Sample-path large deviations for a class of heavy-tailed Markov-additive processes

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Abstract

For a class of additive processes driven by the affine recursion $X_{n+1} = A_{n+1}X_n + B_{n+1}$, we develop a sample-path large deviations principle in the M_1' topology on $D[0,1]$. We allow B_n to have both signs and focus on the case where Kesten's condition holds on A_1, leading to heavy-tailed distributions. The most likely paths in our large deviations results are step functions with both positive and negative jumps.

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1 Introduction

Let $\{X_n,\, n \geq 0\}$ be an affine recursion such that

$$X_{n+1} = A_{n+1}X_n + B_{n+1} \tag{1.1}$$

for a sequence of i.i.d. $\mathbb{R}^2$-valued random vectors (A_n, B_n). The Markov chain driven by (1.1) has been studied extensively in the past several decades and continues to pose new research challenges. A classical result, which can be found in [19] and [23] shows that under certain assumptions (see Assumption 1 below), the Markov chain X_n, $n \geq 0$ has a unique stationary distribution π, for which we have

$$\pi(x,\infty) \sim c_+ x^{-\alpha} \quad \text{and} \quad \pi(-\infty,-x) \sim c_- x^{-\alpha}, \quad \text{as } x \to \infty, \tag{1.2}$$

for some c_-, c_+ satisfying $c_- + c_+ > 0$; see the monograph [7] for a recent comprehensive account.

Define $\bar{X}_n = \{\bar{X}_n(t), t \in [0,1]\}$, with

$$\bar{X}_n(t) = \sum_{i=0}^{\lfloor nt \rfloor - 1} X_i/n. \tag{1.3}$$

The focus of the present paper is on sample-path large deviations of the additive process $\bar{X}_n$, assuming the invariant distribution of X_n has a heavy tail as in (1.2). The study of additive processes of the form (1.3) is less well developed. Classical theory initiated by Donsker and Varadhan (see, for example, [14, 15]) provides powerful tools designed to study large deviations for additive functionals of light-tailed and geometrically ergodic Markov chains. More recent contributions in this area include [24, 25]. Analogues of these sample-path results in a heavy-tailed setting do not seem to be available.

A considerable body of theory has been developed to analyse exceedance probabilities for random walks $\{\hat{S}_n, n \geq 0\}$ with heavy-tailed step sizes. Early papers [31], [32], identified appropriate sequences (x_n) for which

$$\mathbf{P}(\hat{S}_n/n > x_n) = n\mathbf{P}(\hat{S}_1 > x_n)(1 + o(1)), \quad \text{as } n \to \infty, \tag{1.4}$$

holds. For a detailed description, we refer to e.g. [5], [13], and [17]. When (1.4) is valid, the so-called principle of a single big jump is said to hold. As a generalization of (1.4), a functional form has been derived in [22], where random walks with i.i.d. multi-dimensional regularly varying (cf. Definition 1.1 of [22]) step sizes are considered.

Several works have focused on the extension of (1.4) to more general processes where there is a certain dependence structure in the increments. Key references are [18], [21], [28], where stable processes, modulated processes, and stochastic differential equations are considered. Extensions to additive processes of the form considered in this paper have been provided in [29, 30], who also consider more general examples of driving recursion $X_{n+1} = f_{n+1}(X_n)$. The principle of a single big jump is still valid, but an additional constant in the RHS of (1.4) can appear.

Extending the results of [29, 30] to the sample-path level poses several phenomenological and technical challenges. So far, all results cited center around the phenomenon where rare events are caused by a single big jump. However, not all rare events are caused by this relatively simple scenario, for early examples see [16], [37]. In a recent paper, [33] provide sample-path large deviations results for Lévy processes and random walks with regularly varying increments, which deal with a general class of rare events that can especially be caused by multiple jumps. For further examples see [9]. However, the case studied here is considerably harder, as big jumps occur by a condensation phenomenon, through the concatenation of many small jumps. In particular, a large value of the sample mean is not due to a single large value of the A_n or B_n but to large values of the products $A_1 \cdots A_n$, see also [10], [6]. When studying sample-path large deviations, this phenomenon poses nontrivial technical requirements. In particular, an appropriate topology needs to be considered.

Our approach to overcome these challenges is as follows. We first proceed to identify a sequence of regeneration times r_n, $n \geq 1$ (see [2]), and split the Markov chain into i.i.d. cycles. By aggregating the trajectory of $\bar{X}_n$ over regeneration cycles, we obtain a regenerative process with i.i.d. jump distributions and $r_n, n \geq 1$ as renewals. Under a set of assumptions originating from [23] and [19], we establish our first major result, which is that the "area" under a typical regeneration cycle, denoted by $\mathfrak{R}$ (see (3.2) below), has an asymptotic power law. To be precise, we have

$$\mathbf{P}(\mathfrak{R} > x) \sim C_+ x^{-\alpha} \quad \text{and} \quad \mathbf{P}(\mathfrak{R} < -x) \sim C_- x^{-\alpha}, \quad \text{as } x \to \infty, \tag{1.5}$$

for some constants C_-, C_+. This is related to a result of [10] for the case where $X_n \geq 0$. Our argument is different, developed in a two-sided setting, and can be extended to more general recursions, cf. [8].

Using the tail estimates (1.5), we present in Sections 3.2 and 3.3 large deviations results for $\bar{X}_n$ as in (1.3), which constitutes the second major step in our approach. We achieve this by introducing a new asymptotic equivalence concept (see Lemma 2.11 below), which, together with the decomposition in cycles, allows us to build a bridge between our problem and the one studied [33]. In the latter paper, the Skorokhod J_1 topology is used. However, showing that the residual process (i.e. the contribution of the cycle going on at the endpoint of our interval) is negligible in its contribution to $\mathbf{P}(\bar{X}_n \in E)$ is not straightforward, especially when the increments of $\bar{X}_n$ are dependent as in the current setting. To overcome this, we switch to a slightly weaker topology, namely the M_1'-topology on $\mathbb{D}[0,1]$ (as defined in [3], see also Section 3.2 below), and derive asymptotic estimates of events involved with the "area" under the last ongoing cycle. This choice of topology is crucial as it allows many light-tailed jumps, occurring within a cycle, to merge into a single heavy-tailed jump.

Our main sample-path large deviations results are presented in Section 3. For the case where B_n as in (1.1) is nonnegative, our result establishes that

$$C_{\mathcal{J}^*}(E^\circ) \leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in E)}{(n\mathbf{P}(\mathfrak{R} > n))^{\mathcal{J}^*}} \leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in E)}{(n\mathbf{P}(\mathfrak{R} > n))^{\mathcal{J}^*}} \leq C_{\mathcal{J}^*}(E^-). \tag{1.6}$$

Precise details can be found in Section 3.2 below. At this moment, we just mention that C_j is a measure on the Skorokhod space for each j, and $\mathcal{J}^*$ denotes the minimum number of jumps that are required for a nondecreasing, piecewise linear function with drift $\mathbf{E}B_1/(1-\mathbf{E}A_1)$ to be in the set E. In Section 3.3 we develop a two-sided version of this result. While we restrict to the case of affine recursions in (1.1), the methods developed in this paper can be applied to more general recursions of the form $X_{n+1} = f_{n+1}(X_n)$, in which $f_n(z)/z \to A_n$ as $z \to \infty$; we refer to [8] for details.

This paper is organized as follows. In Section 2, we introduce some useful tools for future purposes. We present our main results in Section 3. In Section 4, we present an application of our large deviations result. Sections 5–7 are devoted to the proofs.

2 Preliminaries

In this section, we recall and establish some preliminary results. All the proofs are deferred to Section 5. We start by introducing a regularity condition.

Assumption 1. The random vector (A_1, B_1) satisfies

1. $A_1 \geq 0$ a.s. and the law of $\log A_1$ conditioned on $\{A_1 > 0\}$ is nonarithmetic.
2. There exists an $\alpha \in (1,\infty)$ such that $\mathbf{E}A_1^\alpha = 1$, $\mathbf{E}A_1^\alpha \log^+ A_1 < \infty$ (where $\log^+ x = \max\{\log x, 0\}$), and $\mathbf{E}|B_1|^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$.
3. $\mathbf{P}(A_1 x + B_1 = x) < 1$ for every $x \in \mathbb{R}$.

