1^\circ), \dots, \phi_m^{-1}(A_m^\circ)))$  has Lebesgue measure zero. A similar principle holds for the limit measures  $C_{j,k}$ , defined in the next section where we deal with two-sided Lévy processes.

### 3.2 Two-sided Large Deviations

Consider a two-sided Lévy measure  $\nu$  for which  $\nu[x, \infty)$  is regularly varying with index  $-\alpha$  and  $\nu(-\infty, -x]$  is regularly varying with index  $-\beta$ . Let

$$\bar{X}_n(s) \triangleq \frac{1}{n} X_n(s) - sa - (\mu_1^+ \nu_1^+ - \mu_1^- \nu_1^-)s,$$

where

$$\begin{aligned} \mu_1^+ &\triangleq \frac{1}{\nu_1^+} \int_{[1, \infty)} x \nu(dx), & \nu_1^+ &\triangleq \nu[1, \infty), \\ \mu_1^- &\triangleq \frac{-1}{\nu_1^-} \int_{(-\infty, -1]} x \nu(dx), & \nu_1^- &\triangleq \nu(-\infty, -1]. \end{aligned}$$

Recall the definition of  $\mathbb{D}_{j,k}$  given below Corollary 2.2, and the definition of  $\nu_\alpha^j$  and  $\nu_\beta^k$  as given below (3.1). Let  $C_{0,0}(\cdot) \triangleq \delta_0(\cdot)$  be the Dirac measure concentrated on the zero function. For each  $(j, k) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$ , define a measure

$C_{j,k} \in \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$  concentrated on  $\mathbb{D}_{j,k}$  as  $C_{j,k}(\cdot) \triangleq \mathbf{E} \left[ \nu_\alpha^j \times \nu_\beta^k \{ (x, y) \in (0, \infty)^j \times (0, \infty)^k : \sum_{i=1}^j x_i 1_{[U_i, 1]} - \sum_{i=1}^k y_i 1_{[V_i, 1]} \in \cdot \} \right]$ , where  $U_i$ 's and  $V_i$ 's are i.i.d. uniform on  $[0, 1]$ . Recall that  $\mathbb{D}_{<j,k} = \bigcup_{(l,m) \in \mathbb{I}_{<j,k}} \mathbb{D}_{l,m}$  and  $\mathbb{I}_{<j,k} = \{(l, m) \in \mathbb{Z}_+^2 \setminus \{(j, k)\} : (\alpha - 1)l + (\beta - 1)m \leq (\alpha - 1)j + (\beta - 1)k\}$ .

As in the one-sided case, the proof of the main theorem of this section hinges on the following limit theorem.

**Theorem 3.3.** *For each  $(j, k) \in \mathbb{Z}_+^2$ ,*

$$(n\nu[n, \infty])^{-j} (n\nu(-\infty, -n])^{-k} \mathbf{P}(\bar{X}_n \in \cdot) \rightarrow C_{j,k}(\cdot) \quad (3.7)$$

in  $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$  as  $n \rightarrow \infty$ .

The proof of Theorem 3.3 builds on Theorem 3.1, using Lemma 2.2, Lemma 2.3, and Lemma 2.6 and Theorem 5.1. We provide the full proof in Section 5.2.

Let  $\mathcal{I}(j, k) \triangleq (\alpha - 1)j + (\beta - 1)k$ , and consider a pair of integers  $(\mathcal{J}(A), \mathcal{K}(A))$  such that

$$(\mathcal{J}(A), \mathcal{K}(A)) \in \underset{\substack{(j,k) \in \mathbb{Z}_+^2 \\ \mathbb{D}_{j,k} \cap A \neq \emptyset}}{\arg \min} \mathcal{I}(j, k). \quad (3.8)$$

The next theorem is the first main result of this section.

**Theorem 3.4.** *Suppose that  $A$  is a measurable set. If the argument minimum in (3.8) is non-empty and  $A$  is bounded away from  $\mathbb{D}_{<\mathcal{J}(A), \mathcal{K}(A)}$ , then the argument minimum is unique and*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty])^{\mathcal{J}(A)} (n\nu(-\infty, -n])^{\mathcal{K}(A)}} &\geq C_{\mathcal{J}(A), \mathcal{K}(A)}(A^\circ), \\ \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty])^{\mathcal{J}(A)} (n\nu(-\infty, -n])^{\mathcal{K}(A)}} &\leq C_{\mathcal{J}(A), \mathcal{K}(A)}(\bar{A}). \end{aligned} \quad (3.9)$$

Moreover, if the argument minimum in (3.8) is empty and  $A$  is bounded away from  $\mathbb{D}_{<l,m} \cup \mathbb{D}_{l,m}$  for some  $(l, m) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$ , then

$$\lim_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n, \infty])^l (n\nu(-\infty, -n])^m} = 0. \quad (3.10)$$

The proof of the theorem is provided below as a consequence of the following lemma.

**Lemma 3.1.** *Suppose that a sequence of  $\mathbb{D}$ -valued random elements  $Y_n$  satisfies (3.7) (with  $\bar{X}_n$  replaced with  $Y_n$ ) for each  $(j, k) \in \mathbb{Z}_+^2$ . Then (3.9) (with  $\bar{X}_n$  replaced with  $Y_n$ ) holds if  $A$  is a measurable set for which the argument minimum in (3.8) is non-empty, and  $A$  is bounded away from  $\mathbb{D}_{<\mathcal{J}(A), \mathcal{K}(A)}$ . Moreover, (3.10) (with  $\bar{X}_n$  replaced with  $Y_n$ ) holds if the argument minimum in (3.8) is empty and  $A$  is bounded away from  $\mathbb{D}_{<l,m} \cup \mathbb{D}_{l,m}$  for some  $(l, m) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$ .*

The proof of this lemma is provided in Section 5.2.

*Proof of Theorem 3.4.* The uniqueness of the argument minimum is immediate from the assumption that  $A$  is bounded-away from  $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$ . Since  $\bar{X}_n$  satisfies (3.7) by Theorem 3.3, the conclusion of the theorem follows from applying Lemma 3.1 with  $Y_n = \bar{X}_n$ .  $\square$

In case one is interested in a set for which the argmin of  $\mathcal{I}$  in (3.8) is not unique, a natural approach is to partition  $A$  into smaller sets and analyze each element separately. In the next theorem, we show that this strategy can be successfully employed with a minimal requirement on  $A$ . However, due to the presence of two different slowly varying functions  $n^\alpha\nu[n, \infty)$  and  $n^\beta\nu(-\infty, -n]$ , the limit behavior may not be dominated by a single  $\mathbb{D}_{l,m}$ .

To deal with this case, let  $\mathbb{I}_{=j,k} \triangleq \{(l, m) : (\alpha - 1)l + (\beta - 1)m = (\alpha - 1)j + (\beta - 1)k\}$ ,  $\mathbb{I}_{\ll j,k} \triangleq \{(l, m) : (\alpha - 1)l + (\beta - 1)m < (\alpha - 1)j + (\beta - 1)k\}$ ,  $\mathbb{D}_{=j,k} \triangleq \bigcup_{(l,m) \in \mathbb{I}_{=j,k}} \mathbb{D}_{l,m}$ , and  $\mathbb{D}_{\ll j,k} \triangleq \bigcup_{(l,m) \in \mathbb{I}_{\ll j,k}} \mathbb{D}_{l,m}$ . Denote the slowly varying functions  $n^\alpha\nu[n, \infty)$  and  $n^\beta\nu(-\infty, -n]$  with  $L_+(n)$  and  $L_-(n)$ , respectively.

**Theorem 3.5.** *Let  $A$  be a measurable set and suppose that the argument minimum in (3.8) is non-empty and contains a pair of integers  $(\mathcal{J}(A), \mathcal{K}(A))$ . If  $A_\delta \cap \mathbb{D}_{=\mathcal{J}(A),\mathcal{K}(A)}$  is bounded away from  $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$  for some  $\delta > 0$ , then for any given  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that*

$$\begin{aligned} \mathbf{P}(\bar{X}_n \in A) &\geq \frac{\sum_{(l,m)} (C_{l,m}(A^\circ) - \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)}}, \\ \mathbf{P}(\bar{X}_n \in A) &\leq \frac{\sum_{(l,m)} (C_{l,m}(\bar{A}) + \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)}}, \end{aligned} \quad (3.11)$$

for all  $n \geq N$ , where the summations are over the pairs  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$ . In particular, (3.11) holds if  $A$  is bounded away from  $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$ .

*Proof.* Note first that from Lemma 5.5 (i), there exists a  $\delta' > 0$  such that  $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$  is bounded away from  $A \cap (\mathbb{D}_{l,m})_{\delta'}$  for all  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$ . Moreover, applying Lemma 5.5 (ii) to each  $A \cap (\mathbb{D}_{l,m})_{\delta'}$ , we conclude that there exists  $\rho > 0$  such that  $A \cap (\mathbb{D}_{l,m})_\rho$  is bounded away from  $(\mathbb{D}_{j,k})_\rho$  for any two distinct pairs  $(l, m), (j, k) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$ . This means that  $A \cap (\mathbb{D}_{l,m})_\rho$ 's are all disjoint and bounded away from  $\mathbb{D}_{<l,m}$ .

To derive the lower bound, we apply Theorem 3.4 to  $A^\circ \cap (\mathbb{D}_{l,m})^\rho$  to obtain

$$\begin{aligned} C_{l,m}(A^\circ) &= C_{l,m}(A^\circ \cap \mathbb{D}_{l,m}) = C_{l,m}(A^\circ \cap \mathbb{D}_{l,m} \cap (\mathbb{D}_{l,m})^\rho) \\ &= C_{l,m}(A^\circ \cap (\mathbb{D}_{l,m})^\rho) \leq \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A^\circ \cap (\mathbb{D}_{l,m})^\rho)}{(n\nu[n, \infty))^l (n\nu(-\infty, -n])^m} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A \cap (\mathbb{D}_{l,m})^\rho)}{(n\nu[n, \infty))^l (n\nu(-\infty, -n])^m}, \end{aligned}$$

for each  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$ . That is, for any given  $\epsilon > 0$ , there exists an  $N_{l,m} \in \mathbb{N}$  such that

$$\frac{(C_{l,m}(A^\circ) - \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)l + (\beta-1)m}} \leq \mathbf{P}(\bar{X}_n \in A \cap (\mathbb{D}_{l,m})^\rho), \quad (3.12)$$

for all  $n \geq N_{l,m}$ . Meanwhile, an obvious bound holds for  $A \setminus \bigcup_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho$  ( $\mathbb{D}_{l,m}$ ) $^\rho$ ; i.e.,

$$0 \leq \mathbf{P} \left( \bar{X}_n \in A \setminus \bigcup_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho \right). \quad (3.13)$$

Since  $(\alpha - 1)l + (\beta - 1)m = (\alpha - 1)\mathcal{J}(A) + (\beta - 1)\mathcal{K}(A)$  for  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$ , summing (3.12) over  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$  together with (3.13), we arrive at the lower bound of the theorem, with  $N = \max_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} N_{l,m}$ .

Turning to the upper bound, we apply Theorem 3.4 to  $\bar{A} \cap (\mathbb{D}_{l,m})_\rho$  to get

$$\limsup_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in \bar{A} \cap (\mathbb{D}_{l,m})_\rho)}{(n\nu[n, \infty))^l (n\nu(-\infty, -n])^m} \leq C_{l,m}(\bar{A} \cap (\mathbb{D}_{l,m})_\rho) = C_{l,m}(\bar{A}).$$

for each  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$ . That is, for any given  $\epsilon > 0$ , there exists  $N'_{l,m} \in \mathbb{N}$  such that

$$\mathbf{P}(\bar{X}_n \in A \cap (\mathbb{D}_{l,m})_\rho) \leq \frac{(C_{l,m}(\bar{A}) + \epsilon/2)L_+^l(n)L_-^m(n)}{n^{(\alpha-1)l + (\beta-1)m}}, \quad (3.14)$$

for all  $n \geq N'_{l,m}$ . On the other hand, since  $\bar{A} \setminus \bigcup_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho$  is closed and bounded away from  $\mathbb{D}_{<\mathcal{J}(A), \mathcal{K}(A)}$ ,

$$\limsup_{n \rightarrow \infty} \frac{\mathbf{P} \left( \bar{X}_n \in A \setminus \bigcup_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho \right)}{(n\nu[n, \infty))^{\mathcal{J}(A)} (n\nu(-\infty, -n])^{\mathcal{K}(A)}} \leq C_{\mathcal{J}(A), \mathcal{K}(A)} \left( \bar{A} \setminus \bigcup_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho \right),$$

where the union is over the pairs  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$ . Therefore, there exists  $N'$  such that

$$\begin{aligned} & \mathbf{P} \left( \bar{X}_n \in A \setminus \bigcup_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho \right) \\ & \leq \frac{\left( C_{\mathcal{J}(A), \mathcal{K}(A)} \left( \bar{A} \setminus \bigcup_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho \right) + \epsilon/2 \right) L_+^{\mathcal{J}(A)}(n) L_-^{\mathcal{K}(A)}(n)}{n^{(\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)}} \\ & = \frac{(\epsilon/2) L_+^{\mathcal{J}(A)}(n) L_-^{\mathcal{K}(A)}(n)}{n^{(\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)}}, \end{aligned} \quad (3.15)$$

for  $n \geq N'$  since  $\bar{A} \setminus \bigcup_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (\mathbb{D}_{l,m})^\rho$  is disjoint from the support of  $C_{\mathcal{J}(A), \mathcal{K}(A)}$ . Summing (3.14) over  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$  and (3.15),

$$\mathbf{P}(\bar{X}_n \in A) \leq \frac{\sum_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} (C_{l,m}(\bar{A}) + \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)}}, \quad (3.16)$$

for  $n \geq N$ , where  $N = N' \vee \max_{(l,m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}} N'_{l,m}$ . □

## 4 Implications

This section explores the implications of the large-deviations results in Section 3, and is organized as follows. Section 4.1 proves a result similar to Theorem 3.4,

now focusing on random walks with regularly varying increments. Section 4.2 illustrates that conditional limit theorems can easily be studied by means of the limit theorems established in Section 3. Section 4.3 develops a weak large deviation principle (LDP) of the form (1.3) for the scaled Lévy processes. Finally, Section 4.4 shows that the weak LDP proved in Section 4.3 is the best one can hope for in the presence of regularly varying tails, by showing that a full LDP of the form (1.3) does not exist.

## 4.1 Random Walks

Let  $S_k, k \geq 0$ , be a random walk, set  $\bar{S}_n(t) = S_{[nt]}/n, t \geq 0$ , and define  $\bar{S}_n = \{\bar{S}_n(t), t \in [0, 1]\}$ . Let  $N(t), t \geq 0$ , be an independent unit rate Poisson process. Define the Lévy process  $X(t) \triangleq S_{N(t)}, t \geq 0$ , and set  $\bar{X}_n(t) \triangleq X(nt)/n, t \geq 0$ . The goal is to prove an analogue of Theorem 3.4 for the scaled random walk  $\bar{S}_n$ . Let  $\mathcal{J}(\cdot), \mathcal{K}(\cdot)$ , and  $C_{j,k}(\cdot)$  be defined as in Section 3.2.

**Theorem 4.1.** *Suppose that  $\mathbf{P}(S_1 \geq x)$  is regularly varying with index  $-\alpha$  and  $\mathbf{P}(S_1 \leq -x)$  is regularly varying with index  $-\beta$ . Let  $A$  be a measurable set bounded away from  $\mathbb{D}_{<\mathcal{J}(A), \mathcal{K}(A)}$ . Then*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{(n\mathbf{P}(S_1 \geq n))^{\mathcal{J}(A)}(n\mathbf{P}(S_1 \leq -n))^{\mathcal{K}(A)}} &\geq C_{\mathcal{J}(A), \mathcal{K}(A)}(A^\circ), \\ \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{(n\mathbf{P}(S_1 \geq n))^{\mathcal{J}(A)}(n\mathbf{P}(S_1 \leq -n))^{\mathcal{K}(A)}} &\leq C_{\mathcal{J}(A), \mathcal{K}(A)}(\bar{A}). \end{aligned} \quad (4.1)$$

*Proof.* The idea is to combine our notion of asymptotic equivalence with Theorem 3.4. First, we need to derive the asymptotic behavior of the Lévy measure of the constructed Lévy process. From Example A3.17 in Embrechts et al. (1997), we obtain  $\mathbf{P}(X(1) \geq x) \sim \mathbf{P}(S_1 \geq x)$ . Moreover, Embrechts et al. (1979) implies that  $\nu(x, \infty) \sim \mathbf{P}(X(1) \geq x)$ . Similarly, it follows that  $\nu(-\infty, -x) \sim \mathbf{P}(S_1 \leq -x)$ .

Now, from Lemma 3.1, (4.1) is proved if (3.7) holds for  $\bar{S}_n$ . In view of Corollary 2.1, (3.7) holds—and hence, the proof is completed—if we prove the asymptotic equivalence between  $\bar{X}_n$  and  $\bar{S}_n$  (w.r.t. a geometrically decaying sequence). To prove the asymptotic equivalence, we first argue that the Skorokhod distance between  $\bar{S}_n$  and  $\bar{X}_n$  is bounded by  $\sup_{t \in [0, 1]} |N(tn)/n - t|$ . To see this, define the homeomorphism  $\lambda_n(t)$  as the linear interpolation of the jump points of  $N(nt)/n$ , and observe that  $\bar{X}_n(t) = \bar{S}_n(\lambda_n(t))$ . Thus, the distance between  $\bar{S}_n$  and  $\bar{X}_n$  is bounded by  $\sup_{t \in [0, 1]} |\lambda_n(t) - t|$  which, in itself, is bounded by  $\sup_{t \in [0, 1]} |N(tn)/n - t|$ . From Lemma A.4,

$$\mathbf{P}\left(\sup_{t \in [0, 1]} |N(tn)/n - t| > \delta\right) \leq 3 \sup_{t \in [0, 1]} \mathbf{P}(|N(tn)/n - t| > \delta/3),$$

where  $\mathbf{P}(|N(tn)/n - t| > \delta/3)$  vanishes at a geometric rate w.r.t.  $n$  uniform in  $t \in [0, 1]$ , from which the asymptotic equivalence follows.  $\square$



## 4.2 Conditional Limit Theorems

As before,  $\bar{X}_n$  denotes the scaled Lévy process defined as in Section 3.1 for the one-sided case and Section 3.2 for the two-sided case, respectively. In this section, we present conditional limit theorems which give a precise description of the limit law of  $\bar{X}_n$  conditional on  $\bar{X}_n \in A$ .

The next result, for the one-sided case, follows immediately from the definition of weak convergence and Theorem 3.2.

**Corollary 4.1.** *Suppose that a subset  $B$  of  $\mathbb{D}$  satisfies the conditions in Theorem 3.2 and that  $C_{\mathcal{J}(B)}(B^\circ) = C_{\mathcal{J}(B)}(B) = C_{\mathcal{J}(B)}(\bar{B}) > 0$ . Let  $\bar{X}_n^{|B}$  be a process having the conditional law of  $\bar{X}_n$  given that  $\bar{X}_n \in B$ , then there exists a process  $\bar{X}_\infty^{|B}$  such that*

$$\bar{X}_n^{|B} \Rightarrow \bar{X}_\infty^{|B},$$

in  $\mathbb{D}$ . Moreover, if  $\mathbf{P}^{|B}(\cdot)$  is the law of  $\bar{X}_\infty^{|B}$ , then

$$\mathbf{P}^{|B}(\bar{X}_\infty^{|B} \in \cdot) := \frac{C_{\mathcal{J}(B)}(\cdot \cap B)}{C_{\mathcal{J}(B)}(B)}.$$

Let us provide a more direct probabilistic description of the process  $\bar{X}_\infty^{|B}$ . Directly from the definition of  $\mathbf{P}^{|B}$  we have that

$$\bar{X}_\infty^{|B}(t) = \sum_{n=1}^{\mathcal{J}(B)} \chi_n 1_{[U_n, 1]}(t),$$

where  $U_1, \dots, U_{\mathcal{J}(B)}$  are i.i.d. uniform random variables on  $[0, 1]$  and

$$\begin{aligned} & \mathbf{P}^{|B}(\chi_1 \in dx_1, \dots, \chi_{\mathcal{J}(B)} \in dx_{\mathcal{J}(B)}) \\ &= \frac{\prod_{i=1}^{\mathcal{J}(B)} (\alpha x_i^{-\alpha-1} dx_i) \mathbb{I}(x_{\mathcal{J}(B)} > \dots > x_1 > 0) \mathbf{P}\left(\sum_{n=1}^{\mathcal{J}(B)} x_n 1_{[U_n, 1]}(\cdot) \in B\right)}{C_{\mathcal{J}(B)}(B)}. \end{aligned}$$

An easier to interpret description of  $\mathbf{P}^{|B}$  can be obtained by using the fact that  $\delta_B := d(B, \mathbb{D}_{\leq \mathcal{J}(B)-1}) > 0$ . Define an auxiliary probability measure,  $\mathbf{P}_\#^{|B}$ , under which, not only  $U_1, \dots, U_{\mathcal{J}(B)}$  are i.i.d.  $\text{Uniform}(0, 1)$ , but also  $\chi_1, \dots, \chi_{\mathcal{J}(B)}$  are i.i.d. distributed  $\text{Pareto}(\alpha, \delta_B)$  and independent of the  $U_i$ 's; that is,

$$\begin{aligned} & \mathbf{P}_\#^{|B}(\chi_1 \in dx_1, \dots, \chi_{\mathcal{J}(B)} \in dx_{\mathcal{J}(B)}) \\ &= (\alpha/\delta_B)^{\mathcal{J}(B)} \prod_{i=1}^{\mathcal{J}(B)} (x_i/\delta_B)^{-\alpha-1} dx_i \mathbb{I}(x_i \geq \delta_B). \end{aligned}$$

Then, we have that

$$\mathbf{P}^{|B}(\bar{X}_\infty^{|B} \in \cdot) = \mathbf{P}_\#^{|B}(\bar{X}_\infty^{|B} \in \cdot \mid \bar{X}_\infty^{|B} \in B). \quad (4.2)$$

Moreover, note that

$$\mathbf{P}_\#^{|B} \left( \bar{X}_\infty^{|B} \in B \right) = \delta_B^{-\mathcal{J}(B)(\alpha+2)} C_{\mathcal{J}(B)}(B) > 0. \quad (4.3)$$

In view of (4.2) and (4.3) one can say, at least qualitatively, that the most likely way in which the event  $\bar{X}_n \in B$  is seen to occur is by means of  $\mathcal{J}(B)$  i.i.d. jumps which are suitably Pareto distributed and occurring uniformly throughout the time interval  $[0, 1]$ .