The conditions in Assumption 1 imply that $\mathbf{E}\log A_1 < 0$ and $\mathbf{E}\log^+|B_1| < \infty$, and hence (see e.g. Theorem 2.1.3 of [7]), the Markov chain has a unique stationary distribution, denoted by π. Moreover, [23] and [19] showed there exist constants c_+, c_- satisfying $c_+ + c_- \in (0,\infty)$ such that,

$$\pi(x,\infty) \sim c_+ x^{-\alpha} \quad \text{and} \quad \pi(-\infty,-x) \sim c_- x^{-\alpha}, \qquad \text{as } x \to \infty. \tag{2.1}$$

A natural question is whether $c_+ > 0$ and/or $c_- > 0$ in our setting. In [20], sufficient conditions for $c_+ > 0$ and/or $c_- > 0$ are developed using algebraic methods.

2.1 Background from Markov chain theory

We review some concepts from Markov chain theory. We begin by introducing two conditions on general Markov chains. A Markov chain on some general state space $(\mathbb{S}, \mathcal{S})$ with transition kernel P satisfies a drift condition ($\mathcal{D}$) if

$$\int_{\mathbb{S}} h(y)P(x,dy) \leq \kappa_1 h(x) + \kappa_2 \mathbb{1}_{\mathcal{C}}(x), \tag{$\mathcal{D}$}$$

for some $\kappa_1 \in (0,1)$ and $\kappa_2 > 0$, where h takes values in $[1,\infty)$, and $\mathcal{C}$ is a Borel subset of $\mathbb{R}$. Moreover, we say that a ϕ-irreducible Markov chain on $(\mathbb{S}, \mathcal{S})$ with transition kernel P satisfies a minorization condition ($\mathcal{M}$) if

$$\theta \mathbb{1}_{\mathcal{C}_0}(x)\phi(E \cap E_0) \leq P(x,E), \qquad x \in \mathbb{S}, E \in \mathcal{S}, \tag{$\mathcal{M}$}$$

for some set $E_0 \subseteq \mathbb{S}$, some set $\mathcal{C}_0$ with $\phi(\mathcal{C}_0) > 0$, some constant $\theta > 0$, and some probability measure ϕ on $(\mathbb{S}, \mathcal{S})$.

Remark 2.1. If the minorization condition ($\mathcal{M}$) holds, then there exists a sequence of strictly increasing finite random times r_n, $n \geq 1$ such that $\{X_n\}_{n\geq 0}$ regenerates at each r_n w.r.t. ϕ, that is, $\mathbf{P}(X_{r_i} \in E) = \phi(E \cap E_0)$ for each i. In particular, X_n attempts (independently of everything else) to regenerate each time it enters $\mathcal{C}_0$, and such attemps are successful w.p. θ.

Recall that we say that the Markov chain $\{X_t\}_{n\geq 0}$ is geometrically ergodic if there exists some number $\rho_0 \in (0,1)$ such that

$$\|P^n(x,\cdot) - P_0(\cdot)\|_{TV} = o(\rho_0^n)$$

as $n \to \infty$.

Result 2.2 (Lemma 2.2.3, Proposition 2.2.4, Theorem 2.4.4 of [7]). *Let $\{X_n\}_{n\geq 0}$ be such that $X_{n+1} = A_{n+1}X_n + B_{n+1}$. Supposing that Assumption 1 holds, we have that:*

1. *For any given $\delta \in (0,\alpha)$, $\{X_n\}_{n\geq 0}$ satisfies the drift condition ($\mathcal{D}$) with $h(x) = 1 + |x|^\delta$ and $\mathcal{C} = [-M, M]$ for some constant $M \geq 0$.*
2. *$\{X_n\}_{n\geq 0}$ is π-irreducible.*
3. *$\{X_n\}_{n\geq 0}$ is geometrically ergodic.*

The regeneration scheme described in Remark 2.1 plays an important role in our analysis. Our next assumption guarantees the existence of the regeneration times.

Assumption 2. Condition ($\mathcal{M}$) is satisfied with $\mathcal{C}_0 = [-d, d]$ for some $d > 0$ such that $[-d,d] \cap \text{supp}(\pi) \neq \emptyset$.

For the rest of the paper, we will use r_n to denote the n^{th} regeneration time w.r.t. this particular $\mathcal{C}_0$ and ϕ in Assumption 2. Moreover, we assume $r_0 = 0$, so that $X_0 \in \mathcal{C}_0$. For completeness we mention some sufficient conditions for Assumption 2 to hold in terms of the joint distribution of (A_1, B_1), which is a minor extension of Lemma 2.2.3 of [7]. Let $\mathscr{B}_r(x) = \{x' : |x - x'| < r\}$ for $x \in \mathbb{R}$ and $r > 0$.

Proposition 2.3. *Assume that one of the following conditions hold.*

1. *Let $B_1 \geq b$ a.s. for some $b > 0$. Moreover, there exist intervals $I_1 = (a_1, a_2) \subseteq \mathbb{R}_+$, $I_2 = (b_0 - \delta, b_0 + \delta)$ for some $a_1 < a_2$, b_0, $\delta > 0$, a σ-finite measure ν_0 with b_0 in the support of ν_0, and a constant $c_0 > 0$ such that for any Borel sets D_1, $D_2 \subseteq \mathbb{R}$,*

$$\mathbf{P}((A_1, B_1) \in (D_1 \times D_2)) \geq c_0 |D_1 \cap I_1| \nu_0(D_2 \cap I_2),$$

   where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

2. *There exist intervals $I_1 = (a_0 - \delta, a_0 + \delta) \subseteq \mathbb{R}_+$, $I_2 = (b_1, b_2)$ for some a_0, $b_1 < b_2$, $\delta > 0$, a σ-finite measure ν_0 with a_0 in the support of ν_0, and a constant $c_0 > 0$ such that for any Borel sets D_1, $D_2 \subseteq \mathbb{R}$,*

$$\mathbf{P}((A_1, B_1) \in (D_1 \times D_2)) \geq c_0 \nu_0(D_1 \cap I_1) |D_2 \cap I_2|. \tag{2.2}$$

Then, for any $x_0 \in \mathbb{R}$, there exists $\epsilon = \epsilon(x_0)$, $\theta > 0$, open interval E_0 such that

$$\theta |E \cap E_0| \leq P(x, E), \qquad x \in \mathscr{B}_\epsilon(x_0), E \in \mathcal{B}(\mathbb{R}). \tag{2.3}$$

Our next result implies the geometric decay of $\mathbf{P}(r_1 > k)$ as $k \to \infty$.

Lemma 2.4. *Suppose that Assumption 1 and 2 hold. Let $\{r_n\}_{n\geq 0}$ be the sequence of regeneration times associated with $\mathcal{C}_0$. Let E_1 be a bounded set. There exists $t > 1$ such that*

$$\sup_{x \in E_1} \mathbf{E}[t^{r_1} \,|\, X_0 = x] < \infty.$$

2.2 A useful change of measure

Another helpful tool in our context is the so-called α-shifted change of measure (see e.g. [12, 11]). Let ν denote the distribution of $(\log A_n, B_n)$ and define the α-shifted measure ν^α by

$$\nu^\alpha(E) = \int_E e^{\alpha x} d\nu(x, y), \qquad E \in \mathfrak{B}(\mathbb{R}^2).$$

Let $\mathcal{L}(\log A_n, B_n)$ denote the law of $(\log A_n, B_n)$. For a stopping time T, Let $\mathscr{D}_T^\alpha$ be the dual change of measure such that, under $\mathscr{D}_T^\alpha$,

$$\mathcal{L}(\log A_n, B_n) = \begin{cases} \nu^\alpha, & \text{for } n \leq T, \\ \nu, & \text{for } n > T. \end{cases} \tag{2.4}$$

Let $\mathbf{P}^\alpha$, $\mathbf{P}^{\mathscr{D}_T^\alpha}$, $\mathbf{E}^\alpha$ and $\mathbf{E}^{\mathscr{D}_T^\alpha}$ denote expectation and probability w.r.t. the α-shifted measure ν^α and the dual change of measure $\mathscr{D}_T^\alpha$, respectively. Defining

$$S_n = \sum_{i=1}^n \log A_i, \tag{2.5}$$

we have the following result.

Result 2.5 (Lemma 5.3 of [12]). *Let T and τ be stopping times w.r.t. $\{X_n\}_{n\geq 0}$, let $g\colon \mathbb{R}^\infty \to [0,\infty]$ be a deterministic function, and let g_n denote its projection onto the first $n+1$ coordinates, i.e., $g_n(x_0, \ldots, x_n) = g(x_0, \ldots, x_n, 0, 0, \ldots)$. Then*

$$\begin{aligned} \mathbf{E}[g_{\tau-1}(X_0, \ldots, X_{\tau-1})] = {} & \mathbf{E}^{\mathscr{D}_T^\alpha}\left[g_{\tau-1}(X_0, \ldots, X_{\tau-1}) e^{-\alpha S_T} \mathbb{1}_{\{T<\tau\}}\right] \\ & + \mathbf{E}^{\mathscr{D}_T^\alpha}\left[g_{\tau-1}(X_0, \ldots, X_{\tau-1}) e^{-\alpha S_\tau} \mathbb{1}_{\{T\geq\tau\}}\right]. \end{aligned}$$

Remark 2.6. Note that by the same argument, if a random variable R is measurable w.r.t. the stopped σ-algebra $\mathcal{F}_T$, then

$$\mathbf{E}[R] = \mathbf{E}^{\mathscr{D}_T^\alpha}[Re^{-\alpha S_T}] = \mathbf{E}^\alpha[Re^{-\alpha S_T}].$$

Our analysis relies on the fact that the Markov chain X_n is closely related to a multiplicative random walk, that is,

$$X_{n+1} \approx A_{n+1} X_n, \quad \text{for large } n.$$

Roughly speaking, the process X_n resembles a perturbation of a multiplicative random walk, in an asymptotic sense (for details see [12, 11]). Hence, it is natural to consider the "discrepancy" process between X_n and $\prod_{i=1}^n A_i$, which is defined as

$$Z_n = X_n e^{-S_n} = X_0 + \sum_{k=1}^n B_k e^{-S_k}, \quad n \geq 0, \tag{2.6}$$

where S_n is as in (2.5). Under the α-shifted measure, we have $\mathbf{E}^\alpha \log A_1 = \mathbf{E} A_1^\alpha \log A_1 > 0$ by Assumption 1 and Theorem 2.4.4 of [7]. Consequently, we have the following result.

Lemma 2.7. *Let Assumption 1 hold. Under $\mathbf{P}^\alpha$,*

1. *$|X_n| \uparrow \infty$ a.s. as $n \to \infty$.*
2. *$Z_n \xrightarrow{a.s.} Z$ as $n \to \infty$, where $Z = X_0 + \sum_{k=1}^\infty B_k e^{-S_k}$.*

2.3 $\mathbb{M}$-convergence

We briefly review the notion of $\mathbb{M}$-convergence [26, 33], and introduce a novel asymptotic equivalence concept. Let $(\mathbb{S}, d)$ be a complete separable metric space, and $\mathscr{S}$ be the Borel σ-algebra on $\mathbb{S}$. Given a closed subset $\mathbb{C}$ of $\mathbb{S}$, let $\mathbb{S}\setminus\mathbb{C}$ be equipped with the relative topology as a subspace of $\mathbb{S}$, and consider the associated sub σ-algebra $\mathscr{S}_{\mathbb{S}\setminus\mathbb{C}} = \{E\colon E \subseteq \mathbb{S}\setminus\mathbb{C}, A \in \mathscr{S}\}$ on it. Define $\mathbb{C}^r = \{x \in \mathbb{S}\colon d(x,\mathbb{C}) < r\}$ for $r > 0$, and let $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ be the class of measures defined on $\mathscr{S}_{\mathbb{S}\setminus\mathbb{C}}$ whose restrictions to $\mathbb{S}\setminus\mathbb{C}^r$ are finite for all $r > 0$. Topologize $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ with a sub-basis $\{\{\nu \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})\colon \nu(f) \in G\}\colon f \in \mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}, G \text{ open in } \mathbb{R}_+\}$, where $\mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}$ is the set of real-valued, non-negative, bounded, continuous functions whose support is bounded away from $\mathbb{C}$ (i.e., $f(\mathbb{C}^r) = \{0\}$ for some $r > 0$). A sequence of measures $\nu_n \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ converges to $\nu \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ if $\nu_n(f) \to \nu(f)$ for each $f \in \mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}$. We say that a set $E_1 \subseteq \mathbb{S}$ is bounded away from another set $E_2 \subseteq \mathbb{S}$ if $\inf_{x\in E_1, y\in E_2} d(x,y) > 0$. The following characterization of $\mathbb{M}$-convergence can be considered as a generalization of the classical notion of weak convergence of measures, see e.g. [4].

Result 2.8 (Theorem 2.1 of [26])**.** *Let ν, $\nu_n \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$. We have $\nu_n \to \nu$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ as $n \to \infty$ if and only if*

$$\limsup_{n\to\infty} \nu_n(F) \le \nu(F)$$

for all closed $F \in \mathscr{S}_{\mathbb{S}\setminus\mathbb{C}}$ bounded away from $\mathbb{C}$ and

$$\liminf_{n\to\infty} \nu_n(G) \ge \nu(G)$$

for all open $G \in \mathscr{S}_{\mathbb{S}\setminus\mathbb{C}}$ bounded away from $\mathbb{C}$.

We now introduce a new notion of equivalence between two families of random objects, which will prove to be useful in Section 7. Let $F_\delta = \{x \in \mathbb{S}\colon d(x,F) \le \delta\}$ and $G^{-\delta} = ((G^c)_\delta)^c$. Note that when it comes to the fattening and shaving of sets, we denote open sets with a superscript and closed sets with a subscript.

Definition 2.9. Suppose that X_n and Y_n are random elements taking values in a complete separable metric space $(\mathbb{S}, d)$. Y_n is said to be asymptotically equivalent to X_n with respect to ϵ_n and $\mathbb{C}$, if, for each $\delta > 0$ and $\gamma > 0$,

$$\begin{aligned}&\limsup_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(X_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma}, d(X_n, Y_n) \ge \delta)\\ &= \limsup_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(Y_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma}, d(X_n, Y_n) \ge \delta) = 0.\end{aligned}$$

Remark 2.10. Note that the asymptotic equivalence w.r.t. $\mathbb{C}$ implies the asymptotic equivalence w.r.t. $\mathbb{C}'$ if $\mathbb{C} \subseteq \mathbb{C}'$. In view of this, the strongest notion of asymptotic equivalence w.r.t. a given sequence ϵ_n is the one w.r.t. an empty set. In this case, the conditions for the asymptotic equivalence reduce to a simple condition: $\mathbf{P}(d(X_n, Y_n) \ge \delta) = o(\epsilon_n)$ for any $\delta > 0$. That special case of asymptotic equivalence has been introduced and applied in [33]. In our context, this simple condition suffices for the case of $B_1 \ge 0$ in Section 3.2; however, we have to work with the case that $\mathbb{C}$ is not an empty set when we deal with general B_1 in Section 3.3.

The usefulness of this notion of equivalence comes from the following result.

Lemma 2.11. *Suppose that $\epsilon_n^{-1}\mathbf{P}(X_n \in \cdot) \to \nu(\cdot)$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for some sequence ϵ_n and a closed set $\mathbb{C}$. If Y_n is asymptotically equivalent to X_n with respect to ϵ_n and $\mathbb{C}$, then the law of Y_n has the same normalized limit, i.e., $\epsilon_n^{-1}\mathbf{P}(Y_n \in \cdot) \to \nu(\cdot)$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$.*

3 Main results

This section is organized as follows. In Section 3.1, we analyze the tail estimates of the area under the first return time and regeneration cycle, which are needed to derive the sample-path large deviations of $\bar{X}_n$. In Section 3.2 we derive such results in the case where $B_1 \ge 0$. The two-sided case is more involved and is treated in Section 3.3.

3.1 Tail estimates on the area under the first return time/regeneration cycle

Let

$$\tau_d = \inf\{n \geq 1 \colon |X_n| \leq d\} \tag{3.1}$$

denote the first return time of X_n to the set $[-d, d]$, where d is such that $[-d, d] \cap \mathrm{supp}(\pi) \neq \emptyset$. Recall that $\{r_n\}_{n\geq 0}$ is the sequence of regeneration times of $\{X_n\}_{n\geq 0}$. We denote the area under the first return time and the regeneration cycle by

$$\mathfrak{B} = \sum_{n=0}^{\tau_d - 1} X_n \quad \text{and} \quad \mathfrak{R} = \sum_{n=0}^{r_1 - 1} X_n, \tag{3.2}$$

respectively. Recall that $Z = X_0 + \sum_{k=1}^{\infty} B_k e^{-S_k}$. Finally, note that considering $B_i \equiv 1$, there exists a constant C_∞ from [19] that satisfies

$$\mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > u\right) \sim C_\infty u^{-\alpha}. \tag{3.3}$$

Theorem 3.1. *Suppose that Assumptions 1 and 2 hold.*

1. *We have*

$$\lim_{u\to\infty} u^\alpha \mathbf{P}(\mathfrak{B} > u) = C_\infty \mathbf{E}^\alpha[(Z^+)^\alpha \mathbb{1}_{\{\tau_d = \infty\}}]$$
$$\textit{and} \quad \lim_{u\to\infty} u^\alpha \mathbf{P}(\mathfrak{B} < -u) = C_\infty \mathbf{E}^\alpha[(Z^-)^\alpha \mathbb{1}_{\{\tau_d = \infty\}}].$$

2. *In addition*

$$\lim_{u\to\infty} u^\alpha \mathbf{P}(\mathfrak{R} > u) = C_+ \quad \textit{and} \quad \lim_{u\to\infty} u^\alpha \mathbf{P}(\mathfrak{R} < -u) = C_-,$$

where $C_+ = C_\infty \mathbf{E}^\alpha[(Z^+)^\alpha \mathbb{1}_{\{r_1 = \infty\}}]$ *and* $C_- = C_\infty \mathbf{E}^\alpha[(Z^-)^\alpha \mathbb{1}_{\{r_1 = \infty\}}]$.