We now are ready to provide the corresponding conditional limit theorem for the two-sided case, building on Theorem 3.4. The proof is again immediate, using the definition of weak convergence.

**Corollary 4.2.** *Suppose that a subset  $B$  of  $\mathbb{D}$  satisfies the conditions in Theorem 3.4 and that*

$$C_{\mathcal{J}(B), \mathcal{K}(B)}(B^\circ) = C_{\mathcal{J}(B), \mathcal{K}(B)}(B) = C_{\mathcal{J}(B), \mathcal{K}(B)}(\bar{B}) > 0.$$

Let  $\bar{X}_n^{|B}$  be a process having the conditional law of  $\bar{X}_n$  given that  $\bar{X}_n \in B$ , then

$$\bar{X}_n^{|B} \Rightarrow \bar{X}_\infty^{|B},$$

in  $\mathbb{D}$ . Moreover, if  $\mathbf{P}^{|B}(\cdot)$  is the law of  $\bar{X}_\infty^{|B}$ , then

$$\mathbf{P}^{|B} \left( \bar{X}_\infty^{|B} \in \cdot \right) := \frac{C_{\mathcal{J}(B), \mathcal{K}(B)}(\cdot \cap B)}{C_{\mathcal{J}(B), \mathcal{K}(B)}(B)}.$$

A probabilistic description, completely analogous to that given for the one-sided case, can also be provided in this case. Define  $\delta_B = d(B, \mathbb{D}_{<\mathcal{J}(B), \mathcal{K}(B)}) > 0$  and introduce a probability measure  $\mathbf{P}_\#^{|B}$  under which we have the following: First,  $U_1, \dots, U_{\mathcal{J}(B)}, V_1, \dots, V_{\mathcal{K}(B)}$  are i.i.d.  $U(0, 1)$ ; second,  $\chi_1, \dots, \chi_{\mathcal{J}(B)}$  are i.i.d.  $\text{Pareto}(\alpha, \delta_B)$ , and, finally  $\varrho_1, \dots, \varrho_{\mathcal{K}(B)}$  are i.i.d.  $\text{Pareto}(\beta, \delta_B)$  random variables (all of these random variables are mutually independent). Then, write

$$\bar{X}_\infty^{|B}(t) = \sum_{n=1}^{\mathcal{J}(B)} \chi_n 1_{[U_n, 1]}(t) - \sum_{n=1}^{\mathcal{K}(B)} \varrho_n 1_{[V_n, 1]}(t).$$

Applying the same reasoning as in the one sided case we have that

$$\mathbf{P}^{|B} \left( \bar{X}_\infty^{|B} \in \cdot \right) = \mathbf{P}_\#^{|B} \left( \bar{X}_\infty^{|B} \in \cdot \mid \bar{X}_\infty^{|B} \in B \right)$$

and

$$\mathbf{P}_\#^{|B} \left( \bar{X}_\infty^{|B} \in B \right) = \delta_B^{-\mathcal{J}(B)(\alpha+2) - \mathcal{K}(B)(\beta+2)} C_{\mathcal{J}(B), \mathcal{K}(B)}(B) > 0.$$

We note that these results also hold for random walks, and thus is a significant extension of Theorem 3.1 in Durrett (1980), where it is assumed that  $\alpha > 2$  and  $B = \{\bar{X}_n(1) \geq a\}$ .

### 4.3 Large Deviation Principle

In this section, we show that  $\bar{X}_n$  satisfies a weak large deviation principle with speed  $\log n$ , and a rate function which is piece-wise linear in the number of discontinuities. More specifically, define

$$I(\xi) \triangleq \begin{cases} (\alpha - 1)\mathcal{D}_+(\xi) + (\beta - 1)\mathcal{D}_-(\xi), & \text{if } \xi \text{ is a step function \& } \xi(0) = 0; \\ \infty, & \text{otherwise.} \end{cases} \quad (4.4)$$

where  $\mathcal{D}_-(\xi)$  denotes the number of downward jumps in  $\xi$ .

**Theorem 4.2.** *The scaled process  $\bar{X}_n$  satisfies the weak large deviation principle with rate function  $I$  and speed  $\log n$ , i.e.,*

$$-\inf_{x \in G} I(x) \leq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in G)}{\log n} \quad (4.5)$$

for every open set  $G$ , and

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in K)}{\log n} \leq -\inf_{x \in K} I(x) \quad (4.6)$$

for every compact set  $K$ .

The proof of Theorem 4.2 is provided in Section 5.3. It is based on Theorem 3.4, and a reduction of the case of general  $A$  to open neighborhoods; reminiscent of arguments made in the proof of Cramér's theorem Dembo and Zeitouni (2009).

### 4.4 Nonexistence of Strong Large Deviation Principle

We conclude the current section by showing that the weak LDP presented in the previous section is the best one can hope for in our setting, in the sense that for any Lévy process  $X$  with a regularly varying Lévy measure,  $\bar{X}_n$  cannot satisfy a strong LDP; i.e., (4.6) in Theorem 4.2 cannot be extended to all closed sets.

Consider a mapping  $\pi : \mathbb{D} \rightarrow \mathbb{R}_+^2$  that maps paths in  $\mathbb{D}$  to their largest jump sizes, i.e.,

$$\pi(\xi) \triangleq \left( \sup_{t \in (0,1]} (\xi(t) - \xi(t-)), \sup_{t \in (0,1]} (\xi(t-) - \xi(t)) \right).$$

Note that  $\pi$  is continuous, since each coordinate is continuous: for example, if the first coordinate (the largest upward jump sizes) of  $\pi(\xi)$  and  $\pi(\zeta)$  differ by  $\epsilon$  then  $d(\xi, \zeta) \geq \epsilon/2$ , which implies that the first coordinate is continuous. Now, to derive a contradiction, suppose that  $\bar{X}_n$  satisfies a strong LDP. In particular, suppose (4.6) in Theorem 4.2 is true for all closed sets rather than just compact sets. Since  $\pi$  is continuous w.r.t. the  $J_1$  metric,  $\pi(\bar{X}_n)$  has to satisfy a strong LDP with rate function  $I'(y) = \inf\{I(\xi) : \xi \in \mathbb{D}, y = \pi(x)\}$  by the contraction

principle, in case  $I'$  is a rate function. (Since  $I$  is not a good rate function,  $I'$  is not automatically guaranteed to be a rate function per se; see, for example, Theorem 4.2.1 and the subsequent remarks of [Dembo and Zeitouni, 2009](#).) From the exact form of  $I'$ , given by

$$I'(y_1, y_2) = (\alpha - 1)\mathbb{I}(y_1 > 0) + (\beta - 1)\mathbb{I}(y_2 > 0),$$

one can check that  $I'$  indeed happens to be a rate function. For the sake of simplicity, suppose that  $\alpha = \beta = 2$ , and  $\nu[x, \infty) = \nu(-\infty, -x] = x^{-2}$ . Let  $\hat{J}_n^{\leq 1} \triangleq \frac{1}{n}Q_n^{\leftarrow}(\Gamma_1)1_{[U_1, 1]}$  and  $\hat{K}_n^{\leq 1} \triangleq \frac{1}{n}R_n^{\leftarrow}(\Delta_1)1_{[V_1, 1]}$  where  $Q_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu[s, \infty) < y\} = (n/y)^{1/2}$  and  $R_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu(-\infty, -s] < y\} = (n/y)^{1/2}$ . The random variables  $\Gamma_1$  and  $\Delta_1$  are standard exponential, and  $U_1, V_1$  uniform  $[0, 1]$  (see also Section 5 for similar and more general notational conventions). Note that  $\bar{Y}_n \triangleq (\hat{J}_n^{\leq 1}, \hat{K}_n^{\leq 1})$  is exponentially equivalent to  $\pi(\bar{X}_n)$  if we couple  $\pi(\bar{X}_n)$  and  $(\hat{J}_n^{\leq 1}, \hat{K}_n^{\leq 1})$ , using the representation of  $\bar{X}_n$  as in (5.4): for any  $\delta > 0$ ,  $\mathbf{P}(|\bar{Y}_n - \pi(\bar{X}_n)| > \delta) \leq \mathbf{P}(\bar{Y}_n \neq \pi(\bar{X}_n)) = \mathbf{P}(Q_n^{\leftarrow}(\Gamma_1) \leq 1 \text{ or } R_n^{\leftarrow}(\Delta_1) \leq 1)$ , which decays at an exponential rate. Hence,

$$\frac{\log \mathbf{P}(|\bar{Y}_n - \pi(\bar{X}_n)| > \delta)}{\log n} \rightarrow -\infty,$$

as  $n \rightarrow \infty$ , where  $|\cdot|$  is the Euclidean distance. As a result,  $\bar{Y}_n$  should satisfy the same (strong) LDP as  $\pi(\bar{X}_n)$ . Now, consider the set  $A \triangleq \bigcup_{k=2}^{\infty} [\log k, \infty) \times [k^{-1/2}, \infty)$ . Then, since  $[\log k, \infty) \times [k^{-1/2}, \infty) \subseteq A$  for  $k \geq 2$ ,

$$\begin{aligned} \mathbf{P}(\bar{Y}_n \in A) &\geq \mathbf{P}((\hat{J}_n^{\leq 1}, \hat{K}_n^{\leq 1}) \in [\log n, \infty) \times [n^{-1/2}, \infty)) \\ &= \mathbf{P}(Q_n^{\leftarrow}(\Gamma_1) > n \log n, R_n^{\leftarrow}(\Delta_1) > n^{1/2}) \\ &= \mathbf{P}\left(\left(\frac{n}{\Gamma_1}\right)^{1/2} > n \log n, \left(\frac{n}{\Delta_1}\right)^{1/2} > n^{1/2}\right) \\ &= \mathbf{P}\left(\Gamma_1 < \frac{1}{n(\log n)^2}\right) \mathbf{P}(\Delta_1 < 1) \\ &= (1 - e^{-\frac{1}{n(\log n)^2}})(1 - e^{-1}). \end{aligned}$$

Thus,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{P}(\bar{Y}_n \in A) &\geq \limsup_{n \rightarrow \infty} \frac{\log(1 - e^{-\frac{1}{n(\log n)^2}})(1 - e^{-1})}{\log n} \\ &\geq \limsup_{n \rightarrow \infty} \frac{\log \frac{1}{n(\log n)^2} (1 - \frac{1}{2n(\log n)^2})(1 - e^{-1})}{\log n} \\ &= -1. \end{aligned} \tag{4.7}$$

On the other hand, since  $A \subseteq (0, \infty) \times (0, \infty)$ ,

$$- \inf_{(y_1, y_2) \in A} I'(y_1, y_2) = -2. \tag{4.8}$$

Noting that  $A$  is a closed (but not compact) set, we arrive at a contradiction to the large deviation upper bound for  $\bar{Y}_n$ . This, in turn, proves that  $\bar{X}_n$  cannot satisfy a full LDP.

## 5 Proofs

Section 5.1, Section 5.2, and Section 5.3 provide proofs of the results in Section 2, Section 3, and Section 4, respectively.

### 5.1 Proofs of Section 2

Recall that  $F_\delta = \{x \in \mathbb{S} : d(x, F) \leq \delta\}$  and  $G^{-\delta} = ((G^c)_\delta)^c$ .

*Proof of Lemma 2.1.* Let  $G$  be an open set such that  $G \cap \mathbb{S}_0$  is bounded away from  $\mathbb{C}$ . For a given  $\delta > 0$ , due to the assumed asymptotic equivalence,  $\mathbf{P}(X_n \in G^{-\delta}, d(X_n, Y_n) \geq \delta) = o(\epsilon_n)$ . Therefore,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in G) \\ & \geq \liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta}, d(X_n, Y_n) < \delta) \\ & = \liminf_{n \rightarrow \infty} \epsilon_n^{-1} \{ \mathbf{P}(X_n \in G^{-\delta}) - \mathbf{P}(X_n \in G^{-\delta}, d(X_n, Y_n) \geq \delta) \} \\ & = \liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta}) \end{aligned} \tag{5.1}$$

Pick  $r > 0$  such that  $G^{-\delta} \cap \mathbb{S}_0 \cap \mathbb{C}_r = \emptyset$  and note that  $G^{-\delta} \cap \mathbb{C}_r^c$  is an open set bounded away from  $\mathbb{C}$ . Then,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta}) &= \liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta} \cap \mathbb{S}_0) \\ &= \liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta} \cap \mathbb{S}_0 \cap \mathbb{C}_r^c) \\ &= \liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in G^{-\delta} \cap \mathbb{C}_r^c) \geq \mu(G^{-\delta} \cap \mathbb{C}_r^c) \\ &= \mu(G^{-\delta} \cap \mathbb{C}_r^c \cap \mathbb{S}_0) = \mu(G^{-\delta} \cap \mathbb{S}_0) = \mu(G^{-\delta}). \end{aligned}$$

Since  $G$  is an open set,  $G = \bigcup_{\delta > 0} G^{-\delta}$ . Due to the continuity of measures,  $\lim_{\delta \rightarrow 0} \mu(G^{-\delta}) = \mu(G)$ , and hence, we arrive at the lower bound

$$\liminf_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in G) \geq \mu(G)$$

by taking  $\delta \rightarrow 0$ .

Now, turning to the upper bound, consider a closed set  $F$  such that  $F_\delta \cap \mathbb{S}_0$  is bounded away from  $\mathbb{C}$ . Given a  $\delta > 0$ , by the equivalence assumption,  $\mathbf{P}(Y_n \in$

$F, d(X_n, Y_n) \geq \delta) = o(\epsilon_n)$ . Therefore,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(Y_n \in F) \\
&= \limsup_{n \rightarrow \infty} \epsilon_n^{-1} \{ \mathbf{P}(Y_n \in F, d(X_n, Y_n) < \delta) \\
&\quad + \mathbf{P}(Y_n \in F, d(X_n, Y_n) \geq \delta) \} \\
&= \limsup_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in F_\delta) = \limsup_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in F_\delta \cap \mathbb{S}_0) \\
&\leq \limsup_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in \overline{F_\delta \cap \mathbb{S}_0}) \leq \mu(\overline{F_\delta \cap \mathbb{S}_0}) = \mu(\overline{F_\delta \cap \mathbb{S}_0} \cap \mathbb{S}_0) \\
&\leq \mu(\overline{F_\delta} \cap \mathbb{S}_0) = \mu(\overline{F_\delta}) = \mu(F_\delta).
\end{aligned} \tag{5.2}$$

Note that  $\{F_\delta\}$  is a decreasing sequence of sets,  $F = \bigcap_{\delta > 0} F_\delta$  (since  $F$  is closed), and  $\mu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$  (and hence  $\mu$  is a finite measure on  $\mathbb{S} \setminus \mathbb{C}^r$  for some  $r > 0$  such that  $F_\delta \subseteq \mathbb{S} \setminus \mathbb{C}^r$  for some  $\delta > 0$ ). Due to the continuity (from above) of finite measures,  $\lim_{\delta \rightarrow 0} \mu(F_\delta) = \mu(F)$ . Therefore, we arrive at the upper bound

$$\limsup_{n \rightarrow \infty} \epsilon_n^{-1} \mathbf{P}(X_n \in F) \leq \mu(F)$$

by taking  $\delta \rightarrow 0$ . □

For a measure  $\mu$  on a measurable space  $\mathbb{S}$ , denote the restriction of  $\mu$  to a subspace  $\mathbb{O} \subseteq \mathbb{S}$  with  $\mu|_{\mathbb{O}}$ .

*Proof of Lemma 2.2.* We provide a proof for  $d = 2$  which suffices for the application in this article. The extension to general  $d$  is straightforward, and hence, omitted. In view of the Portmanteau theorem for  $\mathbb{M}$ -convergence—in particular item (v) of Theorem 2.1 of [Lindskog et al. \(2014\)](#)—it is enough to show that for all but countably many  $r > 0$ ,  $(\mu_n^{(1)} \times \mu_n^{(2)})|_{(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r}(\cdot)$  converges to  $(\mu^{(1)} \times \mu^{(2)})|_{(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r}$  weakly on  $(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r$ , which is equipped with the relative topology as a subspace of  $\mathbb{S}_1 \times \mathbb{S}_2$ . From the assumptions of the lemma and again by Portmanteau theorem for  $\mathbb{M}$ -convergence, we note that  $\mu_n^{(1)}|_{\mathbb{S}_1 \setminus \mathbb{C}_1^r}$  converges to  $\mu^{(1)}|_{\mathbb{S}_1 \setminus \mathbb{C}_1^r}$  weakly in  $\mathbb{S}_1 \setminus \mathbb{C}_1^r$ , and  $\mu_n^{(2)}|_{\mathbb{S}_2 \setminus \mathbb{C}_2^r}$  converges to  $\mu^{(2)}|_{\mathbb{S}_2 \setminus \mathbb{C}_2^r}$  weakly in  $\mathbb{S}_2 \setminus \mathbb{C}_2^r$  for all but countably many  $r > 0$ . For such  $r$ 's,  $\mu_n^{(1)}|_{\mathbb{S}_1 \setminus \mathbb{C}_1^r} \times \mu_n^{(2)}|_{\mathbb{S}_2 \setminus \mathbb{C}_2^r}$  converges weakly to  $\mu^{(1)}|_{\mathbb{S}_1 \setminus \mathbb{C}_1^r} \times \mu^{(2)}|_{\mathbb{S}_2 \setminus \mathbb{C}_2^r}$  in  $(\mathbb{S}_1 \setminus \mathbb{C}_1^r) \times (\mathbb{S}_2 \setminus \mathbb{C}_2^r)$ . Noting that  $(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r$  coincides with  $(\mathbb{S}_1 \setminus \mathbb{C}_1^r) \times (\mathbb{S}_2 \setminus \mathbb{C}_2^r)$ , and  $\mu^{(1)}|_{\mathbb{S}_1 \setminus \mathbb{C}_1^r} \times \mu^{(2)}|_{\mathbb{S}_2 \setminus \mathbb{C}_2^r}$  and  $\mu_n^{(1)}|_{\mathbb{S}_1 \setminus \mathbb{C}_1} \times \mu_n^{(2)}|_{\mathbb{S}_2 \setminus \mathbb{C}_2}$  coincide with  $(\mu^{(1)} \times \mu^{(2)})|_{(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r}$  and  $(\mu_n^{(1)} \times \mu_n^{(2)})|_{(\mathbb{S}_1 \times \mathbb{S}_2) \setminus ((\mathbb{C}_1 \times \mathbb{S}_2) \cup (\mathbb{S}_1 \times \mathbb{C}_2))^r}$ , respectively, we reach the conclusion. □

*Proof of Lemma 2.3.* Starting with the upper bound, suppose that  $F$  is a closed set bounded away from  $\bigcap_{i=0}^m \mathbb{C}(i)$ . From the assumption, there exist  $r_0, \dots, r_m$

such that  $F \subseteq \bigcup_{i=0}^m (\mathbb{S} \setminus \mathbb{C}(i)^{r_i})$ , and hence,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(X_n \in F)}{\epsilon_n(0)} &\leq \limsup_{n \rightarrow \infty} \sum_{i=0}^m \frac{\mathbf{P}(X_n \in F \cap (\mathbb{S} \setminus \mathbb{C}(i)^{r_i}))}{\epsilon_n(i)} \frac{\epsilon_n(i)}{\epsilon_n(0)} \\ &\leq \limsup_{n \rightarrow \infty} \sum_{i=0}^m \frac{\mathbf{P}(X_n \in F \setminus \mathbb{C}(i)^{r_i})}{\epsilon_n(i)} \frac{\epsilon_n(i)}{\epsilon_n(0)} \\ &= \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(X_n \in F \setminus \mathbb{C}(0)^{r_0})}{\epsilon_n(0)} \\ &\leq \mu^{(0)}(F \setminus \mathbb{C}(0)^{r_0}) \leq \mu^{(0)}(F) \end{aligned}$$

Turning to the lower bound, if  $G$  is an open set bounded away from  $\bigcap_{i=0}^m \mathbb{C}(i)$ ,

$$\liminf_{n \rightarrow \infty} \frac{\mathbf{P}(X_n \in G)}{\epsilon_n(0)} \geq \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(X_n \in G \setminus \mathbb{C}(0)_r)}{\epsilon_n(0)} \geq \mu^{(0)}(G \setminus \mathbb{C}(0)_r).$$

Taking  $r \rightarrow 0$  yields the lower bound. □

*Proof of Lemma 2.4.* Suppose that  $\mu_n \rightarrow \mu$  in  $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ , and  $\mu(D_h \setminus \mathbb{C}^r) = 0$  and  $\mu(\partial \mathbb{S}_0 \setminus \mathbb{C}^r) = 0$  for each  $r > 0$ . Note that  $\partial h^{-1}(A') \subseteq \mathbb{S} \setminus \mathbb{C}^r$  for some  $r > 0$  due to the assumption, and  $\partial h^{-1}(A') \subseteq h^{-1}(\partial A') \cup D_h \cup \partial \mathbb{S}_0$ . Therefore,  $\mu(\partial h^{-1}(A')) \leq \mu \circ h^{-1}(\partial A') + \mu(D_h \setminus \mathbb{C}^r) + \mu(\partial \mathbb{S}_0 \setminus \mathbb{C}^r) = 0$ . Applying Theorem 2.1 (iv) of [Lindskog et al. \(2014\)](#) for  $h^{-1}(A')$ , we conclude that  $\mu_n(h^{-1}(A')) \rightarrow \mu(h^{-1}(A'))$ . Again, by Theorem 2.1 (iv) of [Lindskog et al. \(2014\)](#), this means that  $\mu_n \circ h^{-1} \rightarrow \mu \circ h^{-1}$  in  $\mathbb{M}(\mathbb{S}' \setminus \mathbb{C}')$ , and hence,  $\hat{h}$  is continuous at  $\mu$ . □