Like in the classical estimates (2.1), it is natural to ask when $C_+, C_- \in (0, \infty)$. A proof of finiteness of $\mathbf{E}^\alpha[|Z|^\alpha]$ is obtained as by-product of the proof of Lemma 6.6. $C_\infty \in (0, \infty)$ by specializing (2.1) to the case $A_1 \leq 0$ and $B_1 \equiv 1$. If B_1 is non-negative and $\mathbf{P}(A_1 = B_1 = 0) = 0$, then $Z > 0$ $\mathbf{P}^\alpha$-a.s. Since also $\mathbf{P}^\alpha(r_1 = \infty) > 0$, $C_+ > 0$.

When B_1 can take both signs, the situation is much more delicate and we sketch how one can deal with this issue. One way is to derive sufficient conditions for the support of Z under $\mathbf{P}^\alpha$ to be the entire real line, from which strict positivity of both C_+ and C_- can be inferred. Such a sufficient condition can be derived from a careful inspection of the proof of Theorem 2.5.5 (1) of [7] (which is a result due to [20]). For example, if the support of (A, B) includes points $(a, b), (a_1, b_1), (a_2, b_2)$ such that $a < 1, a_1, a_2 > 1$ and $b_1/(1 - a_1) < b/(1 - a) < b_2/(1 - a_2)$ the support of Z is the whole real line.

3.2 One-sided large deviations

We first consider the case where B_1 is nonnegative. To deal with the dependence structure of the Markov chain within the regeneration cycle, we consider in this section the M_1' topology. To be precise, define the extended completed graph Γ_ξ' of ξ by

$$\Gamma_\xi' = \{(x, t) \in \mathbb{R} \times [0, 1] \colon x \in [\xi(t^-) \wedge \xi(t), \xi(t^-) \vee \xi(t)]\},$$

where $\xi(0^-) = 0$. Define an order on the graph Γ_ξ' by saying that $(x_1, t_1) \leq (x_2, t_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|\xi(t_1^-) - x_1| \leq |\xi(t_2^-) - x_2|$. We say that a mapping $(u, s) : [0, 1] \to \Gamma_\xi'$ is a parametric representation of ξ if $r \mapsto (u(r), s(r))$ is continuous and non-decreasing. Let $\Pi'(\xi)$ be the set of all parametric representations of $\xi \in \mathbb{D}$. For any $\xi_1, \xi_2 \in \mathbb{D}$, the M_1' metric is defined by

$$d_{M_1'}(\xi_1, \xi_2) = \inf_{\substack{(u_i, s_i) \in \Pi'(\xi_i) \\ i \in \{1,2\}}} \|u_1 - u_2\|_\infty \vee \|s_1 - s_2\|_\infty.$$

For the rest of the paper, we consider the topology w.r.t. this metric, unless specified otherwise.

For the one-sided large deviations result, we need the following elements. We say that a function $\xi \in \mathbb{D}$ is *piecewise constant*, if there exist finitely many time points t_i such that $0 = t_0 < t_1 < \cdots < t_m = 1$ and ξ is constant on the intervals $[t_{i-1}, t_i)$ for all $1 \le i \le m$. For $\xi \in \mathbb{D}$, define the set of discontinuities of ξ by

$$\mathrm{Disc}(\xi) = \{t \in [0,1] \colon \xi(t) \neq \xi(t^-)\}, \tag{3.4}$$

where $\xi(0^-) = 0$. For each integer j, define

$$\underline{\mathbb{D}}_{<j} = \{\xi \in \mathbb{D} \colon \xi \text{ piecewise constant and nondecreasing, } |\mathrm{Disc}(\xi)| < j\}.$$

For $z \in \mathbb{R}$ and each integer j, define

$$\underline{\mathbb{D}}^z_{<j} = \{\xi \in \mathbb{D} \colon \xi = z \cdot id + \xi', \xi' \in \underline{\mathbb{D}}_{<j}\}. \tag{3.5}$$

For each constant $\gamma > 1$, let $\nu_\gamma(x, \infty) = x^{-\gamma}$, and let ν_γ^j denote the restriction (to $\mathbb{R}_+^{j\downarrow} = \{x \in \mathbb{R}^j \colon x_1 \ge \cdots \ge x_j > 0\}$) of the j-fold product measure of ν_γ. Let C_0^z be the Dirac measure concentrated on the linear function zt. For $j \ge 1$, define a sequence of Borel measures $C_j^z \in \mathbb{M}(\mathbb{D} \setminus \underline{\mathbb{D}}_{<j})$ concentrated on $\underline{\mathbb{D}}_{<j+1}$ as

$$C_j^z(\,\cdot\,) = \mathbf{E}\left[\nu_\alpha^j \left\{ x \in (0,\infty)^j \colon z \cdot id + \sum_{i=1}^{j} x_i \mathbb{1}_{U_i} \in \,\cdot\, \right\}\right], \tag{3.6}$$

where α is as in Assumption 1 and the random variables U_i, $i \ge 1$, are i.i.d. uniform distributed on $[0,1]$. For $E \subseteq \mathbb{D}$ and $z \in \mathbb{R}$, define

$$\mathcal{J}_z^{\uparrow}(E) = \inf\{j \colon E \cap \underline{\mathbb{D}}^z_{<j+1} \neq \emptyset\}. \tag{3.7}$$

Setting $\mu = \mathbf{E}B_1/(1 - \mathbf{E}A_1)$, we state below the main theorem for the one-sided case. Recall C_+ defined in Theorem 3.1.

Theorem 3.2. *Suppose that Assumptions 1 and 2 hold. Moreover, let $B_1 \ge 0$, and C_+ be strictly positive.*

1. *For each $j \ge 0$,*
$$n^{j(\alpha-1)} \mathbf{P}(\bar{X}_n \in \,\cdot\,) \to (C_+ \mathbf{E} r_1)^j C_j^\mu(\,\cdot\,),$$
in $\mathbb{M}(\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{<j})$ as $n \to \infty$.

2. *Let E be measurable. If $\mathcal{J}_\mu^{\uparrow}(E) < \infty$ and E is bounded away from $\underline{\mathbb{D}}_{<\mathcal{J}_\mu^{\uparrow}(E)}$, then*
$$\liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in E)}{n^{-\mathcal{J}_\mu^{\uparrow}(E)(\alpha-1)}} \ge (C_+ \mathbf{E} r_1)^{\mathcal{J}_\mu^{\uparrow}(E)} C^\mu_{\mathcal{J}_\mu^{\uparrow}(E)}(E^\circ)$$
$$\text{and} \quad \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in E)}{n^{-\mathcal{J}_\mu^{\uparrow}(E)(\alpha-1)}} \le (C_+ \mathbf{E} r_1)^{\mathcal{J}_\mu^{\uparrow}(E)} C^\mu_{\mathcal{J}_\mu^{\uparrow}(E)}(E^-).$$

3.3 Two-sided large deviations

Similarly as in Section 3.2, we need the following elements. Define the set of step functions with less than j discontinuities by

$$\underline{\mathbb{D}}_{\ll j} = \{\xi \in \mathbb{D} \colon \xi \text{ piecewise constant, } |\mathrm{Disc}(\xi)| < j\}, \quad \text{for } j \ge 0.$$

For $z \in \mathbb{R}$, define

$$\mathbb{D}^z_{\ll j} = \{\xi \in \mathbb{D} \colon \xi = z \cdot id + \xi',\ \xi' \in \mathbb{D}_{\ll j}\}, \quad \text{for } j \geq 0. \tag{3.8}$$

Let $C^z_{0,0}$ be the Dirac measure concentrated on the linear function zt. For each $(j,k) \in \mathbb{Z}^2_+ \setminus \{(0,0)\}$, define a measure $C^z_{j,k}$ as

$$C^z_{j,k}(\,\cdot\,) = \mathbf{E}\left[\nu^{j+k}_\alpha \left\{(x,y) \in (0,\infty)^{j+k} \colon z \cdot id + \sum_{i=1}^{j} x_i \mathbb{1}_{U_i} - \sum_{i=1}^{k} y_i \mathbb{1}_{V_i} \in \,\cdot\right\}\right], \tag{3.9}$$

where U_i, V_i are i.i.d. uniform distributed on $[0,1]$. For $E \subseteq \mathbb{D}$ and $z \in \mathbb{R}$, define

$$\mathcal{J}_z(E) = \inf\{j \colon E \cap \mathbb{D}^z_{\ll j+1} \neq \emptyset\}. \tag{3.10}$$

Recalling $\mu = \mathbf{E}B_1/(1-\mathbf{E}A_1)$, we now state our main theorem for the two-sided case.

Theorem 3.3. *Suppose that Assumptions 1 and 2 hold. Let $\mathbf{E}|B_1|^m < \infty$ for every $m \in \mathbb{Z}_+$. Moreover, let C_+, C_- be as in Theorem 3.1 such that $C_+C_- > 0$.*

1. *For each $j \geq 0$,*
$$n^{j(\alpha-1)}\mathbf{P}(\bar{X}_n \in \,\cdot\,) \to (\mathbf{E}r_1)^j \sum\nolimits_{(l,m)\in I_{=j}} (C_+)^l (C_-)^m C^\mu_{l,m}(\,\cdot\,),$$
in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}^\mu_{\ll j})$ as $n \to \infty$, where $I_{=j} = \{(l,m) \in \mathbb{Z}^2_+ \colon l+m = j\}$.

2. *Let E be measurable. If $\mathcal{J}_\mu(E) < \infty$ and E is bounded away from $\mathbb{D}_{\ll \mathcal{J}_\mu(E)}$, then*
$$\liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in E)}{n^{-\mathcal{J}_\mu(E)(\alpha-1)}} \geq (\mathbf{E}r_1)^{\mathcal{J}_\mu(E)} \sum_{(l,m)\in I_{=\mathcal{J}_\mu(E)}} (C_+)^l (C_-)^m C^\mu_{l,m}(E^\circ)$$
$$\text{and} \quad \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in E)}{n^{-\mathcal{J}_\mu(E)(\alpha-1)}} \leq (\mathbf{E}r_1)^{\mathcal{J}_\mu(E)} \sum_{(l,m)\in I_{=\mathcal{J}_\mu(E)}} (C_+)^l (C_-)^m C^\mu_{l,m}(E^-),$$
where the summations are over all (l,m) that belong to the set $I_{=\mathcal{J}_\mu(E)}$.

4 An application in barrier option pricing

To illustrate how our results can be applied, we consider a problem that arises in the context of financial mathematics. In particular, we consider estimating the value of a down-in barrier option (see Section 11.3 of [34]).

Let the daily log return of some underlying asset be modelled by an AR(1) process X_n, $n \geq 0$, as in (1.1). Let Assumptions 1 and 2 hold. Let $\mathbf{E}|B_1|^m < \infty$ for every $m \in \mathbb{Z}_+$. For real numbers a_- and a_+, we are interested in providing estimates for $\mathbf{P}(E_n)$ as $n \to \infty$, where

$$E_n = \left\{\bar{X}_n \geq a_+, \min_{0\leq k\leq n} \bar{X}_k \leq -a_-\right\},$$

$a_+ > \max\{\mu, 0\}$, and $a_- > \max\{-\mu, 0\}$. This choice of (a_-, a_+) leads to the case where the rare event is caused by two big jumps, and hence, is particularly interesting. Note that the probability of E_n can be interpreted as the chance of exercising a down-in barrier option. Defining

$$E = \left\{\xi \in \mathbb{D} \colon \xi(1) \geq a_+, \inf_{t\in[0,1]} \xi(t) \leq -a_-\right\},$$

we obtain $\mathbf{P}(E_n) = \mathbf{P}(\bar{X}_n \in E)$. Obviously, we have $\mathcal{J}_\mu(E) = 2$, where $\mathcal{J}_\mu$ was defined in (3.10). Hence, to apply Theorem 3.3, we need to show $d_{M_1'}(E, \mathbb{D}^\mu_{\ll 2}) \geq r$ for some $r > 0$. To see this, we assume that

$d_{M_1'}(E, \underline{\mathbb{D}}^{\mu}_{\ll 2}) < r$ for all $r > 0$. Therefore, for any $\epsilon > 0$, there exists $\xi_1 \in E$ and ξ_2 with $\xi_2(t) = \mu t + x\mathbb{1}_{[y,1]}(t)$, $x \in \mathbb{R}$, and $y \in [0, 1]$ such that $d_{M_1'}(\xi_1, \xi_2) < r + \epsilon$. By the definition of the M_1' metric, for any $\delta_1 > 0$, there exists $(u_i, v_i) \in \Pi'(\xi_i)$, $i \in \{1, 2\}$, such that

$$\|u_1 - u_2\|_\infty \vee \|v_1 - v_2\|_\infty < d_{M_1'}(\xi_1, \xi_2) + \delta_1 < r + \epsilon + \delta_1. \tag{4.1}$$

By (4.1), we have that

$$|a_+ - (\mu + x)| = |u_1(1) - u_2(1)| \le \|u_1 - u_2\|_\infty \vee \|v_1 - v_2\|_\infty < r + \epsilon + \delta_1.$$

Letting ϵ, $\delta_1 \to 0$, we obtain that $x \ge (a_+ - \mu) - r > 0$ for sufficiently small r. On th other hand, by the fact $\inf_{t\in[0,1]} \xi_1(t) \le -a_-$, for any $\delta_2 > 0$, there exsits $t' \in [0, 1]$ such that $\xi_1(t') < -a_- + \delta_2$. Let s be such that $v_1(s) = t'$. Let $t'' = v_2(s)$. Again using (4.1), we obtain that

$$|\xi_1(t') - (\mu t'' + x\mathbb{1}_{[y,1]}(t''))| = |u_1(s) - u_2(s)| \le \|u_1 - u_2\|_\infty \vee \|v_1 - v_2\|_\infty < r + \epsilon + \delta_1,$$

and hence,

$$\mu t'' + x\mathbb{1}_{[y,1]}(t'') < \xi_1(t') + (r + \epsilon + \delta_1) < -a_- + r + \epsilon + \delta_1 + \delta_2. \tag{4.2}$$

Combining (4.2) with the fact that $x > 0$, we obtain that

$$\mu\mathbb{1}_{(-\infty,0)}(\mu) \le \mu t'' \le \mu t'' + x\mathbb{1}_{[y,1]}(t'') < -a_- + r + \epsilon + \delta_1 + \delta_2. \tag{4.3}$$

Letting ϵ, δ_1, $\delta_2 \to 0$, we see that (4.3) is contradictory to $a_- > \max\{-\mu, 0\} \ge 0$. Thus, we proved $d_{M_1'}(E, \underline{\mathbb{D}}^{\mu}_{\ll 2}) \ge r$ for some $r > 0$, and hence, we are in the framework of Theorem 3.3.

Next we determine the preconstant in the asymptotics. Define m, $\pi_1 : \mathbb{D} \to \mathbb{R}$ by $m(\xi) = \inf_{t\in[0,1]} \xi(t)$, and $\pi_1(\xi) = \xi(1)$. Note that π_1 and m (cf. [36, Lemma 13.4.1]) are continuous. Thus, $E = m^{-1}(-\infty, -a_-] \cap \pi_1^{-1}[a_+, \infty)$ is a closed set. Recall, for $z \in \mathbb{R}$, that $C^z_{j,k}$ was defined in (3.9). Since $C^{\mu}_{2,0}(E) = C^{\mu}_{0,2}(E) = 0$, it remains to consider $C^{\mu}_{1,1}(E^\circ)$ and $C^{\mu}_{1,1}(E)$. Combining the fact that $m^{-1}(-\infty, -a_-) \cap \pi_1^{-1}(a_+, \infty) \subseteq E$ with the discussion after [33, Theorem 3.2], we conclude that E is a $C^{\mu}_{1,1}$-continuous set. Therefore, applying Theorem 3.3 we obtain

$$\mathbf{P}(E_n) \sim C^{\mu}_{1,1}(E) C_+ C_- n^{-2(\alpha-1)}$$

as $n \to \infty$. In particular, the probability of interest is regularly varying of index $2 - 2\alpha$.