*Proof of Lemma 2.6.* The continuity of  $h$  is well known; see, for example, [Whitt \(1980\)](#). For the second claim, it is enough to prove that for each  $j$  and  $k$ ,  $h^{-1}(A) \subseteq \mathbb{D} \times \mathbb{D}$  is bounded away from  $\mathbb{D}_j \times \mathbb{D}_k$  whenever  $A \subseteq \mathbb{D}$  is bounded away from  $\mathbb{D}_{j,k}$ . Given  $j$  and  $k$ , let  $A \subseteq \mathbb{D}$  be bounded away from  $\mathbb{D}_{j,k}$ . To prove that  $h^{-1}(A)$  is bounded away from  $\mathbb{D}_j \times \mathbb{D}_k$  by contradiction, suppose that it is not. Then, for any given  $\epsilon > 0$ , one can find  $\xi \in \mathbb{D}$  and  $\zeta \in \mathbb{D}$  such that  $d(\xi, \mathbb{D}_j) < \epsilon/2$ ,  $d(\zeta, \mathbb{D}_k) < \epsilon/2$ , and  $\xi - \zeta \in A$ . Since a time-change of a step function doesn't change the number of jumps and jump-sizes, there exist  $\xi' \in \mathbb{D}_j$  and  $\zeta' \in \mathbb{D}_k$  such that  $\|\xi - \xi'\|_\infty < \epsilon/2$  and  $\|\zeta - \zeta'\|_\infty < \epsilon/2$ . Therefore,  $d(\xi - \zeta, \xi' - \zeta') \leq \|(\xi - \zeta) - (\xi' - \zeta')\|_\infty \leq \|\xi - \xi'\|_\infty + \|\zeta - \zeta'\|_\infty < \epsilon$ . From this along with the property  $d(\xi' - \zeta', \mathbb{D}_{j,k}) = 0$ , we conclude that  $d(\xi - \zeta, \mathbb{D}_{j,k}) < \epsilon$ . Taking  $\epsilon \rightarrow 0$ , we arrive at  $d(A, \mathbb{D}_j \times \mathbb{D}_k) = 0$  which is contradictory to the assumption. □

*Proof of Lemma 2.7.* From (i) and the inclusion-exclusion formula,  $\mu_n(\bigcup_{i=1}^m A_i) \rightarrow \mu(\bigcup_{i=1}^m A_i)$  as  $n \rightarrow \infty$  for any finite  $m$  if  $A_i \in \mathcal{A}_p$  is bounded away from  $\mathbb{C}$  for  $i = 1, \dots, m$ . If  $G$  is open and bounded away from  $\mathbb{C}$ , there is a sequence of sets  $A_i, i \geq 1$  in  $\mathcal{A}_p$  such that  $G = \bigcup_{i=1}^{\infty} A_i$ ; note that since  $G$  is bounded away from  $\mathbb{C}$ ,  $A_i$ 's are also bounded away from  $\mathbb{C}$ . For any  $\epsilon > 0$ , one can find  $M_\epsilon$  such that  $\mu(\bigcup_{i=1}^{M_\epsilon} A_i) \geq \mu(G) - \epsilon$ , and hence,

$$\liminf_{n \rightarrow \infty} \mu_n(G) \geq \liminf_{n \rightarrow \infty} \mu_n\left(\bigcup_{i=1}^{M_\epsilon} A_i\right) = \mu\left(\bigcup_{i=1}^{M_\epsilon} A_i\right) \geq \mu(G) - \epsilon.$$

Taking  $\epsilon \rightarrow 0$ , we arrive at the lower bound (2.2). Turning to the upper bound, given a closed set  $F$ , we pick  $A \in \mathcal{A}_p$  bounded away from  $\mathbb{C}$  such that  $F \subseteq A^\circ$ . Then,

$$\begin{aligned} \mu(A) - \limsup_{n \rightarrow \infty} \mu_n(F) &= \lim_{n \rightarrow \infty} \mu_n(A) + \liminf_{n \rightarrow \infty} (-\mu_n(F)) \\ &= \liminf_{n \rightarrow \infty} (\mu_n(A) - \mu_n(F)) = \liminf_{n \rightarrow \infty} \mu_n(A \setminus F) \\ &\geq \liminf_{n \rightarrow \infty} \mu_n(A^\circ \setminus F) \geq \mu(A^\circ \setminus F) \\ &= \mu(A) - \mu(F). \end{aligned}$$

Note that  $\mu(A) < \infty$  since  $A$  is bounded away from  $\mathbb{C}$ , which together with the above inequality establishes the upper bound (2.2).  $\square$

## 5.2 Proofs of Section 3

This section provides the proofs for the limit theorems (Theorem 3.1, Theorem 3.3) presented in Section 3. The proof of Theorem 3.1 is based on

1. The asymptotic equivalence between the target object  $\bar{X}_n$  and the process obtained by keeping its  $j$  largest jumps, which will be denoted as  $J_n^{\leq j}$ : Proposition 5.1 and Proposition 5.2 prove such asymptotic equivalences. Two technical lemmas (Lemma 5.1 and Lemma 5.2) play key roles in Proposition 5.2.
2.  $\mathbb{M}$ -convergence of  $J_n^{\leq j}$ : Lemma 5.3 identifies the convergence of jump size sequences, and Proposition 5.3 deduces the convergence of  $J_n^{\leq j}$  from the convergence of the jump size sequences via the mapping theorem established in Section 2.

For Theorem 3.3, we first establish a general result (Theorem 5.1) for the  $\mathbb{M}$ -convergence of multiple Lévy processes in the associated product space using Lemma 2.2 and 2.3. We then apply Lemma 2.6 to prove Theorem 3.3.

Recall that  $X_n(t) \triangleq X(nt)$  is a scaled process of  $X$ , where  $X$  is a Lévy process with a Lévy measure  $\nu$  supported on  $(0, \infty)$ . Also recall that  $X_n$  has



Itô representation

$$X_n(s) = ns\alpha + B(ns) + \int_{|x| \leq 1} x[N([0, ns] \times dx) - ns\nu(dx)] \quad (5.3)$$

$$+ \int_{|x| > 1} xN([0, ns] \times dx),$$

where  $N$  is the Poisson random measure with mean measure  $\text{Leb} \times \nu$  on  $[0, n] \times (0, \infty)$  and  $\text{Leb}$  denotes the Lebesgue measure. It is easy to see that

$$J_n(s) \triangleq \sum_{l=1}^{\tilde{N}_n} Q_n^{\leftarrow}(\Gamma_l) 1_{[U_l, 1]}(s) \stackrel{\mathcal{D}}{=} \int_{|x| > 1} xN([0, ns] \times dx),$$

where  $\Gamma_l = E_1 + E_2 + \dots + E_l$ ;  $E_i$ 's are i.i.d. and standard exponential random variables;  $U_l$ 's are i.i.d. and uniform variables in  $[0, 1]$ ;  $\tilde{N}_n = N_n([0, 1] \times [1, \infty))$ ;  $N_n = \sum_{l=1}^{\infty} \delta_{(U_l, Q_n^{\leftarrow}(\Gamma_l))}$ , where  $\delta_{(x, y)}$  is the Dirac measure concentrated on  $(x, y)$ ;  $Q_n(x) \triangleq n\nu[x, \infty)$ ,  $Q_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu[s, \infty) < y\}$ . Note that  $\tilde{N}_n$  is the number of  $\Gamma_l$ 's such that  $\Gamma_l \leq n\nu_1^+$ , where  $\nu_1^+ \triangleq \nu[1, \infty)$ , and hence,  $\tilde{N}_n \sim \text{Poisson}(n\nu_1^+)$ . Throughout the rest of this section, we use the following representation for the centered and scaled process  $\bar{X}_n \triangleq \frac{1}{n}X_n$ :

$$\bar{X}_n(s) \stackrel{\mathcal{D}}{=} \frac{1}{n}J_n(s) + \frac{1}{n}B(ns) \quad (5.4)$$

$$+ \frac{1}{n} \int_{|x| \leq 1} x[N([0, ns] \times dx) - ns\nu(dx)] - (\mu_1^+ \nu_1^+)s.$$

*Proof of Theorem 3.1.* We decompose  $\bar{X}_n$  into a centered compound Poisson process  $\bar{J}_n$ , a centered Lévy process with small jumps and continuous increments  $\bar{Y}_n$ , and a residual process that arises due to centering  $\bar{Z}_n$ . After that, we will show that the compound Poisson process determines the limit. More specifically, consider the following decomposition:

$$\bar{X}_n(s) \stackrel{\mathcal{D}}{=} \bar{Y}_n(s) + \bar{J}_n(s) + \bar{Z}_n(s),$$

$$\bar{Y}_n(s) \triangleq \frac{1}{n}B(ns) + \frac{1}{n} \int_{|x| \leq 1} x[N([0, ns] \times dx) - ns\nu(dx)],$$

$$\bar{J}_n(s) \triangleq \frac{1}{n} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+) 1_{[U_l, 1]}(s), \quad (5.5)$$

$$\bar{Z}_n(s) \triangleq \frac{1}{n} \sum_{l=1}^{\tilde{N}_n} \mu_1^+ 1_{[U_l, 1]}(s) - \mu_1^+ \nu_1^+ s,$$

where  $\mu_1^+ \triangleq \frac{1}{\nu_1^+} \int_{[1, \infty)} x\nu(dx)$ . Let  $\hat{J}_n^{\leq j} \triangleq \frac{1}{n} \sum_{l=1}^j Q_n^{\leftarrow}(\Gamma_l) 1_{[U_l, 1]}$  be, roughly speaking, the process obtained by just keeping the  $j$  largest (un-centered) jumps of  $\bar{J}_n$ . In view of Corollary 2.1 and Proposition 5.3, it suffices to show that

$\bar{X}_n$  and  $\hat{J}_n^{\leq j}$  are asymptotically equivalent. Proposition 5.1 along with Proposition 5.2 prove the desired asymptotic equivalence, and hence, conclude the proof of the Theorem 3.1.  $\square$

**Proposition 5.1.** *Let  $\bar{X}_n$  and  $\bar{J}_n$  be as in the proof of Theorem 3.1. Then,  $\bar{X}_n$  and  $\bar{J}_n$  are asymptotically equivalent w.r.t.  $(n\nu[n, \infty))^j$  for any  $j \geq 0$ .*

*Proof.* In view of the decomposition (5.5), we are done if we show that  $\mathbf{P}(\|\bar{Y}_n\| > \delta) = o((n\nu[n, \infty))^{-j})$  and  $\mathbf{P}(\|\bar{Z}_n\| > \delta) = o((n\nu[n, \infty))^{-j})$ . For the tail probability of  $\|\bar{Y}_n\|$ ,

$$\begin{aligned} \mathbf{P} \left[ \sup_{t \in [0,1]} |\bar{Y}_n(t)| > \delta \right] &\leq \mathbf{P} \left[ \sup_{t \in [0,n]} |B(t)| > n\delta/2 \right] \\ &\quad + \mathbf{P} \left[ \sup_{t \in [0,n]} \left| \int_{|x| \leq 1} x [N((0,t] \times dx) - t\nu(dx)] \right| > n\delta/2 \right]. \end{aligned}$$

We have an explicit expression for the first term by the reflection principle, and in particular, it decays at a geometric rate w.r.t.  $n$ . For the second term, let  $Y'(t) \triangleq \int_{|x| \leq 1} x [N((0,t] \times dx) - t\nu(dx)]$ . Using Etemadi's bound for Lévy processes (see Lemma A.4), we obtain

$$\begin{aligned} &\mathbf{P} \left[ \sup_{t \in [0,n]} \left| \int_{|x| \leq 1} x [N([0,t] \times dx) - t\nu(dx)] \right| > n\delta/2 \right] \\ &\leq 3 \sup_{t \in [0,n]} \mathbf{P} \left[ |Y'(t)| > n\delta/6 \right] \\ &\leq 3 \sup_{t \in [0,n]} \left\{ \mathbf{P} \left[ |Y'(\lfloor t \rfloor)| > n\delta/12 \right] + \mathbf{P} \left[ |Y'(t) - Y'(\lfloor t \rfloor)| > n\delta/12 \right] \right\} \\ &\leq 3 \sup_{t \in [0,n]} \mathbf{P} \left[ |Y'(\lfloor t \rfloor)| > n\delta/12 \right] + 3 \sup_{t \in [0,n]} \mathbf{P} \left[ |Y'(t) - Y'(\lfloor t \rfloor)| > n\delta/12 \right] \\ &= 3 \sup_{1 \leq k \leq n} \mathbf{P} \left[ |Y'(k)| > n\delta/12 \right] + 3 \sup_{t \in [0,1]} \mathbf{P} \left[ |Y'(t)| > n\delta/12 \right] \\ &\leq 3 \sup_{1 \leq k \leq n} \mathbf{P} \left[ \left| \sum_{i=1}^k \{Y'(i) - Y'(i-1)\} \right| > n\delta/12 \right] \\ &\quad + 3 \mathbf{P} \left[ \sup_{t \in [0,1]} |Y'(t)|^m > (n\delta/12)^m \right]. \end{aligned}$$

Since  $Y'(i) - Y'(i-1)$  are i.i.d. with  $Y'(i) - Y'(i-1) \stackrel{\mathcal{D}}{=} Y'(1) = \int_{|x| \leq 1} x [N((0,1] \times dx) - \nu(dx)]$  and  $Y'(1)$  has exponential moments, the first term decreases at a geometric rate w.r.t.  $n$  due to the Chernoff bound; on the other hand, since  $Y'(t)$  is a martingale, the second term is bounded by  $3 \frac{\mathbf{E}|Y'(1)|^m}{n^m (\delta/12)^m}$  for any  $m$  by Doob's submartingale maximal inequality. Therefore, by choosing  $m$  large enough, this

term can be made negligible. For the tail probability of  $\|\bar{Z}_n\|$ , note that  $\bar{Z}_n$  is a mean zero Lévy process with the same distribution as  $\mu_1^+(N(ns)/n - \nu_1^+s)$ , where  $N$  is the Poisson process with rate  $\nu_1^+$ . Therefore, again from the continuous-time version of Etemadi's bound, we see that  $\mathbf{P}(\|\bar{Z}_n\| > \delta)$  decays at a geometric rate w.r.t.  $n$  for any  $\delta > 0$ .  $\square$

**Proposition 5.2.** *For each  $j \geq 0$ , let  $\bar{J}_n$  and  $\hat{J}_n^{\leq j}$  be defined as in the proof of Theorem 3.1. Then,  $\bar{J}_n$  and  $\hat{J}_n^{\leq j}$  are asymptotically equivalent w.r.t.  $(n\nu[n, \infty))^j$ .*

*Proof.* With the convention that the summation is 0 in case the superscript is strictly smaller than the subscript, consider the following decomposition of  $\bar{J}_n$ :

$$\begin{aligned}\hat{J}_n^{\leq j} &\triangleq \frac{1}{n} \sum_{l=1}^j Q_n^{\leftarrow}(\Gamma_l) 1_{[U_l, 1]}, & \bar{J}_n^{> j} &\triangleq \frac{1}{n} \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+) 1_{[U_l, 1]}, \\ \check{J}_n^{\leq j} &\triangleq \frac{1}{n} \sum_{l=1}^j -\mu_1^+ 1_{[U_l, 1]}, & \bar{R}_n &\triangleq \frac{1}{n} \mathbb{I}(\tilde{N}_n < j) \sum_{l=\tilde{N}_n+1}^j (Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+) 1_{[U_l, 1]},\end{aligned}$$

so that

$$\bar{J}_n = \hat{J}_n^{\leq j} + \check{J}_n^{\leq j} + \bar{J}_n^{> j} - \bar{R}_n.$$

Note that  $\mathbf{P}(\|\check{J}_n^{\leq j}\| \geq \delta) = 0$  for sufficiently large  $n$  since  $\|\check{J}_n^{\leq j}\| = j\mu_1/n$ . On the other hand,  $\mathbf{P}(\|\bar{R}_n\| \geq \delta)$  decays at a geometric rate since  $\{\|\bar{R}_n\| \geq \delta\} \subseteq \{\tilde{N}_n < j\}$  and  $\mathbf{P}(\tilde{N}_n < j)$  decays at a geometric rate. Since  $\mathbf{P}(\|\bar{J}_n^{> j}\| \geq \delta) \leq \mathbf{P}(\|\bar{J}_n^{> j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma) + \mathbf{P}(\|\bar{J}_n^{> j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma)$ , Lemma 5.1 and Lemma 5.2 given below imply  $\mathbf{P}(\|\bar{J}_n^{> j}\| \geq \delta) = o((n\nu[n, \infty))^j)$  by choosing  $\gamma$  small enough. Therefore,  $\hat{J}_n^{\leq j}$  and  $\bar{J}_n$  are asymptotically equivalent w.r.t.  $(n\nu[n, \infty))^j$ .  $\square$

Define a measure  $\mu_\alpha^{(j)}$  on  $\mathbb{R}_+^{\infty \downarrow}$  by

$$\mu_\alpha^{(j)}(dx_1, dx_2, \dots) \triangleq \prod_{i=1}^j \nu_\alpha(dx_i) \mathbb{I}_{[x_1 \geq x_2 \geq \dots \geq x_j > 0]} \prod_{i=j+1}^{\infty} \delta_0(dx_i),$$

where  $\nu_\alpha(x, \infty) = x^{-\alpha}$ , and  $\delta_0$  is the Dirac measure concentrated at 0.

**Proposition 5.3.** *For each  $j \geq 0$ ,*

$$(n\nu[n, \infty))^{-j} \mathbf{P}(\hat{J}_n^{\leq j} \in \cdot) \rightarrow C_j(\cdot)$$

*in  $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{< j})$  as  $n \rightarrow \infty$ .*

*Proof.* Noting that  $(\mu_\alpha^{(j)} \times \text{Leb}) \circ T_j^{-1} = C_j$  and  $\mathbf{P}(\hat{J}_n^{\leq j} \in \cdot) = \mathbf{P}(\{(Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1), (U_l, l \geq 1)\} \in T_j^{-1}(\cdot))$ , Lemma 5.3 and Corollary 2.2 prove the proposition.  $\square$

**Lemma 5.1.** For any fixed  $\gamma > 0$ ,  $\delta > 0$ , and  $j \geq 0$ ,

$$\mathbf{P} \left\{ \|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma \right\} = o((n\nu[n, \infty))^j). \quad (5.6)$$

*Proof.* (Throughout the proof of this lemma, we use  $\mu_1$  and  $\nu_1$  in place of  $\mu_1^+$  and  $\nu_1^+$  respectively.) We start with the following decomposition of  $\bar{J}_n^{>j}$ : for any fixed  $\lambda \in \left(0, \frac{\delta}{3\nu_1\mu_1}\right)$ ,

$$\begin{aligned} \bar{J}_n^{>j} &= \frac{1}{n} \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l, 1]} \\ &= \tilde{J}_n^{[j+1, n\nu_1(1+\lambda)]} - \tilde{J}_n^{[\tilde{N}_n+1, n\nu_1(1+\lambda)]} \mathbb{I}(\tilde{N}_n < n\nu_1(1+\lambda)) \\ &\quad + \tilde{J}_n^{[n\nu_1(1+\lambda)+1, \tilde{N}_n]} \mathbb{I}(\tilde{N}_n > n\nu_1(1+\lambda)), \end{aligned}$$

where

$$\tilde{J}_n^{[a, b]} \triangleq \frac{1}{n} \sum_{l=[a]}^{\lfloor b \rfloor} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l, 1]}.$$

Therefore,

$$\begin{aligned} &\mathbf{P} \left\{ \|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma \right\} \\ &\leq \mathbf{P} \left( \left\| \tilde{J}_n^{[j+1, n\nu_1(1+\lambda)]} \right\| \geq \delta/3, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma \right) \\ &\quad + \mathbf{P} \left( \left\| \tilde{J}_n^{[\tilde{N}_n+1, n\nu_1(1+\lambda)]} \right\| \geq \delta/3 \right) + \mathbf{P} \left( \tilde{N}_n > n\nu_1(1+\lambda) \right) \\ &= \text{(i)} + \text{(ii)} + \text{(iii)}. \end{aligned}$$

Noting that  $\left\| \tilde{J}_n^{[\tilde{N}_n+1, n\nu_1(1+\lambda)]} \right\| \leq (\nu_1(1+\lambda) - \tilde{N}_n/n)\mu_1$  — recall that  $\tilde{N}_n$  is defined to be the number of  $l$ 's such that  $Q_n^{\leftarrow}(\Gamma_l) \geq 1$ , and hence,  $0 \leq Q_n^{\leftarrow}(\Gamma_l) < 1$  for  $l > \tilde{N}_n$  — we see that (ii) is bounded by

$$\mathbf{P}((\nu_1(1+\lambda) - \tilde{N}_n/n)\mu_1 \geq \delta/3) = \mathbf{P} \left( \frac{\tilde{N}_n}{n\nu_1} \leq 1 + \lambda - \frac{\delta}{3\nu_1\mu_1} \right),$$

which decays at a geometric rate w.r.t.  $n$  since  $\tilde{N}_n$  is Poisson with rate  $n\nu_1$ . For the same reason, (iii) decays at a geometric rate w.r.t.  $n$ . We are done if we prove that (i) is  $o((n\nu[n, \infty))^j)$ . Note that  $Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma$  implies  $Q_n(n\gamma) \geq \Gamma_j$ , and hence,

$$\begin{aligned} &\sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1) 1_{[U_l, 1]} \\ &\leq \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l, 1]} \\ &\leq \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j) - \mu_1) 1_{[U_l, 1]}. \end{aligned}$$

Therefore, if we define

$$\begin{aligned} A_n &\triangleq \{Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma\}, \\ B'_n &\triangleq \left\{ \sup_{t \in [0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j) - \mu_1) 1_{[U_l,1]}(t) \geq n\delta \right\}, \\ B''_n &\triangleq \left\{ \inf_{t \in [0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1) 1_{[U_l,1]}(t) \leq -n\delta \right\}, \end{aligned}$$

then we have that

$$(i) \leq \mathbf{P}(A_n \cap (B'_n \cup B''_n)) \leq \mathbf{P}(A_n \cap B'_n) + \mathbf{P}(A_n \cap B''_n) = \mathbf{P}(A_n)(\mathbf{P}(B'_n) + \mathbf{P}(B''_n))$$

where the last equality is from the independence of  $A_n$  and  $B'_n$  as well as of  $A_n$  and  $B''_n$  (which is, in turn, due to the independence of  $\Gamma_j$  and  $\Gamma_l - \Gamma_j$ ). From Lemma 5.4 (c) and Proposition 5.3,  $\mathbf{P}(A_n) = \mathbf{P}(\hat{J}_n^{\leq j} \in (\mathbb{D} \setminus \mathbb{D}_{< j})^{-\gamma/2}) = O((n\nu[n, \infty))^j)$ , and hence, it suffices to show that the probabilities of the complements of  $B'_n$  and  $B''_n$  converge to 1—i.e., for any fixed  $\gamma > 0$ ,

$$\mathbf{P} \left\{ \sup_{t \in [0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j) - \mu_1) 1_{[U_l,1]}(t) < n\delta \right\} \rightarrow 1, \quad (5.7)$$

and

$$\mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1) 1_{[U_l,1]}(t) > -n\delta \right\} \rightarrow 1. \quad (5.8)$$

Starting with (5.7)

$$\begin{aligned} &\mathbf{P} \left\{ \sup_{t \in [0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j) - \mu_1) 1_{[U_l,1]}(t) < n\delta \right\} \\ &= \mathbf{P} \left\{ \sup_{t \in [0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1 - j} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) < n\delta \right\} \\ &\geq \mathbf{P} \left\{ \sup_{t \in [0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1 - j} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) < n\delta, \tilde{N}_n \leq (1+\lambda)n\nu_1 - j \right\} \\ &\geq \mathbf{P} \left\{ \sup_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) < n\delta, \tilde{N}_n \leq (1+\lambda)n\nu_1 - j \right\} \\ &\geq \mathbf{P} \left\{ \sup_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) < n\delta \right\} - \mathbf{P} \left\{ \tilde{N}_n > (1+\lambda)n\nu_1 - j \right\}. \end{aligned}$$

The second inequality is due to the definition of  $Q_n^{\leftarrow}$  and that  $\mu_1 \geq 1$  (and hence  $Q_n^{\leftarrow}(\Gamma_l) - \mu_1 \leq 0$  on  $l \geq \tilde{N}_n$ ), while the last inequality comes from the generic inequality  $\mathbf{P}(A \cap B) \geq \mathbf{P}(A) - \mathbf{P}(B^c)$ . The second probability converges to 0 since  $\tilde{N}$  is Poisson with rate  $n\nu_1$ . Moving on to the first probability in the last expression, observe that  $\sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(\cdot)$  has the same distribution as the compound Poisson process  $\sum_{i=1}^{J(n)} (D_i - \mu_1)$ , where  $J$  is a Poisson process with rate  $\nu_1$  and  $D_i$ 's are i.i.d. random variables with the distribution  $\nu$  conditioned (and normalized) on  $[1, \infty)$ , i.e.,  $\mathbf{P}\{D_i \geq s\} = 1 \wedge (\nu[s, \infty)/\nu[1, \infty))$ . Using this, we obtain

$$\begin{aligned}
& \mathbf{P} \left\{ \sup_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}(t) < n\delta \right\} \\
&= \mathbf{P} \left\{ \sup_{1 \leq m \leq J(n)} \sum_{l=1}^m (D_l - \mu_1) < n\delta \right\} \tag{5.9} \\
&\geq \mathbf{P} \left\{ \sup_{1 \leq m \leq 2n\nu_1} \sum_{l=1}^m (D_l - \mu_1) < n\delta, J(n) \leq 2n\nu_1 \right\} \\
&\geq \mathbf{P} \left\{ \sup_{1 \leq m \leq 2n\nu_1} \sum_{l=1}^m (D_l - \mu_1) < n\delta \right\} - \mathbf{P}\{J(n) > 2n\nu_1\}
\end{aligned}$$

The second probability vanishes at a geometric rate w.r.t.  $n$  because  $J(n)$  is Poisson with rate  $n\nu_1$ . The first term can be investigated by the generalized Kolmogorov inequality, cf. [Shneer and Wachtel \(2009\)](#) (given as Result A.1 in Appendix A):

$$\mathbf{P} \left( \max_{1 \leq m \leq 2n\nu_1} \sum_{l=1}^m (D_l - \mu_1) \geq n\delta/2 \right) \leq C \frac{2n\nu_1 V(n\delta/2)}{(n\delta/2)^2},$$

where  $V(x) = \mathbf{E}[(D_l - \mu_1)^2; \mu_1 - x \leq D_l \leq \mu_1 + x] \leq \mu_1^2 + \mathbf{E}[D_l^2; D_l \leq \mu_1 + x]$ . Note that

$$\begin{aligned}
\mathbf{E}[D_l^2; D_l \leq \mu_1 + x] &= \int_0^1 2s ds + \int_1^{\mu_1+x} 2s \frac{\nu[s, \infty)}{\nu[1, \infty)} ds \\
&= 1 + \frac{2}{\nu_1} (\mu_1 + x)^{2-\alpha} L(\mu_1 + x),
\end{aligned}$$

for some slowly varying  $L$ . Hence,

$$\mathbf{P} \left( \max_{1 \leq m \leq 2n\nu_1} \sum_{l=1}^m (D_l - \mu_1) < n\delta \right) \geq 1 - \mathbf{P} \left( \max_{1 \leq m \leq 2n\nu_1} \sum_{l=1}^m (D_l - \mu_1) \geq n\delta/2 \right) \rightarrow 1,$$

as  $n \rightarrow \infty$ .

Now, turning to (5.8), let  $\gamma_n \triangleq Q_n(n\gamma)$ .

$$\begin{aligned}
& \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} (Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1) 1_{[U_l,1]}(t) > -n\delta \right\} \\
&= \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1-j} (Q_n^{\leftarrow}(\Gamma_l + \gamma_n) - \mu_1) 1_{[U_l,1]}(t) > -n\delta \right\} \\
&\geq \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1-j} (Q_n^{\leftarrow}(\Gamma_l + \gamma_n) - \mu_1) 1_{[U_l,1]}(t) > -n\delta, E_0 \geq \gamma_n \right\} \\
&\geq \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1-j} (Q_n^{\leftarrow}(\Gamma_l + E_0) - \mu_1) 1_{[U_l,1]}(t) > -n\delta, E_0 \geq \gamma_n \right\} \\
&= \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) > -n\delta, \Gamma_1 \geq \gamma_n \right\} \\
&\geq \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) > -n\delta \right\} - \mathbf{P} \{ \Gamma_1 < \gamma_n \} \\
&= (A) - (B),
\end{aligned}$$

where  $E_0$  is a standard exponential random variable. (Recall that  $\Gamma_l \triangleq E_1 + E_2 + \dots + E_l$ , and hence  $(\Gamma_l + E_0, U_l) \stackrel{\mathcal{D}}{=} (\Gamma_{l+1}, U_l) \stackrel{\mathcal{D}}{=} (\Gamma_{l+1}, U_{l+1})$ .) Since  $(B) = \mathbf{P} \{ \Gamma_1 < \gamma_n \} \rightarrow 0$  (recall that  $\gamma_n = n\nu[n\gamma, \infty)$  and  $\nu$  is regularly varying with index  $-\alpha < -1$ ), we focus on proving that the first term (A) converges to

1:

$$\begin{aligned}
(A) &= \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) > -n\delta \right\} \\
&\geq \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) > -n\delta, \right. \\
&\qquad \qquad \qquad \left. \tilde{N}_n \leq (1+\lambda)n\nu_1 - j + 1 \right\} \\
&\geq \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) \geq -n\delta/3, \right. \\
&\qquad \qquad \qquad \inf_{t \in [0,1]} -(Q_n^{\leftarrow}(\Gamma_1) - \mu_1) 1_{[U_1,1]}(t) > -n\delta/3, \\
&\qquad \qquad \qquad \inf_{t \in [0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) \geq -n\delta/3, \\
&\qquad \qquad \qquad \left. \tilde{N}_n \leq (1+\lambda)n\nu_1 - j + 1 \right\} \\
&\geq \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) \geq -n\delta/3, \right\} \\
&\quad + \mathbf{P} \left\{ Q_n^{\leftarrow}(\Gamma_1) - \mu_1 < n\delta/3 \right\} \\
&\quad + \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) \geq -n\delta/3 \right\} \\
&\quad + \mathbf{P} \left\{ \tilde{N}_n \leq (1+\lambda)n\nu_1 - j + 1 \right\} - 3 \\
&= (\text{AI}) + (\text{AII}) + (\text{AIII}) + (\text{AIV}) - 3.
\end{aligned}$$

The third inequality comes from applying the generic inequality  $\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1$  three times. Since  $\tilde{N}_n$  is Poisson with rate  $n\nu_1$ ,

$$(\text{AIV}) = \mathbf{P} \left\{ \tilde{N}_n \leq (1+\lambda)n\nu_1 - j + 1 \right\} = \mathbf{P} \left\{ \frac{\tilde{N}_n}{n\nu_1} \leq 1 + \lambda - \frac{j-1}{n\nu_1} \right\} \rightarrow 1.$$



For the first term (AI),

$$\begin{aligned}
(\text{AI}) &= \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) \geq -n\delta/3 \right\} \\
&= \mathbf{P} \left\{ \sup_{t \in [0,1]} \sum_{l=1}^{\tilde{N}_n} (\mu_1 - Q_n^{\leftarrow}(\Gamma_l)) 1_{[U_l,1]}(t) \leq n\delta/3 \right\} \\
&= \mathbf{P} \left\{ \sup_{1 \leq m \leq J(n)} \sum_{l=1}^m (\mu_1 - D_l) \leq n\delta/3 \right\},
\end{aligned}$$

where  $D_i$  is defined as before. Note that this is of exactly same form as (5.9) except for the sign of  $D_l$ , and hence, we can proceed exactly the same way using the generalized Kolmogorov inequality to prove that this quantity converges to 1 — recall that the formula only involves the square of the increments, and hence, the change of the sign has no effect. For the second term (AII),

$$(\text{AII}) \geq \mathbf{P}\{Q_n^{\leftarrow}(\Gamma_1) \leq n\delta/3\} \geq \mathbf{P}\{\Gamma_1 > Q_n(n\delta/3)\} \rightarrow 1,$$

since  $Q_n(n\delta/3) \rightarrow 0$ . For the third term (AIII),

$$\begin{aligned}
(\text{AIII}) &= \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]}(t) \geq -n\delta/3 \right\} \\
&\geq \mathbf{P} \left\{ \inf_{t \in [0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} (1 - \mu_1) 1_{[U_l,1]}(t) \geq -n\delta/3 \right\} \\
&\geq \mathbf{P} \left\{ \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} (\mu_1 - 1) \leq n\delta/3 \right\} \\
&\geq \mathbf{P} \left\{ (\mu_1 - 1)((1 + \lambda)n\nu_1 - j - \tilde{N}_n + 1) \leq n\delta/3 \right\} \\
&\geq \mathbf{P} \left\{ 1 + \lambda - \frac{\delta}{3\nu_1(\mu_1 - 1)} \leq \frac{\tilde{N}_n}{n\nu_1} + \frac{j-1}{n\nu_1} \right\} \\
&\rightarrow 1,
\end{aligned}$$

since  $\lambda < \frac{\delta}{3\nu_1(\mu_1-1)}$ . This concludes the proof of the lemma.  $\square$

**Lemma 5.2.** *For any  $j \geq 0$ ,  $\delta > 0$ , and  $m < \infty$ , there is  $\gamma_0 > 0$  such that*

$$\mathbf{P} \left\{ \|\bar{J}_n^{\geq j}\| > \delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma_0 \right\} = o(n^{-m}).$$

*Proof.* (Throughout the proof of this lemma, we use  $\mu_1$  and  $\nu_1$  in place of  $\mu_1^+$  and  $\nu_1^+$  respectively, for the sake of notational simplicity.) Note first that

$Q_n^\leftarrow(\Gamma_j) = \infty$  if  $j = 0$  and hence the claim of the lemma is trivial. Therefore, we assume  $j \geq 1$  throughout the rest of the proof. Since for any  $\lambda > 0$

$$\begin{aligned} & \mathbf{P} \left\{ \|\bar{J}_n^{>j}\| > \delta, Q_n^\leftarrow(\Gamma_j) \leq n\gamma \right\} \\ & \leq \mathbf{P} \left\{ \left\| \sum_{l=j+1}^{\tilde{N}_n} (Q_n^\leftarrow(\Gamma_l) - \mu_1) 1_{[U_l, 1]} \right\| > n\delta, Q_n^\leftarrow(\Gamma_j) \leq n\gamma, \right. \\ & \qquad \qquad \qquad \left. \frac{\tilde{N}_n}{n\nu_1} \in \left[ \frac{j}{n\nu_1}, 1 + \lambda \right] \right\} \\ & + \mathbf{P} \left\{ \frac{\tilde{N}_n}{n\nu_1} \notin \left[ \frac{j}{n\nu_1}, 1 + \lambda \right] \right\}, \end{aligned} \quad (5.10)$$

and  $\mathbf{P} \left\{ \frac{\tilde{N}_n}{n\nu_1} \notin \left[ \frac{j}{n\nu_1}, 1 + \lambda \right] \right\}$  decays at a geometric rate w.r.t.  $n$ , it suffices to show that (5.10) is  $o(n^{-m})$  for small enough  $\gamma > 0$ . First, recall that by the definition of  $Q_n^\leftarrow(\cdot)$ ,

$$Q_n^\leftarrow(x) \geq s \quad \iff \quad x \leq Q_n(s),$$

and

$$n\nu(Q_n^\leftarrow(x), \infty) \leq x \leq n\nu[Q_n^\leftarrow(x), \infty).$$

Let  $L$  be a random variable conditionally (on  $\tilde{N}_n$ ) independent of everything else and uniformly sampled on  $\{j+1, j+2, \dots, \tilde{N}_n\}$ . Recall that given  $\tilde{N}_n$  and  $\Gamma_j$ , the distribution of  $\{\Gamma_{j+1}, \Gamma_{j+2}, \dots, \Gamma_{\tilde{N}_n}\}$  is same as that of the order statistics of  $\tilde{N}_n - j$  uniform random variables on  $[\Gamma_j, n\nu[1, \infty)]$ . Let  $D_l, l \geq 1$ , be i.i.d. random variables whose conditional distribution is the same as the conditional distribution of  $Q_n^\leftarrow(\Gamma_L)$  given  $\tilde{N}_n$  and  $\Gamma_j$ . Then the conditional distribution of  $\sum_{l=j+1}^{\tilde{N}_n} (Q_n^\leftarrow(\Gamma_l) - \mu_1) 1_{[U_l, 1]}$  is the same as that of  $\sum_{l=1}^{\tilde{N}_n - j} (D_l - \mu_1) 1_{[U_l, 1]}$ . Therefore, the conditional distribution of  $\left\| \sum_{l=j+1}^{\tilde{N}_n} (Q_n^\leftarrow(\Gamma_l) - \mu_1) 1_{[U_l, 1]} \right\|_\infty$  is the same as the corresponding conditional distribution of  $\sup_{1 \leq m \leq \tilde{N}_n - j} \left| \sum_{l=1}^m (D_l - \mu_1) \right|$ . To make use of this in the analysis what follows, we make a few observations on the conditional distribution of  $Q_n^\leftarrow(\Gamma_L)$  given  $\Gamma_j$  and  $\tilde{N}_n$ .

(a) *The conditional distribution of  $Q_n^\leftarrow(\Gamma_L)$ :*

Let  $q \triangleq Q_n^\leftarrow(\Gamma_j)$ . Since  $\Gamma_L$  is uniformly distributed on  $[\Gamma_j, Q_n(1)] = [\Gamma_j, n\nu[1, \infty)]$ , the tail probability is

$$\begin{aligned} \mathbf{P}\{Q_n^\leftarrow(\Gamma_L) \geq s | \Gamma_j, \tilde{N}_n\} &= \mathbf{P}\{\Gamma_L \leq Q_n(s) | \Gamma_j, \tilde{N}_n\} \\ &= \mathbf{P}\{\Gamma_L \leq n\nu[s, \infty) | \Gamma_j, \tilde{N}_n\} \\ &= \mathbf{P} \left\{ \frac{\Gamma_L - \Gamma_j}{n\nu[1, \infty) - \Gamma_j} \leq \frac{n\nu[s, \infty) - \Gamma_j}{n\nu[1, \infty) - \Gamma_j} \middle| \Gamma_j, \tilde{N}_n \right\} \\ &= \frac{n\nu[s, \infty) - \Gamma_j}{n\nu[1, \infty) - \Gamma_j} \end{aligned}$$

for  $s \in [1, q]$ ; since this is non-increasing w.r.t.  $\Gamma_j$  and  $n\nu(q, \infty) \leq \Gamma_j \leq n\nu[q, \infty)$ , we have that

$$\frac{\nu[s, q]}{\nu[1, q]} \leq \mathbf{P}\{Q_n^{\leftarrow}(\Gamma_L) \geq s | \Gamma_j, \tilde{N}_n\} \leq \frac{\nu[s, q]}{\nu[1, q]}.$$

- (b) *Difference in mean between conditional and unconditional distribution:*  
From (a), we obtain

$$\tilde{\mu}_n \triangleq \mathbf{E}[Q_n^{\leftarrow}(\Gamma_L) | \Gamma_j, \tilde{N}_n] \in \left[ 1 + \int_1^q \frac{\nu[s, q]}{\nu[1, q]} ds, 1 + \int_1^q \frac{\nu[s, q]}{\nu[1, q]} ds \right],$$

and hence,

$$|\mu_1 - \tilde{\mu}_n| \leq \left| \frac{\nu[1, q] \int_1^\infty \nu[s, \infty) ds - \nu[1, \infty) \int_1^q \nu[s, q] ds}{\nu[1, \infty) \nu[1, q]} \right| \sqrt{\left| \frac{\nu[1, q] \int_1^\infty \nu[s, \infty) ds - \nu[1, \infty) \int_1^q \nu[s, q] ds}{\nu[1, \infty) \nu[1, q]} \right|}.$$

Since

$$\begin{aligned} & \frac{\nu[1, q] \int_1^\infty \nu[s, \infty) ds - \nu[1, \infty) \int_1^q \nu[s, q] ds}{\nu[1, \infty) \nu[1, q]} \\ &= \frac{\nu[q, \infty)}{\nu[1, q]} (q-1) + \frac{\int_q^\infty \nu[s, \infty) ds}{\nu[1, \infty)} - \frac{\nu[q, \infty) \int_1^q \nu[s, \infty) ds}{\nu[1, \infty) \nu[1, q]}, \end{aligned}$$

and

$$\begin{aligned} & \frac{\nu[1, q] \int_1^\infty \nu[s, \infty) ds - \nu[1, \infty) \int_1^q \nu[s, q] ds}{\nu[1, \infty) \nu[1, q]} \\ & \quad - \frac{\nu[1, q] \int_1^\infty \nu[s, \infty) ds - \nu[1, \infty) \int_1^q \nu[s, q] ds}{\nu[1, \infty) \nu[1, q]} \\ &= \frac{\nu\{q\} \left( (q-1)\nu[1, \infty) + \int_q^\infty \nu[s, \infty) ds + \int_1^q \nu[s, \infty) ds \right)}{\nu[1, \infty) (\nu[1, q] + \nu\{q\})}, \end{aligned}$$

we see that  $|\mu_1 - \tilde{\mu}_n|$  is bounded by a regularly varying function with index  $1 - \alpha$  (w.r.t.  $q$ ) from Karamata's theorem.

- (c) *Variance of  $Q_n^{\leftarrow}(\Gamma_L)$ :* Turning to the variance, we observe that, if  $\alpha \leq 2$ ,

$$\begin{aligned} & \mathbf{E}[Q_n^{\leftarrow}(\Gamma_L)^2 | \Gamma_j, \tilde{N}_n] \\ & \leq \int_0^1 2s ds + 2 \int_1^q s \frac{\nu[s, q]}{\nu[1, q]} ds \\ & \leq 1 + \frac{2}{\nu[1, q]} \int_1^q s \nu[s, \infty) ds = 1 + q^{2-\alpha} L(q) \end{aligned} \tag{5.11}$$

for some slowly varying function  $L(\cdot)$ . If  $\alpha > 2$ , the variance is bounded w.r.t.  $q$ .