5 Proofs of Section 2

Proof of Proposition 2.3. Part 1) and 2) [if $x_0 \ne 0$] are in [7, page 22]. Hence, we focus on showing part 2) for the case $x_0 = 0$.

Note that for any Borel set E, (2.2) implies that

$$P(x, E) = \mathbf{E}[\mathbb{1}_{\{A_1 x + B_1 \in E\}}] \ge c_0 \int_{a_0-\epsilon}^{a_0+\epsilon} \int_{I_2} \mathbb{1}_{\{ax+b\in E\}}\, db\, \nu_0(da) = c_0 \int_{a_0-\epsilon}^{a_0+\epsilon} \int \mathbb{1}_{\{z-ax\in I_2\}} \mathbb{1}_{\{z\in E\}}\, dz\, \nu_0(da).$$

Let

$$E_0 = \begin{cases} \big(b_1 + \epsilon(a_0 + \epsilon),\ b_2 - \epsilon(a_0 + \epsilon)\big) & \text{if } x_0 = 0 \\ \big(b_1 + (x_0 + \epsilon)(a_0 + \epsilon),\ b_2 + (x_0 - \epsilon)(a_0 - \epsilon)\big) & \text{if } x_0 > 0 \\ \big(b_1 + (x_0 + \epsilon)(a_0 - \epsilon),\ b_2 + (x_0 - \epsilon)(a_0 + \epsilon)\big) & \text{if } x_0 < 0 \end{cases},$$

and pick an $\epsilon > 0$ sufficiently small so that E_0 is non-empty and $\epsilon < |x_0| \wedge a_0$. Note that if $x \in \mathscr{B}_\epsilon(x_0)$, $z \in E_0$, and $a \in (a_0 - \epsilon, a_0 + \epsilon)$, then $z \in E_0$ implies $z - ax \in I_2$; that is, $\mathbb{1}_{\{z\in E_0\}} \le \mathbb{1}_{\{z-ax\in I_2\}}$. Therefore, we have that

$$P(x, E) \ge c_0 \int_{a_0-\epsilon}^{a_0+\epsilon} \int \mathbb{1}_{\{z\in E_0\}} \mathbb{1}_{\{z\in E\}}\, dz\, \nu_0(da) \ge c_0 \nu_0((a_0 - \epsilon, a_0 + \epsilon))|E \cap E_0|, \qquad x \in \mathscr{B}_\epsilon(x_0).$$

The constant $\theta = c_0 \nu_0((a_0 - \epsilon, a_0 + \epsilon))$ is strictly positive since a_0 belongs to the support of ν_0. □

Proof of Lemma 2.4. By Theorem 15.2.6 of [27] and Result 2.2, any bounded set is h-geometrically regular with $h(x) = |x|^\delta + 1$, $\delta \in (0, \alpha)$. Thus, from the definition of h-geometrical regularity (cf. page 373 of [27]), there exists $t > 1$ such that $\sup_{x\in E_1} \mathbf{E}[\sum_{k=0}^{\tau_d - 1} h(X_k)t^k \,|\, X_0 = x] < \infty$ and $\sup_{x\in \mathcal{C}_0} \mathbf{E}[\sum_{k=0}^{\tau_d - 1} h(X_k)t^k \,|\, X_0 = x] < \infty$. Since $h \geq 1$,

$$\chi_0(t) \triangleq \sup_{x\in E_1} \mathbf{E}[t^{\tau_d} \,|\, X_0 = x] < \sup_{x\in E_1} \mathbf{E}\left[\sum\nolimits_{k=0}^{\tau_d-1} h(X_k)t^k \,\middle|\, X_0 = x\right] < \infty. \tag{5.1}$$

Likewise, $\chi_1(t) \triangleq \sup_{x\in\mathcal{C}_0} \mathbf{E}[t^{\tau_d} \,|\, X_0 = x] < \infty$. Note that for any $s \in (1, t)$, by Jensen's inequality, we get

$$\chi_1(s) = \sup\nolimits_{x\in\mathcal{C}_0} \mathbf{E}[s^{\tau_d} \,|\, X_0 = x] = \sup\nolimits_{x\in\mathcal{C}_0} \mathbf{E}[t^{\frac{\log s}{\log t}\tau_d} \,|\, X_0 = x] \leq \sup\nolimits_{x\in\mathcal{C}_0} \mathbf{E}[t^{\tau_d} \,|\, X_0 = x]^{\frac{\log s}{\log t}} = \chi_1(t)^{\frac{\log s}{\log t}} \to 1$$

as $s \to 1$. Now let $t > 1$ be sufficiently close to 1 so that $(1-\theta)\chi_1(t) < 1$. From the regeneration scheme as described in Remark 2.1, we obtain

$$\sup_{x\in E_1} \mathbf{E}[t^{r_1} \,|\, X_0 = x] \leq \chi_0(t) \left(\theta + \sum_{n=1}^{\infty} \theta(1-\theta)^n (\chi_1(t))^n\right) < \infty. \tag{5.2}$$

□

Proof of Lemma 2.7. By Assumption 1, the set $[M, \infty)$ is attainable by $\{|X_n|\}_{n\geq 0}$ for sufficiently large M. Hence, by Theorem 8.3.6 of [27], Lemma 2.7 is proved once we show

$$\mathbf{P}^\alpha(|X_n| \geq M, \text{ for all } n \geq 1 \,|\, |X_0| \geq 2M) > 0.$$

Note that

$$|X_n| = e^{S_n}\left|X_0 + \sum_{i=1}^n B_i e^{-S_i}\right| \geq e^{S_n}\left(|X_0| - \sum_{i=1}^n |B_i| e^{-S_i}\right) \geq e^{S_n}\left(|X_0| - \sum_{i=1}^\infty |B_i| e^{-S_i}\right).$$

Combining this with the fact that $\mathbf{E}^\alpha \log A_1 > 0$, we conclude that $\mathbf{P}(\exp(S_n) \geq 1, \text{ for all } n \geq 1) = \mathbf{P}(S_n \geq 0, \text{ for all } n \geq 1) > 0$, and hence, the first statement is proved. The second statement follows from the fact that the random walk $-S_n$ has a negative drift under $\mathbf{P}^\alpha$. □

Proof of Lemma 2.11. Let G be an open set bounded away from $\mathbb{C}$ so that $G \subseteq (\mathbb{S}\setminus\mathbb{C})^{-\gamma}$ for some $\gamma > 0$. For a given $\delta > 0$, due to the assumed asymptotic equivalence, $\mathbf{P}(X_n \in (S\setminus\mathbb{C})^{-\gamma}, d(X_n, Y_n) \geq \delta) = o(\epsilon_n)$. Therefore,

$$\begin{aligned}
\liminf_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(Y_n \in G) &\geq \liminf_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(X_n \in G^{-\delta}, d(X_n, Y_n) < \delta) \\
&= \liminf_{n\to\infty} \epsilon_n^{-1}\{\mathbf{P}(X_n \in G^{-\delta}) - \mathbf{P}(X_n \in G^{-\delta}, d(X_n, Y_n) \geq \delta)\} \\
&\geq \liminf_{n\to\infty} \epsilon_n^{-1}\{\mathbf{P}(X_n \in G^{-\delta}) - \mathbf{P}(X_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma}, d(X_n, Y_n) \geq \delta)\} \\
&= \liminf_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(X_n \in G^{-\delta}) \geq \nu(G^{-\delta}).
\end{aligned}$$

Since G is an open set, $G = \bigcup_{\delta>0} G^{-\delta}$. Due to the continuity of measures, $\lim_{\delta\to 0}\nu(G^{-\delta}) = \nu(G)$, and hence, we arrive at the lower bound

$$\liminf_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(Y_n \in G) \geq \nu(G)$$

by taking $\delta \to 0$. Now, turning to the upper bound, consider a closed set F bounded away from $\mathbb{C}$ so that $F \subseteq (\mathbb{S}\setminus\mathbb{C})^{-\gamma}$ for some $\gamma > 0$. Given a $\delta > 0$, by the equivalence assumption, $\mathbf{P}(Y_n \in F, d(X_n, Y_n) \geq \delta) \leq$