Now, with (b) and (c) in hand, we can proceed with an explicit bound since all the randomness is contained in  $q$ . Namely, we infer

$$\begin{aligned}
& \mathbf{P} \left( \left\| \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l, 1]} \right\|_{\infty} > n\delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma, \frac{\tilde{N}_n}{n\nu_1} \in \left[ \frac{j}{n\nu_1}, 1 + \lambda \right] \right) \\
&= \mathbf{P} \left( \left\| \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l, 1]} \right\|_{\infty} > n\delta, \Gamma_j \geq Q_n(n\gamma), \frac{\tilde{N}_n}{n\nu_1} \in \left[ \frac{j}{n\nu_1}, 1 + \lambda \right] \right) \\
&= \mathbf{E} \left[ \mathbf{P} \left( \left\| \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l, 1]} \right\|_{\infty} > n\delta \middle| \Gamma_j, \tilde{N}_n \right); \Gamma_j \geq Q_n(n\gamma), \right. \\
&\qquad\qquad\qquad \left. \frac{\tilde{N}_n}{n\nu_1} \in \left[ \frac{j}{n\nu_1}, 1 + \lambda \right] \right) \\
&= \mathbf{E} \left[ \mathbf{P} \left( \max_{1 \leq m \leq \tilde{N}_n - j} \left| \sum_{l=1}^m (D_l - \mu_1) \right| > n\delta \middle| \Gamma_j, \tilde{N}_n \right); \Gamma_j \geq Q_n(n\gamma), \right. \\
&\qquad\qquad\qquad \left. \frac{\tilde{N}_n}{n\nu_1} \in \left[ \frac{j}{n\nu_1}, 1 + \lambda \right] \right).
\end{aligned}$$

By Etemadi's bound (Result A.2 in Appendix),

$$\begin{aligned}
& \mathbf{P} \left( \max_{1 \leq m \leq \tilde{N}_n - j} \left| \sum_{l=1}^m (D_l - \mu_1) \right| \geq n\delta \middle| \Gamma_j, \tilde{N}_n \right) \\
&\leq 3 \max_{1 \leq m \leq \tilde{N}_n} \mathbf{P} \left( \left| \sum_{l=1}^m (D_l - \mu_1) \right| \geq n\delta \middle| \Gamma_j, \tilde{N}_n \right) \\
&\leq 3 \max_{1 \leq m \leq \tilde{N}_n} \left\{ \mathbf{P} \left( \sum_{l=1}^m (D_l - \mu_1) \geq n\delta \middle| \Gamma_j, \tilde{N}_n \right) \right. \\
&\qquad\qquad\qquad \left. + \mathbf{P} \left( \sum_{l=1}^m (\mu_1 - D_l) \geq n\delta \middle| \Gamma_j, \tilde{N}_n \right) \right\} \\
&\qquad\qquad\qquad (5.12)
\end{aligned}$$

and as  $|D_l - \tilde{\mu}_n|$  is bounded by  $q$ , we can apply Prokhorov's bound (Result A.3

in Appendix) to get

$$\begin{aligned}
& \mathbf{P} \left( \sum_{l=1}^m (\mu_1 - D_l) \geq n\delta \middle| \Gamma_j, \tilde{N}_n \right) \\
&= \mathbf{P} \left( \sum_{l=1}^m (\tilde{\mu}_n - D_l) \geq n\delta - m(\mu_1 - \tilde{\mu}_n) \middle| \Gamma_j, \tilde{N}_n \right) \\
&\leq \mathbf{P} \left( \sum_{l=1}^m (\tilde{\mu}_n - D_l) \geq n\delta - n\nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n) \middle| \Gamma_j, \tilde{N}_n \right) \\
&\leq \left( \frac{qn(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{m\mathbf{var}(Q_n^{\leftarrow}(\Gamma_L))} \right)^{-\frac{n(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{2q}} \\
&\leq \left( \frac{n\nu_1(1+\lambda)\mathbf{var}(Q_n^{\leftarrow}(\Gamma_L))}{qn(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))} \right)^{\frac{n(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{2q}} \\
&= \begin{cases} \left( \frac{\nu_1(1+\lambda)(1+q^{2-\alpha}L_1(q))}{q(\delta - \nu_1(1+\lambda)q^{1-\alpha}L_2(q))} \right)^{\frac{n(\delta - \nu_1(1+\lambda)q^{1-\alpha}L_2(q))}{2q}} & \text{if } \alpha \leq 2, \\ \left( \frac{\nu_1(1+\lambda)C}{q(\delta - \nu_1(1+\lambda)q^{1-\alpha}L_2(q))} \right)^{\frac{n(\delta - \nu_1(1+\lambda)q^{1-\alpha}L_2(q))}{2q}} & \text{otherwise,} \end{cases}
\end{aligned}$$

for some  $C > 0$  if  $m \leq (1+\lambda)n\nu_1$ . Therefore, there exist constants  $M$  and  $c$  such that  $q \geq M$  (i.e.,  $\Gamma_j \leq Q_n(M)$ ) implies

$$\mathbf{P} \left( \sum_{l=1}^m (\mu_1 - D_l) \geq n\delta \middle| \Gamma_j \right) \leq c(q^{1-\alpha\wedge 2})^{\frac{n\delta}{8q}},$$

and since we are conditioning on  $q = Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma$ ,

$$c(q^{1-\alpha\wedge 2})^{\frac{n\delta}{8q}} \leq c(q^{1-\alpha\wedge 2})^{\frac{\delta}{8\gamma}}.$$

Hence,

$$\mathbf{P} \left( \sum_{l=1}^m (\mu_1 - D_l) \geq n\delta \middle| \Gamma_j \right) \leq c(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2})^{\frac{\delta}{8\gamma}}.$$

With the same argument, we also get

$$\mathbf{P} \left( \sum_{l=1}^m (D_l - \mu_1) \geq n\delta \middle| \Gamma_j \right) \leq c(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2})^{\frac{\delta}{8\gamma}}.$$

Combining (5.12) with the two previous estimates, we obtain

$$\mathbf{P} \left( \max_{1 \leq m \leq \tilde{N}_n - j} \left| \sum_{l=1}^m (D_l - \mu_1) \right| \geq n\delta \middle| \Gamma_j, \tilde{N}_n \right) \leq 6c(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2})^{\frac{\delta}{8\gamma}},$$

on  $\Gamma_j \geq Q_n(n\gamma)$ ,  $\tilde{N}_n - j \leq n\nu_1(1 + \lambda)$ , and  $\Gamma_j \leq Q_n(M)$ . Now,

$$\begin{aligned}
& \mathbf{E} \left[ \mathbf{P} \left( \max_{1 \leq m \leq \tilde{N}_n - j} \left| \sum_{l=1}^m (D_l - \mu_1) \right| > n\delta \middle| \Gamma_j, \tilde{N}_n \right); \Gamma_j \geq Q_n(n\gamma) \right. \\
& \qquad \qquad \qquad \left. \& \frac{\tilde{N}_n}{n\nu_1} \in \left[ \frac{j}{n\nu_1}, 1 + \lambda \right] \right] \\
& \leq \mathbf{E} \left[ \mathbf{P} \left( \max_{1 \leq m \leq \tilde{N}_n - j} \left| \sum_{l=1}^m (D_l - \mu_1) \right| > n\delta \middle| \Gamma_j, \tilde{N}_n \right); \Gamma_j \geq Q_n(n\gamma); \right. \\
& \qquad \qquad \qquad \left. \frac{\tilde{N}_n}{n\nu_1} \in \left[ \frac{j}{n\nu_1}, 1 + \lambda \right]; \Gamma_j \leq Q_n(M) \right] \\
& \quad + \mathbf{P}(\Gamma_j > Q_n(M)) \\
& \leq \mathbf{E} \left[ 6c \left( Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2} \right)^{\frac{\delta}{8\gamma}} \right] + \mathbf{P}(\Gamma_j > Q_n(M)) \\
& \leq \mathbf{E} \left[ 6c \left( Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2} \right)^{\frac{\delta}{8\gamma}}; Q_n^{\leftarrow}(\Gamma_j) \geq n^\beta \right] + \mathbf{P} \left( Q_n^{\leftarrow}(\Gamma_j) < n^\beta \right) \\
& \qquad \qquad \qquad + \mathbf{P}(\Gamma_j > Q_n(M)) \\
& \leq 6c \left( n^{\beta(1-\alpha\wedge 2)} \right)^{\frac{\delta}{8\gamma}} + \mathbf{P} \left( \Gamma_j > Q_n(n^\beta) \right) + \mathbf{P}(\Gamma_j > Q_n(M)) \\
& \leq 6c \left( n^{\beta(1-\alpha\wedge 2)} \right)^{\frac{\delta}{8\gamma}} + \mathbf{P} \left( \Gamma_j > (n^{1-\alpha\beta} L(n)) \right) + \mathbf{P}(\Gamma_j > Q_n(M)),
\end{aligned}$$

for any  $\beta > 0$ . If one chooses  $\beta$  so that  $1 - \alpha\beta > 0$  (for example,  $\beta = \frac{1}{2\alpha}$ ), the second and third terms vanish at a geometric rate w.r.t.  $n$ . On the other hand, we can pick  $\gamma$  small enough compared to  $\delta$ , so that the first term is decreasing at an arbitrarily fast polynomial rate. This concludes the proof of the lemma.  $\square$

Recall that we denote the Lebesgue measure on  $[0, 1]^\infty$  with  $\text{Leb}$  and defined measures  $\mu_\alpha^{(j)}$  and  $\mu_\beta^{(j)}$  on  $\mathbb{R}_+^{\infty\downarrow}$  as

$$\mu_\alpha^{(j)}(dx_1, dx_2, \dots) \triangleq \prod_{i=1}^j \nu_\alpha(dx_i) \mathbb{I}_{[x_1 \geq x_2 \geq \dots \geq x_j > 0]} \prod_{i=j+1}^{\infty} \delta_0(dx_i),$$

and  $\nu_\alpha(x, \infty) = x^{-\alpha}$ , where  $\delta_0$  is the Dirac measure concentrated at 0.

**Lemma 5.3.** *For each  $j \geq 0$ ,*

$$(n\nu[n, \infty])^{-j} \mathbf{P}[\left( (Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1), (U_l, l \geq 1) \right) \in \cdot] \rightarrow (\mu_\alpha^{(j)} \times \text{Leb})(\cdot)$$

in  $\mathbb{M}((\mathbb{R}_+^{\infty\downarrow} \times [0, 1]^\infty) \setminus (\mathbb{H}_{< j} \times [0, 1]^\infty))$  as  $n \rightarrow \infty$ .

*Proof.* We first prove that

$$(n\nu[n, \infty])^{-j} \mathbf{P}[\left( Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1 \right) \in \cdot] \rightarrow \mu_\alpha^{(j)}(\cdot) \quad (5.13)$$

in  $\mathbb{M}(\mathbb{R}_+^{\infty\downarrow} \setminus \mathbb{H}_{<j})$  as  $n \rightarrow \infty$ . To show this, we only need to check that

$$(n\nu[n, \infty))^{-j} \mathbf{P}[(Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1) \in A] \rightarrow \mu_\alpha^{(j)}(A) \quad (5.14)$$

for  $A$ 's that belong to the convergence-determining class  $\mathcal{A}_j \triangleq \{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1, \dots, x_l \leq z_l : l \geq j, x_1 \geq \dots \geq x_l > 0\}$ . To see that  $\mathcal{A}_j$  is a convergence-determining class for  $\mathbb{M}(\mathbb{R}_+^{\infty\downarrow} \setminus \mathbb{H}_{<j})$ -convergence, note that  $\mathcal{A}'_j \triangleq \{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1 < y_1, \dots, x_l \leq z_l < y_l : l \geq j, x_1, \dots, x_l \in (0, \infty), y_1, \dots, y_l \in (0, \infty)\}$  satisfies conditions (i), (ii), and (iii) of Lemma 2.7, and hence, is a convergence-determining class. Now define  $\mathcal{A}_j(i)$ 's recursively as  $\mathcal{A}_j(i+1) \triangleq \{B \setminus A : A, B \in \mathcal{A}_j(i), A \subseteq B\}$  for  $i \geq 0$ , and  $\mathcal{A}_j(0) = \mathcal{A}'_j \triangleq \{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1, \dots, x_l \leq z_l : l \geq j, x_1, \dots, x_l > 0\}$ . Since we restrict the set-difference operation between nested sets, the limit associated with the sets in  $\mathcal{A}_j(i+1)$  is determined by the sets in  $\mathcal{A}_j(i)$ , and eventually,  $\mathcal{A}'_j$ . Noting that  $\mathcal{A}'_j \subseteq \bigcup_{i=0}^{\infty} \mathcal{A}_j(i)$ , we see that  $\mathcal{A}'_j$  is a convergence-determining class. Now, since both  $\mathbf{P}[(Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1) \in \cdot]$  and  $\mu_\alpha^{(j)}(\cdot)$  are supported on  $\mathbb{R}_+^{\infty\downarrow}$ , one can further reduce the convergence determining class from  $\mathcal{A}'_j$  to  $\mathcal{A}_j$ .

To check the desired convergence for the sets in  $\mathcal{A}_j$ , we first characterize the limit measure. Let  $l \geq j$  and  $x_1 \geq \dots \geq x_l > 0$ . By the change of variables  $v_i = x_i^\alpha y_i^{-\alpha}$  for  $i = 1, \dots, j$ ,

$$\begin{aligned} & \mu_\alpha^{(j)}(\{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 \leq z_1, \dots, x_l \leq z_l\}) \\ &= \mathbb{I}(j = l) \cdot \int_{x_j}^{\infty} \cdots \int_{x_1}^{\infty} \mathbb{I}(y_1 \geq \cdots \geq y_j) d\nu_\alpha(y_1) \cdots d\nu_\alpha(y_j) \\ &= \mathbb{I}(j = l) \cdot \left( \prod_{i=1}^j x_i \right)^{-\alpha} \cdot \int_0^1 \cdots \int_0^1 \mathbb{I}(x_1^{-\alpha} v_1 \leq \cdots \leq x_j^{-\alpha} v_j) dv_1 \cdots dv_j. \end{aligned}$$

Next, we find a similar representation for the distribution of  $\Gamma_1, \dots, \Gamma_l$ . Let  $U_{(1)}, \dots, U_{(l-1)}$  be the order statistics of  $l-1$  iid uniform random variables on  $[0, 1]$ . Recall first that the conditional distribution of  $(\Gamma_1/\Gamma_l, \dots, \Gamma_{l-1}/\Gamma_l)$  given  $\Gamma_j$  does not depend on  $\Gamma_j$  and coincides with the distribution of  $(U_{(1)}, \dots, U_{(l-1)})$ ; see, for example, Pyke (1965). Suppose that  $l \geq j$  and  $0 \leq y_1 \leq \dots \leq y_l$ . By

the change of variables  $u_i = \gamma^{-1}y_i v_i$  for  $i = 1, \dots, l-1$ , and  $\gamma = y_l v_l$ ,

$$\begin{aligned}
& \mathbf{P}(\Gamma_1 \leq y_1, \dots, \Gamma_l \leq y_l) \\
&= \mathbf{E} \left[ \mathbf{P}(\Gamma_1/\Gamma_l \leq y_1/\Gamma_l, \dots, \Gamma_{l-1}/\Gamma_l \leq y_{l-1}/\Gamma_l | \Gamma_l) \cdot \mathbb{I}(\Gamma_l \leq y_l) \right] \\
&= \int_0^{y_l} \mathbf{P}(U_{(1)} \leq y_1/\gamma, \dots, U_{(l-1)} \leq y_{l-1}/\gamma) \frac{e^{-\gamma} \gamma^{l-1}}{(l-1)!} d\gamma \\
&= \int_0^{y_l} e^{-\gamma} \gamma^{l-1} \int_0^{y_{l-1}/\gamma} \dots \int_0^{y_1/\gamma} \mathbb{I}(u_1 \leq \dots \leq u_{l-1} \leq 1) du_1 \dots du_{l-1} d\gamma \\
&= \left( \prod_{i=1}^{l-1} y_i \right) \int_0^{y_l} e^{-\gamma} \int_0^1 \dots \int_0^1 \mathbb{I}(y_1 v_1 \leq \dots \leq y_{l-1} v_{l-1} \leq \gamma) dv_1 \dots dv_{l-1} d\gamma \\
&= \left( \prod_{i=1}^l y_i \right) \cdot \int_0^1 \dots \int_0^1 e^{-y_l v_l} \mathbb{I}(y_1 v_1 \leq \dots \leq y_l v_l) dv_1 \dots dv_l.
\end{aligned}$$

Since  $0 \leq Q_n(nx_1) \leq \dots \leq Q_n(nx_l)$  for  $x_1 \geq \dots \geq x_l > 0$ ,

$$\begin{aligned}
& (n\nu[n, \infty])^{-j} \mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n \geq x_1, \dots, Q_n^{\leftarrow}(\Gamma_l) \geq x_l] \\
&= (n\nu[n, \infty])^{-j} \mathbf{P}[\Gamma_1 \leq Q_n(nx_1), \dots, \Gamma_l \leq Q_n(nx_l)] \\
&= (n\nu[n, \infty])^{-j} \cdot \left( \prod_{i=1}^l Q_n(nx_i) \right) \\
&\quad \cdot \int_0^1 \dots \int_0^1 e^{-Q_n(nx_i) v_i} \mathbb{I}(Q_n(nx_1) v_1 \leq \dots \leq Q_n(nx_l) v_l) dv_1 \dots dv_l \\
&= \left( \prod_{i=1}^j \frac{Q_n(nx_i)}{n\nu[n, \infty]} \right) \cdot \left( \prod_{i=j+1}^l Q_n(nx_i) \right) \\
&\quad \cdot \int_0^1 \dots \int_0^1 e^{-Q_n(nx_i) v_i} \mathbb{I}\left( \frac{Q_n(nx_i)}{n\nu[n, \infty]} v_1 \leq \dots \leq \frac{Q_n(nx_i)}{n\nu[n, \infty]} v_l \right) dv_1 \dots dv_l.
\end{aligned}$$

Note that  $Q_n(nx_i) \rightarrow 0$  and  $\frac{Q_n(nx_i)}{n\nu[n, \infty]} \rightarrow x_i^{-\alpha}$  as  $n \rightarrow \infty$  for each  $i = 1, \dots, l$ . Therefore, by bounded convergence,

$$\begin{aligned}
& (n\nu[n, \infty])^{-j} \mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n \geq x_1, \dots, Q_n^{\leftarrow}(\Gamma_l) \geq x_l] \\
&\rightarrow \mathbb{I}(j = l) \left( \prod_{i=1}^j x_i \right)^{-\alpha} \cdot \int_0^1 \dots \int_0^1 \mathbb{I}(x_1^{-\alpha} v_1 \leq \dots \leq x_j^{-\alpha} v_j) dv_1 \dots dv_j \\
&= \mu_\alpha^{(j)}(\{z \in \mathbb{R}_+^{\infty \downarrow} : x_1 \leq z_1, \dots, x_l \leq z_l\}),
\end{aligned}$$

which concludes the proof of (5.13). The conclusion of the lemma follows from the independence of  $(Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1)$  and  $(U_l, l \geq 1)$  and Lemma 2.2.  $\square$

**Lemma 5.4.** *Suppose that  $x_1 \geq \dots \geq x_j \geq 0$ ;  $u_i \in (0, 1)$  for  $i = 1, \dots, j$ ;  $y_1 \geq \dots \geq y_k \geq 0$ ;  $v_i \in (0, 1)$  for  $i = 1, \dots, k$ ;  $u_1, \dots, u_j, v_1, \dots, v_k$  are all distinct.*



(a) For any  $\epsilon > 0$ ,

$$\begin{aligned} & \{x \in G : d(x, y) < (1 + \epsilon)\delta \text{ implies } y \in G\} \\ & \subseteq G^{-\delta} \\ & \subseteq \{x \in G : d(x, y) < \delta \text{ implies } y \in G\}. \end{aligned}$$

Also,  $(A \cap B)_\delta \subseteq A_\delta \cap B_\delta$  and  $A^{-\delta} \cup B^{-\delta} \subseteq (A \cup B)^{-\delta}$  for any  $A$  and  $B$ .

(b)  $\sum_{i=1}^j x_i 1_{[u_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{< j})^{-\delta}$  implies  $x_j \geq \delta$ .

(c)  $\sum_{i=1}^j x_i 1_{[u_i, 1]} \notin (\mathbb{D} \setminus \mathbb{D}_{< j})^{-\delta}$  implies  $x_j \leq 2\delta$ .

(d)  $\sum_{i=1}^j x_i 1_{[u_i, 1]} - \sum_{i=1}^k y_i 1_{[v_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{< j, k})^{-\delta}$  implies  $x_j \geq \delta$  and  $y_k \geq \delta$ .

(e) Suppose that  $\xi \in \mathbb{D}_{j, k}$ . If  $l < j$  or  $m < k$ , then  $\xi$  is bounded away from  $\mathbb{D}_{l, m}$ .

(f) If  $I(\xi) > (\alpha - 1)j + (\beta - 1)k$ , then  $\xi$  is bounded away from  $\mathbb{D}_{< j, k} \cup \mathbb{D}_{j, k}$ .

*Proof.* (a) Immediate consequences of the definition.

(b) From (a), we see that  $\sum_{i=1}^j x_i 1_{[u_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{< j})^{-\delta}$  and  $\sum_{i=1}^{j-1} x_i 1_{[u_i, 1]} \in \mathbb{D}_{< j}$  implies  $d\left(\sum_{i=1}^j x_i 1_{[u_i, 1]}, \sum_{i=1}^{j-1} x_i 1_{[u_i, 1]}\right) \geq \delta$ , which is not possible if  $x_j < \delta$ .

(c) We prove that for any  $\epsilon > 0$ ,  $\sum_{i=1}^j x_i 1_{[u_i, 1]} \notin (\mathbb{D} \setminus \mathbb{D}_{< j})^{-\delta}$  implies  $x_j \leq (2 + \epsilon)\delta$ . To show this, in turn, we work with the contrapositive. Suppose that  $x_j > (2 + \epsilon)\delta$ . If  $d(\sum_{i=1}^j x_i 1_{[u_i, 1]}, \zeta) < (1 + \epsilon/2)\delta$ , by the definition of the Skorokhod metric, there exists a non-decreasing homeomorphism  $\phi$  of  $[0, 1]$  onto itself such that  $\|\sum_{i=1}^j x_i 1_{[u_i, 1]} - \zeta \circ \phi\|_\infty < (1 + \epsilon/2)\delta$ . Note that at each discontinuity point of  $\sum_{i=1}^j x_i 1_{[u_i, 1]}$ ,  $\zeta \circ \phi$  should also be discontinuous. Otherwise, the supremum distance between  $\sum_{i=1}^j x_i 1_{[u_i, 1]}$  and  $\zeta \circ \phi$  has to be greater than  $(1 + \epsilon/2)\delta$ , since the smallest jump size of  $\sum_{i=1}^j x_i 1_{[u_i, 1]}$  is greater than  $(2 + \epsilon)\delta$ . Hence, there has to be at least  $j$  discontinuities in the path of  $\zeta$ ; i.e.,  $\zeta \in \mathbb{D} \setminus \mathbb{D}_{< j}$ . We have shown that  $d(\sum_{i=1}^j x_i 1_{[u_i, 1]}, \zeta) < (1 + \epsilon/2)\delta$  implies  $\zeta \in \mathbb{D} \setminus \mathbb{D}_{< j}$ , which in turn, along with (a), shows that  $\sum_{i=1}^j x_i 1_{[u_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{< j})^{-\delta}$ .

(d) Suppose that  $\sum_{i=1}^j x_i 1_{[u_i, 1]} - \sum_{i=1}^k y_i 1_{[v_i, 1]} \in (\mathbb{D} \setminus \mathbb{D}_{< j, k})^{-\delta}$ . Since  $\sum_{i=1}^{j-1} x_i 1_{[u_i, 1]} - \sum_{i=1}^k y_i 1_{[v_i, 1]} \notin \mathbb{D} \setminus \mathbb{D}_{< j, k}$ ,

$$x_j \geq d\left(\sum_{i=1}^j x_i 1_{[u_i, 1]} - \sum_{i=1}^k y_i 1_{[v_i, 1]}, \sum_{i=1}^{j-1} x_i 1_{[u_i, 1]} - \sum_{i=1}^k y_i 1_{[v_i, 1]}\right) \geq \delta.$$

Similarly, we get  $y_k \geq \delta$ .

(e) Let  $\xi = \sum_{i=1}^j x_i 1_{[u_i, 1]} - \sum_{i=1}^k y_i 1_{[v_i, 1]}$ . First, we claim that  $d(\zeta, \xi) \geq x_j/2$  for any  $\zeta \in \mathbb{D}_{l, m}$  with  $l < j$ . Suppose not, i.e.,  $d(\zeta, \xi) < x_j/2$ . Then there exists a non-decreasing homeomorphism  $\phi$  of  $[0, 1]$  onto itself such that

$\|\sum_{i=1}^j x_i 1_{[u_i, 1]} - \zeta \circ \phi\|_\infty < x_j/2$ . Note that this implies that at each discontinuity point  $s$  of  $\xi$ ,  $\zeta \circ \phi$  should also be discontinuous. Otherwise,  $|\zeta \circ \phi(s) - \xi(s)| + |\zeta \circ \phi(s-) - \xi(s-)| \geq |\xi(s) - \xi(s-)| \geq x_j$ , and hence it is contradictory to the bound on the supremum distance between  $\xi$  and  $\zeta \circ \phi$ . However, this implies that  $\zeta$  has  $j$  upward jumps and hence, contradictory to the assumption  $\zeta \in \mathbb{D}_{l,m}$ , proving the claim. Likewise,  $d(\zeta, \xi) \geq y_k/2$  for any  $\xi \in \mathbb{D}_{l,m}$  with  $m < k$ .

(f) Note that in case  $I(\xi)$  is finite,  $\mathcal{D}_+(\xi) > j$  or  $\mathcal{D}_-(\xi) > k$ . In this case, the conclusion is immediate from (e). In case  $I(\xi) = \infty$ , either  $\mathcal{D}_+(\zeta) = \infty$ ,  $\mathcal{D}_-(\zeta) = \infty$ ,  $\xi(0) \neq 0$ , or  $\xi$  contains a continuous non-constant piece. By containing a continuous non-constant piece, we refer to the case that there exist  $t_1$  and  $t_2$  such that  $t_1 < t_2$ ,  $\xi(t_1) \neq \xi(t_2-)$  and  $\xi$  is continuous on  $(t_1, t_2)$ . For the first two cases where the number of jumps is infinite, the conclusion is an immediate consequence of (e). The case  $\xi(0) \neq 0$  is also obvious. Now we are left with dealing with the last case, where  $\xi$  has a continuous non-constant piece. To discuss this case, assume w.l.o.g. that  $\xi(t_1) < \xi(t_2-)$ . We claim that  $d(\xi, \mathbb{D}_{j,k}) \geq \frac{\xi(t_2-) - \xi(t_1)}{2(j+1)}$ . Note that for any step function  $\zeta$ ,

$$\begin{aligned} \|\xi - \zeta\| &\geq |\xi(t_2-) - \zeta(t_2-)| \vee |\xi(t_1) - \zeta(t_1)| \\ &\geq (\xi(t_2-) - \zeta(t_2-)) \vee (\zeta(t_1) - \xi(t_1)) \\ &\geq \frac{1}{2} \left\{ (\xi(t_2-) - \xi(t_1)) - (\zeta(t_2-) - \zeta(t_1)) \right\} \\ &\geq \frac{1}{2} \left\{ (\xi(t_2-) - \xi(t_1)) - \sum_{t \in (t_1, t_2)} (\zeta(t) - \zeta(t-)) \right\} \\ &\geq \frac{1}{2} \left\{ (\xi(t_2-) - \xi(t_1)) - 2\mathcal{D}_+(\zeta) \|\xi - \zeta\| \right\}, \end{aligned}$$

where the fourth inequality is due to the fact that  $\|\xi - \zeta\| \geq \frac{\zeta(t) - \zeta(t-)}{2}$  for all  $t \in (t_1, t_2)$ . From this, we get

$$\|\xi - \zeta\| \geq \frac{\xi(t_2-) - \xi(t_1)}{2(\mathcal{D}_+(\zeta) + 1)} \geq \frac{\xi(t_2-) - \xi(t_1)}{2(j+1)},$$

for  $\zeta \in \mathbb{D}_{j,k}$ . Now, suppose that  $\zeta \in \mathbb{D}_{j,k}$ . Since  $\zeta \circ \phi$  is again in  $\mathbb{D}_{j,k}$  for any non-decreasing homeomorphism  $\phi$  of  $[0, 1]$  onto itself,

$$d(\xi, \zeta) \geq \frac{\xi(t_2-) - \xi(t_1)}{2(j+1)},$$

which proves the claim.  $\square$

Now we move on to the proof of Theorem 3.3. We first establish Theorem 5.1, which plays a key role in the proof. Recall that  $\mathbb{D}_{< j} = \bigcup_{0 \leq l < j} \mathbb{D}_l$  and let  $\mathbb{D}_{< (j_1, \dots, j_d)} \triangleq \bigcup_{(l_1, \dots, l_d) \in \mathbb{I}_{< (j_1, \dots, j_d)}} \prod_{i=1}^d \mathbb{D}_{l_i}$  where  $\mathbb{I}_{< (j_1, \dots, j_d)} \triangleq \{(l_1, \dots, l_d) \in \mathbb{Z}_+^d \setminus \{(j_1, \dots, j_d)\} : (\alpha_1 - 1)l_1 + \dots + (\alpha_d - 1)l_d \leq (\alpha_1 - 1)j_1 + \dots + (\alpha_d - 1)j_d\}$ . For

each  $l \in \mathbb{Z}_+$  and  $i = 1, \dots, d$ , let  $C_l^{(i)}(\cdot) \triangleq \mathbf{E} \left[ \nu_{\alpha_i}^l \{x \in (0, \infty)^l : \sum_{j=1}^l x_j 1_{[U_j, 1]} \in \cdot\} \right]$  where  $U_1, \dots, U_l$  are iid uniform on  $[0, 1]$ , and  $\nu_{\alpha_i}^l$  is as defined right below (3.1).

**Theorem 5.1.** *Consider independent 1-dimensional Lévy processes  $X^{(1)}, \dots, X^{(d)}$  with spectrally positive Lévy measures  $\nu_1(\cdot), \dots, \nu_d(\cdot)$ , respectively. Suppose that each  $\nu_i$  is regularly varying (at infinity) with index  $-\alpha_i < -1$ , and let  $\bar{X}_n^{(i)}$  be centered and scaled scaled version of  $X^{(i)}$  for each  $i = 1, \dots, d$ . Then, for each  $(j_1, \dots, j_d) \in \mathbb{Z}_+^d$ ,*

$$\frac{\mathbf{P}((\bar{X}_n^{(1)}, \dots, \bar{X}_n^{(d)}) \in \cdot)}{\prod_{i=1}^d (n\nu_i[n, \infty])^{j_i}} \rightarrow C_{j_1}^{(1)} \times \dots \times C_{j_d}^{(d)}(\cdot)$$

in  $\mathbb{M} \left( \prod_{i=1}^d \mathbb{D} \setminus \mathbb{D}_{<(j_1, \dots, j_d)} \right)$ .

*Proof.* From Theorem 3.1, we know that  $(n\nu_i[n, \infty])^{-j} \mathbf{P}(\bar{X}_n^{(i)} \in \cdot) \rightarrow C_j(\cdot)$  in  $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j})$  for  $i = 1, \dots, d$  and any  $j \geq 0$ . This along with Lemma 2.2, for each  $(l_1, \dots, l_d) \in \mathbb{Z}_+^d$  we obtain

$$\prod_{i=1}^d (n\nu_i[n, \infty])^{-l_i} \mathbf{P}((\bar{X}_n^{(1)}, \dots, \bar{X}_n^{(d)}) \in \cdot) \rightarrow C_{l_1}^{(1)} \times \dots \times C_{l_d}^{(d)}(\cdot)$$

in  $\mathbb{M} \left( \prod_{i=1}^d \mathbb{D} \setminus \mathbb{C}_{(l_1, \dots, l_d)} \right)$  where  $\mathbb{C}_{(l_1, \dots, l_d)} \triangleq \bigcup_{i=1}^d (\mathbb{D}^{i-1} \times \mathbb{D}_{<l_i} \times \mathbb{D}^{d-i})$ . Since  $\mathbb{D}_{<(j_1, \dots, j_d)} = \bigcap_{(l_1, \dots, l_d) \notin \mathbb{I}_{<(j_1, \dots, j_d)}} \mathbb{C}_{(l_1, \dots, l_d)}$ , our strategy is to proceed with Lemma 2.3 to obtain the desired  $\mathbb{M} \left( \prod_{i=1}^d \mathbb{D} \setminus \mathbb{D}_{<(j_1, \dots, j_d)} \right)$ -convergence by combining the  $\mathbb{M} \left( \prod_{i=1}^d \mathbb{D} \setminus \mathbb{C}_{(l_1, \dots, l_d)} \right)$ -convergences for  $(l_1, \dots, l_d) \notin \mathbb{I}_{<(j_1, \dots, j_d)}$ . We first rewrite the infinite intersection over  $\mathbb{Z}_+^d \setminus \mathbb{I}_{<(j_1, \dots, j_d)}$  as a finite one to facilitate the application of the lemma. Consider a partial order  $\prec$  on  $\mathbb{Z}_+^d$  such that  $(l_1, \dots, l_d) \prec (m_1, \dots, m_d)$  if and only if  $\mathbb{C}_{(l_1, \dots, l_d)} \subsetneq \mathbb{C}_{(m_1, \dots, m_d)}$ . Note that this is equivalent to  $l_i \leq m_i$  for  $i = 1, \dots, d$  and  $l_i < m_i$  for at least one  $i = 1, \dots, d$ . Let  $\mathbb{J}_{j_1, \dots, j_d}$  be the subset of  $\mathbb{Z}_+^d$  consisting of the minimal elements of  $\mathbb{Z}_+^d \setminus \mathbb{I}_{<(j_1, \dots, j_d)}$ , i.e.,  $\mathbb{J}_{j_1, \dots, j_d} \triangleq \{(l_1, \dots, l_d) \in \mathbb{Z}_+^d \setminus \mathbb{I}_{<(j_1, \dots, j_d)} : (m_1, \dots, m_d) \prec (l_1, \dots, l_d) \text{ implies } (m_1, \dots, m_d) \in \mathbb{I}_{<(j_1, \dots, j_d)}\}$ . Figure 1 illustrates how the sets  $\mathbb{I}_{<(j_1, \dots, j_d)}$  and  $\mathbb{J}_{j_1, \dots, j_d}$  look when  $d = 2$ ,  $j_1 = 2$ ,  $j_2 = 2$ ,  $\alpha_1 = 2$ ,  $\alpha_2 = 3$ . It is straightforward to show that  $|\mathbb{J}_{j_1, \dots, j_d}| < \infty$ , and that  $(m_1, \dots, m_d) \notin \mathbb{I}_{<(j_1, \dots, j_d)}$  implies  $\mathbb{C}_{(l_1, \dots, l_d)} \subseteq \mathbb{C}_{(m_1, \dots, m_d)}$  for some  $(l_1, \dots, l_d) \in \mathbb{J}_{j_1, \dots, j_d}$ ; therefore,  $\mathbb{D}_{<(j_1, \dots, j_d)} = \bigcap_{(l_1, \dots, l_d) \in \mathbb{J}_{j_1, \dots, j_d}} \mathbb{C}_{(l_1, \dots, l_d)}$ . In view of this and the fact that  $\limsup \frac{\prod_{i=1}^d (n\nu_i[n, \infty])^{-l_i}}{\prod_{i=1}^d (n\nu_i[n, \infty])^{-j_i}} \rightarrow 0$  for  $(l_1, \dots, l_d) \in \mathbb{J}_{j_1, \dots, j_d} \setminus \{(j_1, \dots, j_d)\}$ , the conclusion of the theorem follows from Lemma 2.3 if we show that for each  $r > 0$ ,  $\xi \triangleq (\xi_1, \dots, \xi_d) \notin \left( \bigcup_{(l_1, \dots, l_d) \in \mathbb{I}_{<(j_1, \dots, j_d)}} \prod_{i=1}^d \mathbb{D}_{l_i} \right)^r$

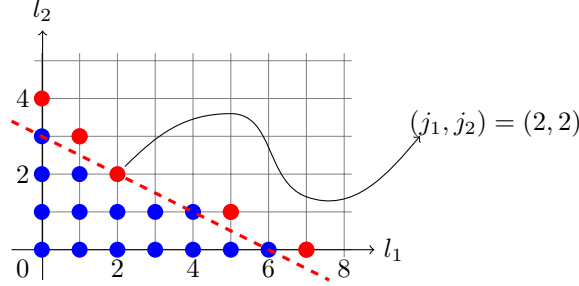


Figure 1: An example of  $\mathbb{I}_{<(j_1, \dots, j_d)}$  and  $\mathbb{J}_{j_1, \dots, j_d}$  where  $d = 2$ ,  $j_1 = 2$ ,  $j_2 = 2$ ,  $\alpha_1 = 2$ , and  $\alpha_2 = 3$ . The blue dots represent the elements of  $\mathbb{I}_{<(j_1, j_2)}$ , and the red dots represent the elements of  $\mathbb{J}_{j_1, j_2}$ . The dashed red line represents  $(l_1, l_2)$  such that  $(\alpha_1 - 1)l_1 + (\alpha_2 - 1)l_2 = (\alpha_1 - 1)j_1 + (\alpha_2 - 1)j_2$ .

implies  $\xi \notin (\mathbb{C}_{(l_1, \dots, l_d)})^r$  for some  $(l_1, \dots, l_d) \in \mathbb{J}_{j_1, \dots, j_d}$ . To see that this is the case, suppose that  $\xi$  is bounded away from  $\bigcup_{(l_1, \dots, l_d) \in \mathbb{I}_{<(j_1, \dots, j_d)}} \prod_{i=1}^d \mathbb{D}_{l_i}$  by  $r > 0$ . Let  $m_i \triangleq \inf\{k \geq 0 : \xi_i \in (\mathbb{D}_{\leq k})^r\}$ . In case  $m_i = \infty$  for some  $i$ , one can pick a large enough  $M \in \mathbb{Z}_+$  such that  $M\mathbf{e}_i \notin \mathbb{I}_{<(j_1, \dots, j_d)}$  where  $\mathbf{e}_i$  is the unit vector with 0 entries except for the  $i$ -th coordinate. Letting  $(l_1, \dots, l_d) \in \mathbb{J}_{j_1, \dots, j_d}$  be an index such that  $\mathbb{C}_{(l_1, \dots, l_d)} \subseteq \mathbb{C}_{M\mathbf{e}_i}$ , we find that  $\xi \notin (\mathbb{C}_{(l_1, \dots, l_d)})^r \subseteq (\mathbb{C}_{M\mathbf{e}_i})^r$  verifying the premise. If  $\max_{i=1, \dots, d} m_i < \infty$ ,  $\xi \in (\prod_{i=1}^d \mathbb{D}_{m_i})^r$  and hence,  $(m_1, \dots, m_d) \notin \mathbb{I}_{<(j_1, \dots, j_d)}$ , which, in turn, implies that there exists  $(l_1, \dots, l_d) \in \mathbb{J}_{j_1, \dots, j_d}$  such that  $\mathbb{C}_{(l_1, \dots, l_d)} \subseteq \mathbb{C}_{(m_1, \dots, m_d)}$ . However, due to the construction of  $m_i$ 's, each  $\xi_i$  is bounded away from  $\mathbb{D}_{<m_i}$  by  $r$ , and hence,  $\xi$  is bounded away from  $\mathbb{D}^{i-1} \times \mathbb{D}_{<m_i} \times \mathbb{D}^{d-i}$  by  $r$  for each  $i$ . Therefore,  $\xi \notin (\mathbb{C}_{(l_1, \dots, l_d)})^r \subseteq (\mathbb{C}_{(m_1, \dots, m_d)})^r$ , and hence, the premise is verified. Now we can apply Lemma 2.3 to reach the conclusion of the theorem.  $\square$

*Proof of Theorem 3.3.* Let  $X^{(+)}$  and  $X^{(-)}$  be Lévy processes with spectrally positive Lévy measures  $\nu_+$  and  $\nu_-$  respectively, where  $\nu_+[x, \infty) = \nu[x, \infty)$  and  $\nu_-[x, \infty) = \nu(-\infty, -x]$  for each  $x > 0$ , and denote the corresponding scaled processes as  $\bar{X}_n^{(+)}(\cdot) \triangleq X^{(+)}(n\cdot)/n$  and  $\bar{X}_n^{(-)}(\cdot) \triangleq X^{(-)}(n\cdot)/n$ . More specifically, let

$$\begin{aligned} \bar{X}_n^{(+)}(s) &= sa + B(ns)/n + \frac{1}{n} \int_{|x| \leq 1} x[N([0, ns] \times dx) - ns\nu(dx)] \\ &\quad + \frac{1}{n} \int_{x > 1} xN([0, ns] \times dx), \\ \bar{X}_n^{(-)}(s) &= \frac{1}{n} \int_{x < -1} xN([0, ns] \times dx). \end{aligned}$$

From Theorem 5.1, we know that  $(n\nu[n, \infty))^{-j}(n\nu(-\infty, -n])^{-k} \mathbf{P}((\bar{X}_n^{(+)}, \bar{X}_n^{(-)}) \in \cdot) \rightarrow C_j^+ \times C_k^-(\cdot)$  in  $\mathbb{M}((\mathbb{D} \times \mathbb{D}) \setminus \mathbb{D}_{<(j, k)})$  where  $C_j^+(\cdot) \triangleq \mathbf{E}[\nu_\alpha^j\{x \in (0, \infty)^j\} :$

$\sum_{i=1}^j x_i 1_{[U_i, 1]} \in \cdot\}$  and  $C_k^-(\cdot) \triangleq \mathbf{E}\left[\nu_\beta^k\{y \in (0, \infty)^k : \sum_{i=1}^k y_i 1_{[U_i, 1]} \in \cdot\}\right]$ . In view of Lemma 2.6 and that  $C_j^+ \times C_k^- \{(\xi, \zeta) \in \mathbb{D} \times \mathbb{D} : (\xi(t) - \xi(t-))(\zeta(t) - \zeta(t-)) \neq 0 \text{ for some } t \in (0, 1]\} = \emptyset$ , we can apply Lemma 2.4 for  $h(\xi, \zeta) = \xi - \zeta$ . Noting that  $C_{j,k}(\cdot) = (C_j^+ \times C_k^-) \circ h^{-1}(\cdot)$ , we conclude that  $(n\nu[n, \infty))^{-j} (n\nu(-\infty, -n])^{-k} \mathbf{P}(\bar{X}_n^{(+)} - \bar{X}_n^{(-)} \in \cdot) \rightarrow C_{j,k}(\cdot)$  in  $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{< j, k})$ . Since  $\bar{X}_n$  has the same distribution as  $\bar{X}_n^{(+)} - \bar{X}_n^{(-)}$ , the desired  $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{< j, k})$ -convergence for  $\bar{X}_n$  follows.  $\square$

*Proof of Lemma 3.1.* In general,

$$\min_{\substack{(j,k) \in \mathbb{Z}_+^2 \\ \mathbb{D}_{j,k} \cap \bar{A} \neq \emptyset}} \mathcal{I}(j, k) \leq \mathcal{I}(\mathcal{J}(A), \mathcal{K}(A)) \leq \min_{\substack{(j,k) \in \mathbb{Z}_+^2 \\ \mathbb{D}_{j,k} \cap A^\circ \neq \emptyset}} \mathcal{I}(j, k),$$

and the left inequality cannot be strict since  $A$  is bounded away from  $\mathbb{D}_{< \mathcal{J}(A), \mathcal{K}(A)}$ . On the other hand, in case the right inequality is strict, then  $\mathbb{D}_{\mathcal{J}(A), \mathcal{K}(A)} \cap A^\circ = \emptyset$ , which in turn implies  $C_{\mathcal{J}(A), \mathcal{K}(A)}(A^\circ) = 0$  since  $C_{\mathcal{J}(A), \mathcal{K}(A)}$  is supported on  $\mathbb{D}_{\mathcal{J}(A), \mathcal{K}(A)}$ . Therefore, the lower bound is trivial if the right inequality is strict. In view of these observations, we can assume w.l.o.g. that  $(\mathcal{J}(A), \mathcal{K}(A))$  is also in both  $\arg \min_{\substack{(j,k) \in \mathbb{Z}_+^2 \\ \mathbb{D}_{j,k} \cap A^\circ \neq \emptyset}} \mathcal{I}(j, k)$  and  $\arg \min_{\substack{(j,k) \in \mathbb{Z}_+^2 \\ \mathbb{D}_{j,k} \cap \bar{A} \neq \emptyset}} \mathcal{I}(j, k)$ . Since  $A^\circ$  and  $\bar{A}$  are also bounded-away from  $\mathbb{D}_{< \mathcal{J}(A), \mathcal{K}(A)}$ , the upper bound of (3.9) is obtained from (2.1) and Theorem 3.3 for  $\bar{A}$ ,  $j = \mathcal{J}(\bar{A}) = \mathcal{J}(A)$ , and  $k = \mathcal{K}(\bar{A}) = \mathcal{K}(A)$ ; the lower bound of (3.9) is obtained from (2.2) and Theorem 3.3 for  $A^\circ$ ,  $j = \mathcal{J}(A^\circ) = \mathcal{J}(A)$ , and  $k = \mathcal{K}(A^\circ) = \mathcal{K}(A)$ . Finally, we obtain (3.10) from Theorem 3.3 and (2.1) with  $j = l$ ,  $k = m$ ,  $F = \bar{A}$  along with the fact that  $C_{l,m}(\bar{A}) = 0$  since  $A$  is bounded away from  $\mathbb{D}_{l,m}$ .  $\square$

**Lemma 5.5.** *Let  $A$  be a measurable set and suppose that the argument minimum in (3.8) is non-empty and contains a pair of integers  $(\mathcal{J}(A), \mathcal{K}(A))$ . Let  $(l, m) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$ .*

- (i) *If  $A_\delta \cap \mathbb{D}_{l,m}$  is bounded away from  $\mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}$  for some  $\delta > 0$ , then  $A \cap (\mathbb{D}_{l,m})_\gamma$  is bounded away from  $\mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}$  for some  $\gamma > 0$ .*
- (ii) *If  $A$  is bounded away from  $\mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}$ , then there exists  $\delta > 0$  such that  $A \cap (\mathbb{D}_{l,m})_\delta$  is bounded away from  $\mathbb{D}_{j,k}$  for any  $(j, k) \in \mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)} \setminus \{(l, m)\}$ .*

*Proof.* For (i), we prove that if  $d(A_{2\delta} \cap \mathbb{D}_{l,m}, \mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}) > 3\delta$  then  $d(A \cap (\mathbb{D}_{l,m})_\delta, \mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}) \geq \delta$ . Suppose that  $d(A \cap (\mathbb{D}_{l,m})_\delta, \mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}) < \delta$ . Then, there exists  $\xi \in A \cap (\mathbb{D}_{l,m})_\delta$  and  $\zeta \in \mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}$  such that  $d(\xi, \zeta) < \delta$ . Note that we can find  $\xi' \in \mathbb{D}_{l,m}$  such that  $d(\xi, \xi') \leq 2\delta$ , which means that  $\xi' \in A_{2\delta} \cap \mathbb{D}_{l,m}$ . Therefore,  $d(A_{2\delta} \cap \mathbb{D}_{l,m}, \mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}) \leq d(\xi', \zeta) \leq d(\xi', \xi) + d(\xi, \zeta) \leq 2\delta + \delta \leq 3\delta$ .