$\mathbf{P}(Y_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma}, d(X_n,Y_n)\geq\delta) = o(\epsilon_n)$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\epsilon_n^{-1}\mathbf{P}(Y_n\in F) &= \limsup_{n\to\infty}\epsilon_n^{-1}\{\mathbf{P}(Y_n\in F, d(X_n,Y_n)<\delta)+\mathbf{P}(Y_n\in F, d(X_n,Y_n)\geq\delta)\}\\ &\leq \limsup_{n\to\infty}\epsilon_n^{-1}\{\mathbf{P}(X_n\in F_\delta)+\mathbf{P}(Y_n\in F, d(X_n,Y_n)\geq\delta)\}\\ &= \limsup_{n\to\infty}\epsilon_n^{-1}\mathbf{P}(X_n\in F_\delta)\leq\nu(F_\delta)\end{aligned}$$

as long as δ is small enough so that F_δ is bounded away from $\mathbb{C}$. Note that $\{F_\delta\}$ is a decreasing sequence of sets, $F=\bigcap_{\delta>0}F_\delta$ (since F is closed), and $\nu\in\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ (and hence ν is a finite measure on $\mathbb{S}\setminus\mathbb{C}^r$ for some $r>0$ such that $F_\delta\subseteq\mathbb{S}\setminus\mathbb{C}^r$ for some $\delta>0$). Due to the continuity (from above) of finite measures, $\lim_{\delta\to 0}\nu(F_\delta)=\nu(F)$. Therefore, we arrive at the upper bound $\limsup_{n\to\infty}\epsilon_n^{-1}\mathbf{P}(Y_n\in F)\leq\nu(F)$ by taking $\delta\to 0$. □

6 Proofs of Section 3.1

This section provides the proof of Theorem 3.1. Before turning to technical details, we briefly describe our strategy for proving the tail asymptotics of $\mathfrak{B}$. A similar idea is behind the proof for $\mathfrak{R}$. Let

$$T(u)=\inf\{n\geq 0\colon |X_n|>u\}\qquad\text{and}\qquad K_\beta^\gamma(u)=\inf\{n>T(u^\beta)\colon |X_n|\leq u^\gamma\}\tag{6.1}$$

where $0<\gamma<\beta<1$. We can then write

$$\mathfrak{B}=\sum_{n=0}^{T(u^\beta)-1}X_n+\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1}X_n+\sum_{n=K_\beta^\gamma(u)}^{\tau_d-1}X_n.\tag{6.2}$$

We will choose β close enough to 1 and γ far enough from 1 so that $\beta+\gamma>1$ and we can find $\rho\in(\gamma,\beta)$ such that $\beta-\gamma+\rho>1$. The proof of Theorem 3.1 (1) is based on the following fact.

- On the event $\{T(u^\beta)<\tau_d\}$, the first and the last term on the right hand side (r.h.s.) of (6.2) are negligible in contributing to the tail asymptotics. Proposition 6.1 below proves such properties. Lemma 6.5 is useful in showing Proposition 6.1.

- In view of the previous bullet, the second term on the r.h.s. of (6.2) plays the key role in $\mathbf{P}(\mathfrak{B}>u)$. Our analysis relies on the fact that the Markov chain X_n resembles a multiplicative random walk in the corresponding regime. Proposition 6.2 below proves such asymptotics. Lemmas 6.6, 6.7 are helpful for proving Proposition 6.2.

Similarly, the proof of Theorem 3.1 (2) hinges on Propositions 6.3 and 6.4, which play the similar roles as Proposition 6.1 and 6.2, respectively.

Proposition 6.1. *Suppose that Assumptions 1 and 2 hold. There exist a $\beta<1$ and $0<\gamma<\beta$ such that*

$$\mathbf{P}\left(\left|\sum_{n=0}^{T(u^\beta)-1}X_n\right|>u, T(u^\beta)<\tau_d\right)\quad\textit{and}\quad\mathbf{P}\left(\left|\sum_{n=K_\beta^\gamma(u)}^{\tau_d-1}X_n\right|>u, T(u^\beta)<\tau_d\right)$$

are of order $o(u^{-\alpha})$ as $u\to\infty$.

Proposition 6.2. *Suppose that Assumptions 1 and 2 hold. There exist* $0 < \gamma < \beta < 1$ *(identical to those in Proposition 6.1) such that*

$$\lim_{u\to\infty} u^\alpha \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > u, T(u^\beta) < \tau_d\right) = C_\infty \mathbf{E}^\alpha[(Z^+)^\alpha \mathbb{1}_{\{\tau_d=\infty\}}]$$

$$\text{and} \quad \lim_{u\to\infty} u^\alpha \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n < -u, T(u^\beta) < \tau_d\right) = C_\infty \mathbf{E}^\alpha[(Z^-)^\alpha \mathbb{1}_{\{\tau_d=\infty\}}].$$

Proof of Theorem 3.1 (1). Recalling $T(u^\beta) = \inf\{n \geq 0 \colon |X_n| > u^\beta\}$ for $\beta \in (0,1)$, write

$$\mathbf{P}(\pm\mathfrak{B} > u) = \mathbf{P}(\pm\mathfrak{B} > u, T(u^\beta) < \tau_d) + \mathbf{P}(\pm\mathfrak{B} > u, T(u^\beta) \geq \tau_d). \tag{6.3}$$

Since $\mathbf{P}(\tau_d > n)$ decays geometrically in n since $|X_0| \leq d$, and $|X_n| \leq u^\beta$ for $n \leq \tau_d - 1$ on $T(u^\beta) \geq \tau_d$, we have that

$$\begin{aligned}\mathbf{P}(\pm\mathfrak{B} > u, T(u^\beta) \geq \tau_d) &\leq \mathbf{P}(|\mathfrak{B}| > u, T(u^\beta) \geq \tau_d) \leq \mathbf{P}\left(\sum_{n=0}^{\tau_d-1} |X_n| > u, T(u^\beta) \geq \tau_d\right) \\ &\leq \mathbf{P}(u^\beta \tau_d \geq u) = \mathbf{P}(\tau_d \geq u^{1-\beta}) = o(u^{-\alpha}). \end{aligned} \tag{6.4}$$

Using (6.3) and (6.4), we can focus on analyzing the first term on the r.h.s. of (6.3). For $0 < \gamma < \beta < 1$, recall $K_\beta^\gamma(u) = \inf\{n > T(u^\beta) \colon |X_n| \leq u^\gamma\}$. Using the decomposition in (6.2) and Proposition 6.1, we obtain that, for $\epsilon \in (0,1)$,

$$\begin{aligned}\mathbf{P}(\mathfrak{B} > u, T(u^\beta) < \tau_d) &\leq \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > (1-\epsilon)u, T(u^\beta) < \tau_d\right) \\ &\quad + \mathbf{P}\left(\left|\sum_{n=0}^{T(u^\beta)-1} X_n\right| > \frac{\epsilon u}{2}, T(u^\beta) < \tau_d\right) + \mathbf{P}\left(\left|\sum_{n=K_\beta^\gamma(u)}^{\tau_d-1} X_n\right| > \frac{\epsilon u}{2}, T(u^\beta) < \tau_d\right) \\ &= \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > (1-\epsilon)u, T(u^\beta) < \tau_d\right) + o(u^{-\alpha}), \end{aligned} \tag{6.5}$$

and

$$\begin{aligned}\mathbf{P}(\mathfrak{B} > u, T(u^\beta) < \tau_d) &\geq \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > (1+\epsilon)u, T(u^\beta) < \tau_d\right) \\ &\quad - \mathbf{P}\left(\left|\sum_{n=0}^{T(u^\beta)-1} X_n\right| > \frac{\epsilon u}{2}, T(u^\beta) < \tau_d\right) - \mathbf{P}\left(\left|\sum_{n=K_\beta^\gamma(u)}^{\tau_d-1} X_n\right| > \frac{\epsilon u}{2}, T(u^\beta) < \tau_d\right) \\ &= \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > (1+\epsilon)u, T(u^\beta) < \tau_d\right) + o(u^{-\alpha}). \end{aligned} \tag{6.6}$$

From (6.5), (6.6), and Proposition 6.2,

$$(1+\epsilon)^{-\alpha} C_\infty \mathbf{E}^\alpha[(Z^+)^\alpha \mathbb{1}_{\{\tau_d=\infty\}}] + o(1) \leq u^\alpha \mathbf{P}(\mathfrak{B} > u, T(u^\beta) < \tau_d) \leq (1-\epsilon)^{-\alpha} C_\infty \mathbf{E}^\alpha[(Z^+)^\alpha \mathbb{1}_{\{\tau_d=\infty\}}] + o(1).$$

Since ϵ is arbitrary,

$$u^\alpha \mathbf{P}(\mathfrak{B} > u, T(u^\beta) < \tau_d) = C_\infty \mathbf{E}^\alpha[(Z^+)^\alpha \mathbb{1}_{\{\tau_d = \infty\}}] + o(1).$$

Along with (6.3), (6.4), this proves the first limit of Theorem 3.2 (1):

$$u^\alpha \mathbf{P}(\mathfrak{B} > u) = C_\infty \mathbf{E}^\alpha[(Z^+)^\alpha \mathbb{1}_{\{\tau_d = \infty\}}] + o(1).$$

We can use similar estimates to "sandwich" the quantity $\mathbf{P}(\mathfrak{B} < -u)$ and establish the second limit of Theorem 3.2 (1). □

Now we move on to proving Theorem 3.1 (2). We first need the following propositions.