For (ii), suppose that  $d(A, \mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}) > \gamma$  for some  $\gamma > 0$  and  $(l, m)$  and  $(j, k)$  are two distinct pairs that belong to  $\mathbb{I}_{=\mathcal{J}(A), \mathcal{K}(A)}$ . Assume w.l.o.g. that

$j < l$ . (If  $j > l$ , it should be the case that  $k < m$ , and hence one can proceed in the same way by switching the roles of upward jumps and downward jumps in the following argument.) Let  $c$  be a positive number such that  $c > 8(l-j)+2$  and set  $\delta = \gamma/c$ . We will show that  $A \cap (\mathbb{D}_{l,m})_\delta$  and  $(\mathbb{D}_{j,k})_\delta$  are bounded away from each other. Let  $\xi$  be an arbitrary element of  $A \cap (\mathbb{D}_{l,m})_\delta$ . Then, there exists a  $\zeta \in \mathbb{D}_{l,m}$  such that  $d(\zeta, \xi) \leq 2\delta$ . Note that  $d(\zeta, \mathbb{D}_{\ll \mathcal{J}(A), \mathcal{K}(A)}) \geq (c-2)\delta$ ; in particular,  $d(\zeta, \mathbb{D}_{j,m}) \geq (c-2)\delta$ . If we write  $\zeta \triangleq \sum_{i=1}^l x_i 1_{[u_i, 1]} - \sum_{i=1}^m y_i 1_{[v_i, 1]}$ , this implies that  $x_{j+1} \geq \frac{(c-2)\delta}{l-j}$ . Otherwise,  $(c-2)\delta > \sum_{i=j+1}^l x_i = \|\zeta - \zeta'\| \geq d(\zeta, \zeta')$ , where  $\zeta' \triangleq \zeta - \sum_{i=j+1}^l x_i 1_{[u_i, 1]} \in \mathbb{D}_{j,m}$ . Therefore,  $d(\zeta, \mathbb{D}_{j,k}) \geq \frac{(c-2)\delta}{2(l-j)}$ , which in turn implies  $d(\xi, \mathbb{D}_{j,k}) \geq \frac{(c-2)\delta}{2(l-j)} - 2\delta > 2\delta$ . Since  $\xi$  was arbitrary, we conclude that  $A \cap (\mathbb{D}_{l,m})_\delta$  bounded away from  $(\mathbb{D}_{j,k})_\delta$ .  $\square$

### 5.3 Proofs for Section 4

Recall that

$$I(\xi) \triangleq \begin{cases} (\alpha-1)\mathcal{D}_+(\xi) + (\beta-1)\mathcal{D}_-(\xi) & \text{if } \xi \text{ is a step function with } \xi(0) = 0 \\ \infty & \text{otherwise} \end{cases}.$$

*Proof of Theorem 4.2.* Observe first that  $I(\cdot)$  is a rate function. The level sets  $\{\xi : I(\xi) \leq x\}$  equal  $\bigcup_{\substack{(l,m) \in \mathbb{Z}_+^2 \\ (\alpha-1)l + (\beta-1)m \leq \lfloor x \rfloor}} \mathbb{D}_{l,m}$  and are therefore closed—note the level sets are not compact so  $I(\cdot)$  is not a good rate function (see, for example, [Dembo and Zeitouni \(2009\)](#) for the definition and properties of good rate functions).

Starting with the lower bound, suppose that  $G$  is an open set. We assume w.l.o.g. that  $\inf_{\xi \in G} I(\xi) < \infty$ , since the inequality is trivial otherwise. Due to the discrete nature of  $I(\cdot)$ , there exists a  $\xi^* \in G$  such that  $I(\xi^*) = \inf_{\xi \in G} I(\xi)$ . Set  $j \triangleq \mathcal{D}_+(\xi^*)$  and  $k \triangleq \mathcal{D}_-(\xi^*)$ . Let  $u_1^+, \dots, u_j^+$  be the sorted (from the earliest to the latest) upward jump times of  $\xi^*$ ;  $x_1^+, \dots, x_j^+$  be the sorted (from the largest to the smallest) upward jump sizes of  $\xi^*$ ;  $u_1^-, \dots, u_k^-$  be the sorted downward jump times of  $\xi^*$ ;  $x_1^-, \dots, x_k^-$  be the sorted downward jump sizes of  $\xi^*$ . Also, let  $x_{j+1}^+ = x_{k+1}^- = 0$ ,  $u_0^+ = u_0^- = 0$ , and  $u_{j+1}^+ = u_{k+1}^- = 1$ . Note that if  $\zeta \in \mathbb{D}_{l,m}$  for  $l < j$ , then  $d(\xi^*, \zeta) \geq x_j^+/2$  since at least one of the  $j$  upward jumps of  $\xi^*$  cannot be matched by  $\zeta$ . Likewise, if  $\zeta \in \mathbb{D}_{l,m}$  for  $m < k$ , then  $d(\xi^*, \zeta) \geq x_k^-/2$ . Therefore,  $d(\mathbb{D}_{< j, k}, \xi^*) \geq (x_j^+ \wedge x_k^-)/2$ . On the other hand, since  $G$  is an open set, we can pick  $\delta_0 > 0$  so that the open ball  $B_{\xi^*, \delta_0} \triangleq \{\zeta \in \mathbb{D} : d(\zeta, \xi^*) < \delta_0\}$  centered at  $\xi^*$  with radius  $\delta_0$  is a subset of  $G$ —i.e.,  $B_{\xi^*, \delta_0} \subset G$ . Let  $\delta = (\delta_0 \wedge x_j^+ \wedge x_k^-)/4$ . If  $j = k = 0$ , then  $\xi^* \equiv 0$ , and hence,  $\{\bar{X}_n \in G\}$  contains  $\{\|\bar{X}_n\| \leq \delta\}$  which is a subset of  $B_{\xi^*, \delta}$ . One can apply [Lemma A.4](#) to show that  $\mathbf{P}(X_n \in G)$  converges to 1, which, in turn, proves the inequality. Now, suppose that either  $j \geq 1$  or  $k \geq 1$ . Then,  $d(B_{\xi^*, \delta}, \mathbb{D}_{< j, k}) \geq \delta$ . As  $d(B_{\xi^*, \delta}, \mathbb{D}_{< j, k}) > 0$  and  $B_{\xi^*, \delta}$  is open, we see from our sharp asymptotics

(Theorem 3.1) that

$$C_{j,k}(B_{\xi^*,\delta}) \leq \liminf_{n \rightarrow \infty} (n\nu[n, \infty))^{-j} (n\nu(-\infty, -n])^{-k} P(\bar{X}_n \in B_{\xi^*,\delta}).$$

From the definition of  $C_{j,k}$ , it follows that  $C_j(B_{\xi^*,\delta}) > 0$ . To see this, note first that we can assume w.l.o.g. that  $x_i^\pm$ 's are all distinct since  $G$  is open (because, if some of the jump sizes are identical, we can pick  $\epsilon$  such that  $B_{\xi^*,\epsilon} \subseteq G$ , and then perturb those jump sizes by  $\epsilon$  to get a new  $\xi^*$  which still belongs to  $G$  while whose jump sizes are all distinct.) Suppose that  $\xi^* = \sum_{l=1}^j x_{i_l^+}^+ 1_{[u_i^+,1]} - \sum_{l=1}^k x_{i_l^-}^- 1_{[u_i^-,1]}$ , where  $\{i_1^\pm, \dots, i_j^\pm\}$  are permutations of  $\{1, \dots, j\}$ . Let  $2\delta' \triangleq \delta \wedge \underline{\Delta}_u^+ \wedge \underline{\Delta}_x^+ \wedge \underline{\Delta}_u^- \wedge \underline{\Delta}_x^-$ , where  $\underline{\Delta}_u^+ = \min_{i=1, \dots, j+1} (u_i^+ - u_{i-1}^+)$ ,  $\underline{\Delta}_x^+ = \min_{i=1, \dots, j} (x_{i-1}^+ - x_i^+)$ ,  $\underline{\Delta}_u^- = \min_{i=1, \dots, k+1} (u_i^- - u_{i-1}^-)$ , and  $\underline{\Delta}_x^- = \min_{i=1, \dots, k} (x_{i-1}^- - x_i^-)$ . Consider a subset  $B'$  of  $B_{\xi^*,\delta}$ :

$$B' \triangleq \left\{ \sum_{l=1}^j y_{i_l^+}^+ 1_{[v_l^+,1]} - \sum_{l=1}^k y_{i_l^-}^- 1_{[v_l^-,1]} : \begin{aligned} v_i^+ &\in (u_i^+ - \delta', u_i^+ + \delta'), y_i^+ \in (x_i^+ - \delta', x_i^+ + \delta'), i = 1, \dots, j; \\ v_i^- &\in (u_i^- - \delta', u_i^- + \delta'), y_i^- \in (x_i^- - \delta', x_i^- + \delta'), i = 1, \dots, k \end{aligned} \right\}.$$

Then,

$$\begin{aligned} & C_{j,k}(B_{\xi^*,\delta}) \\ & \geq C_{j,k}(B') \\ & = \int_{(u_1^+ - \delta', u_1^+ + \delta') \times \dots \times (u_j^+ - \delta', u_j^+ + \delta')} d\text{Leb} \cdot \int_{(x_1^+ - \delta', x_1^+ + \delta') \times \dots \times (x_j^+ - \delta', x_j^+ + \delta')} d\nu_\alpha \\ & \quad \cdot \int_{(u_1^- - \delta', u_1^- + \delta') \times \dots \times (u_k^- - \delta', u_k^- + \delta')} d\text{Leb} \cdot \int_{(x_1^- - \delta', x_1^- + \delta') \times \dots \times (x_k^- - \delta', x_k^- + \delta')} d\nu_\beta \\ & \geq (2\delta')^j (2\delta' (x_1^+)^{\alpha})^j (2\delta')^k (2\delta' (x_1^-)^{\beta})^k > 0. \end{aligned}$$

We conclude that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in G)}{\log n} \geq \liminf_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in B_{\xi^*,\delta})}{\log n} \\ & \geq \liminf_{n \rightarrow \infty} \frac{\log(C_{j,k}(B_{\xi^*,\delta})(n\nu[n, \infty))^j (n\nu(-\infty, -n])^k (1 + o(1)))}{\log n} \quad (5.15) \\ & = -((\alpha - 1)j + (\beta - 1)k), \end{aligned}$$

which is the lower bound. Turning to the upper bound, suppose that  $K$  is a compact set. We first consider the case where  $\inf_{\xi \in K} I(\xi) < \infty$ . Pick  $\xi^*$ ,  $j$  and  $k$  as in the lower bound, i.e.,  $I(\xi^*) \triangleq \inf_{\xi \in K} I(\xi)$ ,  $j \triangleq \mathcal{D}_+(\xi^*)$ , and  $k \triangleq \mathcal{D}_-(\xi^*)$ . Here we can assume w.l.o.g. either  $j \geq 1$  or  $k \geq 1$  since the inequality is trivial in case  $j = k = 0$ . For each  $\zeta \in K$ , either  $I(\zeta) > I(\xi^*)$ , or  $I(\zeta) = I(\xi^*)$ . We construct an open cover of  $K$  by considering these two cases separately:

- If  $I(\zeta) > I(\xi^*)$ ,  $\zeta$  is bounded away from  $\mathbb{D}_{<j,k} \cup \mathbb{D}_{j,k}$  (Lemma 5.4 (f)). For each such  $\zeta$ 's, pick a  $\delta_\zeta > 0$  in such a way that  $d(\zeta, \mathbb{D}_{<j,k} \cup \mathbb{D}_{j,k}) > \delta_\zeta$ . Set  $j_\zeta \triangleq j$  and  $k_\zeta \triangleq k$ . Note that in this case  $C_{j_\zeta, k_\zeta}(\bar{B}_{\zeta, \delta_\zeta}) = 0$ .
- If  $I(\zeta) = I(\xi^*)$ , set  $j_\zeta \triangleq \mathcal{D}_+(\zeta)$  and  $k_\zeta \triangleq \mathcal{D}_-(\zeta)$ . Since they are bounded away from  $\mathbb{D}_{<j_\zeta, k_\zeta}$  (Lemma 5.4 (e)), we can choose  $\delta_\zeta > 0$  such that  $d(\zeta, \mathbb{D}_{<j_\zeta, k_\zeta}) > \delta_\zeta$  and  $C_{j_\zeta, k_\zeta}(\bar{B}_{\zeta, \delta_\zeta}) < \infty$ .

Consider an open cover  $\{B_{\zeta_i; \delta_{\zeta_i}} : \zeta \in K\}$  of  $K$  and its finite subcover  $\{B_{\zeta_i; \delta_{\zeta_i}}\}_{i=1, \dots, m}$ . For each  $\zeta_i$ , we apply the sharp asymptotics (Theorem 3.3) to  $\bar{B}_{\zeta_i; \delta_{\zeta_i}}$  to get

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in \bar{B}_{\zeta_i; \delta_{\zeta_i}})}{\log n} \leq (\alpha - 1)j_{\zeta_i} + (\beta - 1)k_{\zeta_i} = -I(\xi^*). \quad (5.16)$$

Therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in \bar{F})}{\log n} &\leq \limsup_{n \rightarrow \infty} \frac{\log \sum_{i=1}^m \mathbf{P}(\bar{X}_n \in \bar{B}_{\zeta_i; \delta_{\zeta_i}})}{\log n} \\ &= \max_{i=1, \dots, m} \limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in \bar{B}_{\zeta_i; \delta_{\zeta_i}})}{\log n} \\ &\leq -I(\xi^*) = -\inf_{\xi \in K} I(\xi), \end{aligned} \quad (5.17)$$

completing the proof of the upper bound in case the right-hand side is finite.

Now, turning to the case  $\inf_{\xi \in K} I(\xi) = \infty$ , fix an arbitrary positive integer  $l$ . Since  $\mathbb{D}_{<l,l}$  is closed and disjoint with a compact set  $K$ , it is also bounded away from each  $\zeta \in K$ . Now picking  $\delta_\zeta > 0$  so that  $\bar{B}_{\zeta, \delta_\zeta}$  is disjoint with  $K$  for each  $\zeta$ , one can construct an open cover  $\{B_{\zeta; \delta_\zeta} : \zeta \in K\}$  of  $K$ . Let  $\{B_{\zeta_i; \delta_{\zeta_i}}\}_{i=1, \dots, m}$  its finite subcover, then from the same calculation as (5.16) and (5.17),

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{P}(\bar{X}_n \in K)}{\log n} \leq -(\alpha + \beta - 2)m.$$

Taking  $m \rightarrow \infty$ , we arrive at the desired upper bound.  $\square$

## 6 Applications

In this section, we illustrate the use of our main results, established in Section 3, in several problem contexts that arise in control, insurance, and finance. In all examples, we assume that  $\bar{X}_n(t) = X(nt)/n$ , where  $X(\cdot)$  is a centered Lévy process satisfying (1.1).

### 6.1 Crossing High Levels with Moderate Jumps

We are interested in level crossing probabilities of Lévy processes where the jumps are conditioned to be moderate. More precisely, we are interested in probabilities of the form  $\mathbf{P}(\sup_{t \in [0,1]} [\bar{X}_n(t) - ct] \geq a; \sup_{t \in [0,1]} [\bar{X}_n(t) - \bar{X}_n(t-)] \leq$



b). We make a technical assumption that  $a$  is not a multiple of  $b$  and focus on the case where the Lévy process  $\bar{X}_n$  is spectrally positive.

The setting of this example is relevant in, for example, insurance, where huge claims may be reinsured and therefore do not play a role in the ruin of an insurance company. [Asmussen and Pihlsgård \(2005\)](#) focus on obtaining various estimates of infinite-time ruin probabilities using analytic methods. Here, we provide complementary sharp asymptotics for the finite-time ruin probability, using probabilistic techniques.

Set  $A \triangleq \{\xi \in \mathbb{D} : \sup_{t \in [0,1]} [\xi(t) - ct] \geq a; \sup_{t \in [0,1]} [\xi(t) - \xi(t-)] \leq b\}$  and define  $j \triangleq \lceil a/b \rceil$ . Intuitively,  $j$  should be the key parameter, as it takes at least  $j$  jumps of size  $b$  to cross level  $a$ . Our goal is to make this intuition rigorous by applying [Theorem 3.2](#) and by showing that the upper and lower bounds are tight.

We first check that  $A_\delta \cap \mathbb{D}_j$  is bounded away from the closed set  $\mathbb{D}_{\leq j-1}$  for some  $\delta > 0$ . To see this, it suffices to show that

- 1)  $\sup_{t \in [0,1]} [\xi(t) - \xi(t-)] \leq b$  and  $\sup_{t \in [0,1]} [\zeta(t) - \zeta(t-)] > b'$  imply  $d(\xi, \zeta) > \frac{b'-b}{3}$ ; and
- 2)  $\sup_{t \in [0,1]} [\xi(t) - ct] < a'$  and  $\sup_{t \in [0,1]} [\zeta(t) - ct] \geq a$  imply  $d(\xi, \zeta) \geq \frac{a-a'}{c+1}$ .

It is straightforward to check 1). To see 2), note that for any  $\epsilon > 0$ , one can find  $t^*$  such that  $\zeta(t^*) - ct^* \geq a - \epsilon$ . Of course,  $\xi(\lambda(t^*)) - c\lambda(t^*) < a'$  for any homeomorphism  $\lambda(\cdot)$ . Subtracting the latter inequality from the former inequality, we obtain

$$\zeta(t^*) - \xi(\lambda(t^*)) \geq a - a' - \epsilon + c(t^* - \lambda(t^*)). \quad (6.1)$$

One can choose  $\lambda$  so that  $d(\xi, \zeta) + \epsilon \geq \|\lambda - e\| \geq \lambda(t^*) - t^*$  and  $d(\zeta, \xi) + \epsilon \geq \|\zeta - \xi \circ \lambda\| \geq \zeta(t^*) - \xi(\lambda(t^*))$ , which together with [\(6.1\)](#) yields

$$d(\xi, \zeta) > a - a' - (c+1)\epsilon - cd(\xi, \zeta).$$

This leads to  $d(\xi, \zeta) \geq \frac{a-a'}{c+1}$  by taking  $\epsilon \rightarrow 0$ . With 1) and 2) in hand, it follows that  $\phi_1(\xi) \triangleq \sup_{t \in [0,1]} [\xi(t) - \xi(t-)]$  and  $\phi_2(\xi) \triangleq \sup_{t \in [0,1]} [\xi(t) - ct]$  are continuous functionals and  $A_\delta \subseteq A(\delta)$ , where  $A(\delta) \triangleq \{\xi \in \mathbb{D} : \sup_{t \in [0,1]} [\xi(t) - ct] \geq a - (c+1)\delta; \sup_{t \in [0,1]} [\xi(t) - \xi(t-)] \leq b + 3\delta\}$ . Since  $\xi \in A(\delta) \cap \mathbb{D}_j$  implies that the jump size of  $\xi$  is bounded from below by  $(b+3\delta)j - (a - (c+1)\delta)$ , one can choose  $\delta > 0$  so that  $A(\delta) \cap \mathbb{D}_j$  is bounded away from  $\mathbb{D}_{\leq j-1}$ . This implies that  $A_\delta \cap \mathbb{D}_j$  is also bounded away from  $\mathbb{D}_{\leq j-1}$  for sufficiently small  $\delta > 0$ . Hence, [Theorem 3.2](#) applies with  $\mathcal{J}(A) = j$ .

Next, to identify the limit, recall the discussion at the end of [Section 3.1](#).

Note that  $A = \phi_1^{-1}[a, \infty) \cap \phi_2^{-1}(-\infty, b]$  and

$$\begin{aligned} & \hat{T}_j^{-1}(\phi_1^{-1}[a, \infty) \cap \phi_2^{-1}(-\infty, b]) \\ &= \left\{ (x, u) \in \hat{S}_j : \sum_{i=1}^j x_i \geq a + c \max_{i=1, \dots, j} u_i, \max_{i=1, \dots, j} x_i \leq b \right\}, \\ & \hat{T}_j^{-1}(\phi_1^{-1}(a, \infty) \cap \phi_2^{-1}(-\infty, b)) \\ &= \left\{ (x, u) \in \hat{S}_j : \sum_{i=1}^j x_i > a + c \max_{i=1, \dots, j} u_i, \max_{i=1, \dots, j} x_i < b \right\}. \end{aligned} \quad (6.2)$$

We see that  $\hat{T}_j^{-1}(\phi_1^{-1}[a, \infty) \cap \phi_2^{-1}(-\infty, b]) \setminus \hat{T}_j^{-1}(\phi_1^{-1}(a, \infty) \cap \phi_2^{-1}(-\infty, b))$  has Lebesgue measure 0, and hence,  $A$  is  $C_j$ -continuous. Thus, (3.6) holds with

$$C_j(A) = \mathbf{E} \left[ \nu_\alpha^j \left\{ (0, \infty)^j : \sum_{i=1}^j x_i 1_{[U_i, 1]} \in A \right\} \right] = \int_{(x, u) \in \hat{T}_j^{-1}(A)} \prod_{i=1}^j [\alpha x_i^{-\alpha-1} dx_i du_i] > 0.$$

Therefore, we conclude that

$$\mathbf{P} \left( \sup_{t \in [0, 1]} [\bar{X}_n(t) - ct] \geq a; \sup_{t \in [0, 1]} [\bar{X}_n(t) - \bar{X}_n(t-)] \leq b \right) \sim C_j(A) (n\nu[n, \infty))^j. \quad (6.3)$$

In particular, the probability of interest is regularly varying with index  $-(\alpha - 1)[a/b]$ .

## 6.2 A Two-sided Barrier Crossing Problem

We consider a Lévy-driven Ornstein-Uhlenbeck process of the form

$$d\bar{Y}_n(t) = -\kappa d\bar{Y}_n(t) + d\bar{X}_n(t), \quad \bar{Y}_n(0) = 0.$$

We apply our results to provide sharp large-deviations estimates for

$$b(n) = \mathbf{P} \left( \inf\{\bar{Y}_n(t) : 0 \leq t \leq 1\} \leq -a_-, \bar{Y}_n(1) \geq a_+ \right)$$

as  $n \rightarrow \infty$ , where  $a_-, a_+ > 0$ . This probability can be interpreted as the price of a barrier digital option (see [Cont and Tankov, 2004](#), Section 11.3). In order to apply our results it is useful to represent  $\bar{Y}_n$  as an explicit function of  $\bar{X}_n$ . In particular, we have that

$$\bar{Y}_n(t) = \exp(-\kappa t) \left( \bar{Y}_n(0) + \int_0^t \exp(\kappa s) d\bar{X}_n(s) \right) \quad (6.4)$$

$$= \bar{X}_n(t) - \kappa \exp(-\kappa t) \int_0^t \exp(\kappa s) \bar{X}_n(s) ds. \quad (6.5)$$

Hence, if  $\phi : \mathbb{D}([0, 1], \mathbb{R}) \rightarrow \mathbb{D}([0, 1], \mathbb{R})$  is defined via

$$\phi(\xi)(t) = \xi(t) - \kappa \exp(-\kappa t) \int_0^t \exp(\kappa s) \xi(s) ds,$$

then  $\bar{Y}_n = \phi(\bar{X}_n)$ . Moreover, if we let

$$A = \left\{ \xi \in \mathbb{D} : \inf_{0 \leq t \leq 1} \phi(\xi)(t) \leq -a_-, \phi(\xi)(1) \geq a_+ \right\},$$

then we obtain

$$b(n) = \mathbf{P}(\bar{X}_n \in A).$$

In order to easily verify topological properties of  $A$ , let us define  $m, \pi_1 : \mathbb{D}([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$  by  $m(\xi) = \inf_{0 \leq t \leq 1} \xi(t)$ , and  $\pi_1(\xi) = \xi(1)$ . Note that  $\pi_1$  is continuous (see [Billingsley, 2013](#), Theorem 12.5), that  $m$  is continuous as well, and so is  $\phi$ . Thus,  $m \circ \phi$  and  $\pi_1 \circ \phi$  are continuous. We can therefore write

$$A = (m \circ \phi)^{-1}(-\infty, -a_-] \cap (\pi_1 \circ \phi)^{-1}[a_+, \infty),$$

concluding that  $A$  is a closed set. We now apply Theorem 3.4. To show that  $\mathbb{D}_{i,0}$  is bounded away from  $(m \circ \phi)^{-1}(-\infty, -a_-]$ , select  $\theta$  such that  $d(\theta, \mathbb{D}_{i,0}) < r$  with  $r < a_- / (1 + \kappa \exp(\kappa))$ . There exists a  $\xi \in \mathbb{D}_{i,0}$  such that  $d(\theta, \xi) < r$  and  $\xi$  satisfies  $\xi(t) = \sum_{j=1}^i x_j I_{[u_j, 1]}(t)$ , with  $i \geq 1$ . There also exists a homeomorphism  $\lambda : [0, 1] \rightarrow [0, 1]$  such that

$$\sup_{t \in [0, 1]} |\lambda(t) - t| \vee |(\xi \circ \lambda)(t) - \theta(t)| < r. \quad (6.6)$$

Now, define  $\psi = \theta - (\xi \circ \lambda)$ . Due to the linearity of  $\phi$ , and representations (6.4) and (6.5), we obtain that

$$\begin{aligned} \phi(\theta)(t) &= \phi((\xi \circ \lambda))(t) + \phi(\psi)(t) \\ &= \exp(-\kappa t) \sum_{j=1}^i \exp(\kappa \lambda^{-1}(u_j)) x_j I_{[\lambda^{-1}(u_j), 1]}(t) + \psi(t) \\ &\quad - \kappa \exp(-\kappa t) \int_0^t \exp(\kappa s) \psi(s) ds. \end{aligned}$$

Since  $x_j \geq 0$ , applying the triangle inequality and inequality (6.6) we conclude (by our choice of  $r$ ), that

$$\inf_{0 \leq t \leq 1} \phi(\theta)(t) \geq -r(1 + \kappa \exp(\kappa)) > -a_-.$$

A similar argument allows us to conclude that  $\mathbb{D}_{0,i}$  is bounded away from  $(\pi_1 \circ \phi)^{-1}[a_+, \infty)$ . Hence, in addition to being closed,  $A$  is bounded away from  $\mathbb{D}_{0,i} \cup \mathbb{D}_{i,0}$  for any  $i \geq 1$ . Moreover, let  $\xi \in A \cap \mathbb{D}_{1,1}$ , with

$$\xi(t) = x I_{[u, 1]}(t) - y I_{[v, 1]}(t), \quad (6.7)$$

where  $x > 0$  and  $y > 0$ . Using (6.4), we obtain that  $\xi \in A \cap \mathbb{D}_{1,1}$ , is equivalent to

$$y \geq a_-, \quad u > v, \quad \text{and} \quad x \geq a_+ \exp(\kappa(1-u)) + y \exp(-\kappa(u-v)).$$

Now, we claim that

$$\begin{aligned} A^\circ &= \left\{ \xi \in \mathbb{D} : \inf_{0 \leq t \leq 1} \phi(\xi)(t) < -a_-, \phi(\xi)(1) > a_+ \right\} \\ &= (m \circ \phi)^{-1}(-\infty, -a_-) \cap (\pi_1 \circ \phi)^{-1}(a_+, \infty). \end{aligned} \quad (6.8)$$

It is clear that  $A^\circ$  contains the open set in the right hand side. We now argue that such a set is actually maximal, so that equality holds. Suppose that  $\phi(\xi)(1) = a_+$ , while  $\min_{0 \leq t \leq 1} \phi(\xi)(t) < -a_-$ . We then consider  $\psi = -\delta I_{\{1\}}(t)$  with  $\delta > 0$ , and note that  $d(\xi, \xi + \psi) \leq \delta$ , and

$$\phi(\xi + \psi)(t) = \phi(\xi)(t) I_{[0,1)}(t) + (a_+ - \delta) I_{\{1\}}(t),$$

so that  $\xi + \psi \notin A$ . Similarly, we can see that the other inequality (involving  $a_-$ ) must also be strict, hence concluding that (6.8) holds.

We deduce that, if  $\xi \in A^\circ \cap \mathbb{D}_{1,1}$  with  $\xi$  satisfying (6.7), then

$$y > a_-, \quad u > v, \quad x > a_+ \exp(\kappa(1-u)) + y \exp(-\kappa(u-v)).$$

Thus, we can see that  $A$  is  $C_{1,1}(\cdot)$ -continuous, either directly or by invoking our discussion in Section 3.1 regarding continuity of sets. Therefore, applying Theorem 3.4, we conclude that

$$b(n) \sim n\nu[n, \infty)n\nu(-\infty, -n]C_{1,1}(A)$$

as  $n \rightarrow \infty$ , where

$$C_{1,1}(A) = \int_0^1 \int_{a_-}^\infty \int_v^1 \int_{a_+ \exp(\kappa(1-u)) + y \exp(-\kappa(u-v))}^\infty \nu_\alpha(dx) du \nu_\beta(dy) dv.$$

In particular, the probability of interest is regularly varying with index  $2 - \alpha - \beta$ .

### 6.3 Identifying the Optimal Number of Jumps for Sets of the Form $A = \{\xi : l \leq \xi \leq u\}$

The sets that appeared in the examples in Section 6.1 and Section 6.2 lend themselves to a direct characterization of the optimal numbers of jumps  $(\mathcal{J}(A), \mathcal{K}(A))$ . However, in more complicated problems, deciding what kind of paths the most probable limit behaviors consist of may not be as obvious. In this section, we show that for sets of a certain form, we can identify an optimal path. Consider continuous real-valued functions  $l$  and  $u$ , which satisfy  $l(t) < u(t)$  for every  $t \in [0, 1]$ , and suppose that  $l(0) < 0 < u(0)$ . Define  $A = \{\xi : l(t) \leq \xi(t) \leq u(t)\}$ . We assume that both  $\alpha, \beta < \infty$ , which is the most interesting case.

The goal of this section is to construct an algorithm which yields an expression for  $\mathcal{J}(A)$  and  $\mathcal{K}(A)$ . In fact, we can completely identify a function  $h$  that solves the optimization problem defining  $(\mathcal{J}(A), \mathcal{K}(A))$ . This function will be a step function with both positive and negative steps. We first construct such a

function, and then verify its optimality. The first step is to identify the times at which this function jumps. Define the sets

$$A_t \triangleq \{x : l(t) \leq x \leq u(t)\}, \quad A_{s,t}^* \triangleq \bigcap_{s \leq r \leq t} A_r,$$

and the times  $(t_n, n \geq 1)$  by

$$t_{n+1} \triangleq 1 \wedge \inf\{t > t_n : A_{\tau_n, t} = \emptyset\} \quad \text{for } n \geq 2, \quad t_1 \triangleq 1 \wedge \inf\{t > 0 : 0 \notin A_t\}.$$

Let  $n^* = \inf\{n \geq 1 : t_n = 1\}$ . Assume that  $n^* > 1$ , since the zero function is the obvious optimal path in case  $n^* = 1$ . Due to the construction of the times  $t_n, n \geq 1$ , we have the following properties:

- Either  $l(t_1) = 0$  or  $u(t_1) = 0$ .
- For every  $n = 1, \dots, n^* - 2$ ,  $\sup_{t \in [t_n, t_{n+1}]} l(t) = \inf_{t \in [t_n, t_{n+1}]} u(t)$ .
- $H_{fin} \triangleq [\sup_{t \in [t_{n^*-1}, t_{n^*}]} l(t), \inf_{t \in [t_{n^*-1}, t_{n^*}]} u(t)]$  is nonempty.

Set  $h_n \triangleq \sup_{t \in [t_n, t_{n+1}]} l(t)$  for  $n = 1, \dots, n^* - 1$ , and set  $h_{n^*-1} \triangleq h_{fin}$  for any  $h_{fin} \in H_{fin}$ . Define now  $h(t)$  as 0 on  $t \in [0, t_1]$ ,  $h(t) = h_n$  on  $t \in [t_n, t_{n+1})$  for  $n = 1, \dots, n^* - 2$ , and  $h(t) = h_{n^*-1}$  on  $t \in [t_{n^*-1}, 1]$ . We claim now that  $(\mathcal{J}(A), \mathcal{K}(A)) = (\mathcal{J}(\{h\}), \mathcal{K}(\{h\}))$ . In fact, we can prove that if  $g \in A$  is a step function,  $\mathcal{D}_+(g) \geq \mathcal{D}_+(h)$  and  $\mathcal{D}_-(g) \geq \mathcal{D}_-(h)$ , which implies the optimality of  $h$ . The proof is based on the following observation. At each  $t_{n+1}$ , either

- 1) for any  $\epsilon > 0$  one can find  $t \in [t_{n+1}, t_{n+1} + \epsilon]$  such that  $u(t) < h_n$ , or
- 2) for any  $\epsilon > 0$  one can find  $t \in [t_{n+1}, t_{n+1} + \epsilon]$  such that  $l(t) > h_n$ .

Otherwise, there exists  $\epsilon > 0$  such that  $h_n \in A_{t_n, t_{n+1} + \epsilon}$ , contradicting the definition of  $t_n$ , which requires  $A_{t_n, t_{n+1} + \epsilon} = \emptyset$ . From this observation, we can prove that on each interval  $(t_n, t_{n+1}]$ , any feasible path must jump at least once in the same direction as that of the jump of  $h$ . To see this, first suppose that 1) is the case at  $t_{n+1}$ , and  $g \in A$  is a step function. Note that due to its continuity,  $l(\cdot)$  should have achieved its supremum at  $t_{sup} \in [t_n, t_{n+1}]$ , i.e.,  $l(t_{sup}) = h_n$ , and hence,  $g(t_{sup}) \geq h_n$ . On the other hand, due to the right continuity of  $g$  and 1),  $g$  has to be strictly less than  $h_n$  at  $t_{n+1}$ , i.e.,  $g(t_{n+1}) < h_n$ . Therefore,  $g$  must have a downward jump on  $(t_{sup}, t_{n+1}] \subseteq (t_n, t_{n+1}]$ . Note that the direction of the jump of  $h$  in the interval  $(t_n, t_{n+1}]$  (more specifically at  $t_{n+1}$ ) also has to be downward. Since  $g$  is an arbitrary feasible path, this means that whenever  $h$  jumps downward on  $(t_n, t_{n+1})$ , any feasible path in  $A$  should also jump downward. Hence, any feasible path must have either equal or a greater number of downward jumps as  $h$ 's on  $[0, 1]$ . Case 2) leads to a similar conclusion about the number of upward jumps of feasible paths. The number of upward jumps of  $h$  is optimal, proving that  $h$  is indeed the optimal path.

## 6.4 Multiple Optima

This section illustrates how to handle a case where we require Theorem 3.5, and consider an illustrative example where a rare event can be caused by two different configurations of big jumps. Suppose that the regularly varying indices  $-\alpha$  and  $-\beta$  for positive and negative parts of the Lévy measure  $\nu$  of  $X$  are equal, and consider the set  $A \triangleq \{\xi \in \mathbb{D} : |\xi(t)| \geq t - 1/2\}$ . Then,  $\arg \min_{(j,k) \in \mathbb{Z}_+^2} \mathcal{I}(j,k) = \{(1,0), (0,1)\}$ , and  $\mathbb{D}_{\ll 1,0} = \mathbb{D}_{\ll 0,1} = \mathbb{D}_{0,0}$ . Since  $|\xi(1)| \geq 1/2$  for any  $\xi \in A$ ,  $d(A, \mathbb{D}_{0,0}) = 1/2 > 0$ . Theorem 3.5 therefore applies, and for each  $\epsilon > 0$ , there exists  $N$  such that

$$\begin{aligned} \mathbf{P}(\bar{X}_n \in A) &\geq \frac{(C_{l,m}(A^\circ \cap \mathbb{D}_{1,0}) - \epsilon)L_+(n) + (C_{l,m}(A^\circ \cap \mathbb{D}_{0,1}) - \epsilon)L_-(n)}{n^{\alpha-1}}, \\ \mathbf{P}(\bar{X}_n \in A) &\leq \frac{(C_{l,m}(A^- \cap \mathbb{D}_{1,0}) + \epsilon)L_+(n) + (C_{l,m}(A^- \cap \mathbb{D}_{0,1}) + \epsilon)L_-(n)}{n^{\alpha-1}}, \end{aligned}$$

for all  $n \geq N$ . Note that  $A$  is closed, since if there is  $\xi \in \mathbb{D}$  and  $s \in [0,1]$  such that  $|\xi(s)| < s - 1/2$ , then  $B(\xi, \frac{s-1/2-\xi(s)}{2}) \subseteq A^c$ . Therefore,  $A^- \cap \mathbb{D}_{1,0} = A \cap \mathbb{D}_{1,0} = \{\xi = x1_{[u,1]} : x \geq 1/2, 0 < u \leq 1/2\}$ , and hence,  $C_{1,0}(A^- \cap \mathbb{D}_{1,0}) = \mathbf{P}(U_1 \in (0, 1/2])\nu_\alpha[1/2, \infty) = (1/2)^{1-\alpha}$ . Noting that  $A^\circ \cap \mathbb{D}_{1,0} \supseteq (A \cap \mathbb{D}_{1,0})^\circ = \{\xi = x1_{[u,1]} : x > 1/2, 0 < u < 1/2\}$ , we deduce  $C_{1,0}(A^\circ \cap \mathbb{D}_{1,0}) \geq \mathbf{P}(U_1 \in (0, 1/2))\nu_\alpha(1/2, \infty) = (1/2)^{1-\alpha}$ . Therefore,  $C_{1,0}(A^\circ \cap \mathbb{D}_{1,0}) = C_{1,0}(A^- \cap \mathbb{D}_{1,0}) = (1/2)^{1-\alpha}$ . Similarly, we can check that  $C_{0,1}(A^\circ \cap \mathbb{D}_{0,1}) = C_{0,1}(A^- \cap \mathbb{D}_{0,1}) = (1/2)^{1-\beta} (= (1/2)^{1-\alpha})$ . Therefore, for  $n \geq N$ ,

$$\begin{aligned} ((1/2)^{1-\alpha} - \epsilon)(L_+(n) + L_-(n))n^{1-\alpha} \\ \leq \mathbf{P}(\bar{X}_n \in A) \\ \leq ((1/2)^{1-\alpha} + \epsilon)(L_+(n) + L_-(n))n^{1-\alpha}. \end{aligned}$$

This is equivalent to

$$\begin{aligned} \left(\frac{1}{2}\right)^{1-\alpha} &\leq \liminf_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(L_+(n) + L_-(n))n^{1-\alpha}} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(L_+(n) + L_-(n))n^{1-\alpha}} \leq \left(\frac{1}{2}\right)^{1-\alpha}. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(L_+(n) + L_-(n))n^{1-\alpha}} = \left(\frac{1}{2}\right)^{1-\alpha}.$$

## A Inequalities

**Lemma A.1** (Generalized Kolmogorov inequality; [Shneer and Wachtel \(2009\)](#)). *Let  $S_n = X_1 + \dots + X_n$  be a random walk with mean zero increments, i.e.,  $\mathbf{E}X_i = 0$ . Then,*

$$\mathbf{P}(\max_{k \leq n} S_k \geq x) \leq C \frac{nV(x)}{x^2},$$

where  $V(x) = \mathbf{E}(X_1^2; |X_1| \leq x)$ , for all  $x > 0$ .

**Lemma A.2** (Etemadi's inequality). *Let  $X_1, \dots, X_n$  be independent real-valued random variables defined on some common probability space, and let  $\alpha \geq 0$ . Let  $S_k$  denote the partial sum  $S_k = X_1 + \dots + X_k$ . Then*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| \geq 3x\right) \leq 3 \max_{1 \leq k \leq n} \mathbb{P}(|S_k| \geq x).$$

**Lemma A.3** (Prokhorov's inequality; Prokhorov (1959)). *Suppose that  $\xi_i$ ,  $i = 1, \dots, n$  are independent, zero-mean random variables such that there exists a constant  $c$  for which  $|\xi_i| \leq c$  for  $i = 1, \dots, n$ , and  $\sum_{i=1}^n \mathbf{var} \xi_i < \infty$ . Then*

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i > x\right) \leq \exp\left\{-\frac{x}{2c} \operatorname{arcsinh} \frac{xc}{2 \sum_{i=1}^n \mathbf{var} \xi_i}\right\},$$

which, in turn, implies

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i > x\right) \leq \left(\frac{cx}{\sum_{i=1}^n \mathbf{var} \xi_i}\right)^{-\frac{x}{2c}}.$$

We extend the Etemadi's inequality to Lévy processes in the following lemma.

**Lemma A.4.** *Let  $Z$  be a Lévy process. Then,*

$$\mathbb{P}\left(\sup_{t \in [0, n]} |Z(t)| \geq x\right) \leq 3 \sup_{t \in [0, n]} \mathbf{P}(|Z(t)| \geq x/3).$$

*Proof.* Since  $Z$  (and hence  $|Z|$  also) is in  $\mathbb{D}$ ,  $\sup_{0 \leq k \leq 2^m} |Z(\frac{nk}{2^m})|$  converges to  $\sup_{t \in [0, n]} |Z(t)|$  almost surely as  $m \rightarrow \infty$ . To see this, note that one can choose  $t_i$ 's such that  $|Z(t_i)| \geq \sup_{t \in [0, n]} |Z(t)| - i^{-1}$ . Since  $\{t_i\}$ 's are in a compact set  $[0, n]$ , there is a subsequence, say,  $t'_i$ , such that  $t'_i \rightarrow t_0$  for some  $t_0 \in [0, n]$ . The supremum has to be achieved at either  $t_0^-$  or  $t_0$ . Either way, with large enough  $m$ ,  $\sup_{0 \leq k \leq 2^m} |Z(\frac{nk}{2^m})|$  becomes arbitrarily close to the supremum. Now, by bounded convergence,

$$\begin{aligned} & \mathbf{P}\left\{\sup_{t \in [0, n]} |Z(t)| > x\right\} \\ &= \lim_{m \rightarrow \infty} \mathbf{P}\left\{\sup_{0 \leq k \leq 2^m} \left|Z\left(\frac{nk}{2^m}\right)\right| > x\right\} \\ &= \lim_{m \rightarrow \infty} \mathbf{P}\left\{\sup_{0 \leq k \leq 2^m} \left|\sum_{i=0}^k \left(Z\left(\frac{ni}{2^m}\right) - Z\left(\frac{n(i-1)}{2^m}\right)\right)\right| > x\right\} \\ &\leq \lim_{m \rightarrow \infty} 3 \sup_{0 \leq k \leq 2^m} \mathbf{P}\left\{\left|\sum_{i=0}^k \left(Z\left(\frac{ni}{2^m}\right) - Z\left(\frac{n(i-1)}{2^m}\right)\right)\right| > x/3\right\} \\ &= \lim_{m \rightarrow \infty} 3 \sup_{0 \leq k \leq 2^m} \mathbf{P}\left\{\left|Z\left(\frac{nk}{2^m}\right)\right| > x/3\right\} \\ &\leq 3 \sup_{t \in [0, n]} \mathbf{P}\{|Z(t)| > x/3\}, \end{aligned}$$

where  $Z(t) \triangleq 0$  for  $t < 0$ . □

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