Proposition 6.3. *Let Assumptions 1 and 2 hold. There exist $0 < \gamma < \beta < 1$ such that*

$$\mathbf{P}\left(\left|\sum_{n=0}^{T(u^\beta)-1} X_n\right| > u, T(u^\beta) < r_1\right) \qquad \text{and} \qquad \mathbf{P}\left(\left|\sum_{n=K_\beta^\gamma(u)}^{r_1-1} X_n\right| > u, T(u^\beta) < r_1\right)$$

are of order $o(u^{-\alpha})$ as $u \to \infty$.

Proposition 6.4. *Let Assumptions 1 and 2 hold. There exist $0 < \gamma < \beta < 1$ such that*

$$\lim_{u\to\infty} u^\alpha \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > u, T(u^\beta) < r_1\right) = C_+$$

$$\text{and} \quad \lim_{u\to\infty} u^\alpha \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n < -u, T(u^\beta) < r_1\right) = C_-,$$

where $C_+ = C_\infty \mathbf{E}^\alpha[(Z^+)^\alpha \mathbb{1}_{\{r_1=\infty\}}]$ and $C_+ = C_\infty \mathbf{E}^\alpha[(Z^-)^\alpha \mathbb{1}_{\{r_1=\infty\}}]$.

Proof of Theorem 3.1 (2). Using similar arguments as in (6.3) and (6.4), we can focus on $\mathbf{P}(\pm\mathfrak{R} > u, T(u^\beta) < r_1)$. Combining the similar "sandwich" technique as in (6.5)–(6.6) with Proposition 6.3, it remains to analyze

$$u^\alpha \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > u, T(u^\beta) < r_1\right).$$

Using Proposition 6.4, we conclude the proof. □

Next we prove Proposition 6.1. For this, we need the following lemma. Let $\{Y_n\}_{n\geq 0}$ be the $\mathbb{R}_+$-valued Markov chain defined by $Y_{n+1} = A_{n+1}Y_n + |B_{n+1}|$, for $n \geq 0$, and $\tau = \inf\{n \geq 1 : Y_n \leq d\}$.

Lemma 6.5. *Suppose that Assumptions 1 and 2 hold. Let $L > 0$ be given, and let $\epsilon > 0$ be such that $\lfloor \alpha - \epsilon \rfloor \geq 1$. Then there exists a positive constant c such that, for sufficiently large x,*

$$\mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = x] \leq c x^{\lfloor \alpha - \epsilon \rfloor}.$$

In particular $\mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = x] = \mathcal{O}(x)$.

Proof of Proposition 6.1. To begin with, note that

$$\mathbf{P}\left(\left|\sum_{n=0}^{T(u^\beta)-1} X_n\right| > u, T(u^\beta) < \tau_d\right) \leq \mathbf{P}\left(\sum_{n=0}^{T(u^\beta)-1} |X_n| > u, T(u^\beta) < \tau_d\right)$$
$$\leq \mathbf{P}(u^\beta \tau_d > u) = \mathbf{P}(\tau_d > u^{1-\beta}),$$

which decays exponentially. It remains to show the second claim. Let ρ be a number such that $\rho \in (\gamma, \beta)$ and $\beta - \gamma + \rho > 1$, and define

$$\mathfrak{E}_1(u) = \{\exists\, n \text{ such that } K_\beta^\gamma(u) \le n \le \tau_d \text{ and } |X_n| \ge u^\rho\}.$$

Note that

$$\mathbf{P}\left(\left|\sum_{n=K_\beta^\gamma(u)}^{\tau_d-1} X_n\right| > u, T(u^\beta) < \tau_d\right) \le \mathbf{P}\left(\sum_{n=K_\beta^\gamma(u)}^{\tau_d-1} |X_n| > u, T(u^\beta) < \tau_d, \mathfrak{E}_1(u)\right) + \mathbf{P}\left(\sum_{n=K_\beta^\gamma(u)}^{\tau_d-1} |X_n| > u, T(u^\beta) < \tau_d, (\mathfrak{E}_1(u))^c\right),$$

where the second term in the last equation is bounded by $\mathbf{P}(\tau_d > u^{1-\rho})$, and hence is of order $o(u^{-\alpha})$. It remains to analyze the first term, which is bounded by $\mathbf{P}(T(u^\beta) < \tau_d, \mathfrak{E}_1(u))$. Our goal here is to show that

$$\mathbf{P}(T(u^\beta) < \tau_d, \mathfrak{E}_1(u)) = o(u^{-\alpha}), \quad \text{as } u \to \infty. \tag{6.7}$$

To begin with, note that, on $\mathfrak{E}_1(u)$, where $K_\beta^\gamma(u) < \infty$ almost surely, we can define $\{Y_n'\}_{n\ge 0}$ as follows

$$Y_0' = u^\gamma, \qquad Y_{n+1}' = A_{K_\beta^\gamma(u)+n+1} Y_n' + |B_{K_\beta^\gamma(u)+n+1}|, \qquad \text{for } n \ge 0.$$

so that $|X_{K_\beta^\gamma(u)+n}| \le Y_n'$, for all $n \ge 0$. Let $\tau' \triangleq \inf\{n \ge 1 \colon Y_n' \le d\}$. Since $\mathbb{1}_{\{T(u^\beta)<\tau_d\}} \in m\mathcal{F}_{T(u^\beta)}$, and Y_n' is well-defined on $\{T(u^\beta) < \tau_d\}$ (since $K_\beta^\gamma(u) < \infty$ $\mathbf{P}$-almost surely there), we have that

$$\begin{aligned}\mathbf{P}(T(u^\beta) < \tau_d, \mathfrak{E}_1(u)) &\le \mathbf{P}(T(u^\beta) < \tau_d, \exists n \le \tau' \text{ such that } Y_n' \ge u^\rho)\\ &= \mathbf{P}(T(u^\beta) < \tau_d)\mathbf{P}(\exists\, n \le \tau' \text{ such that } Y_n' \ge u^\rho \,\big|\, T(u^\beta) < \tau_d),\end{aligned} \tag{6.8}$$

where $\mathbf{P}(T(u^\beta) < \tau_d) = \mathcal{O}(u^{-\alpha\beta})$ (cf. Corollary 4.2 of [11]). Since we have chosen β, γ, and ρ in such a way that $\beta - \gamma + \rho > 1$, it remains to show that the second term on the r.h.s. is $\mathcal{O}(u^{-\alpha(\rho-\gamma)})$. Recall the definition of Y_n and τ, and note that from the strong Markov property,

$$\mathbf{P}(\exists\, n \le \tau' \text{ such that } Y_n' \ge u^\rho \,\big|\, T(u^\beta) < \tau_d) = \mathbf{P}(\exists\, n \le \tau \text{ such that } Y_n \ge u^\rho \,|\, Y_0 = u^\gamma)$$

as $u \to \infty$. Recall Remark 2.6 and consider $T = \inf\{n \ge 1 \colon Y_n \ge u^\rho\}$. We obtain

$$\begin{aligned}\mathbf{P}(\exists\, n \le \tau \text{ such that } Y_n \ge u^\rho \,|\, Y_0 = u^\gamma) &= \mathbf{P}(T < \tau \,|\, Y_0 = u^\gamma) = \mathbf{E}^\alpha\left[e^{-\alpha S_T}\mathbb{1}_{\{T<\tau\}} \,\middle|\, Y_0 = u^\gamma\right]\\ &= u^{-\alpha(\rho-\gamma)}\mathbf{E}^\alpha\left[\left(\frac{Y_T}{u^\rho}\right)^{-\alpha}\left(\frac{Y_T}{e^{S_T}u^\gamma}\right)^{\alpha}\mathbb{1}_{\{T<\tau\}}\,\middle|\, Y_0 = u^\gamma\right]\\ &\le u^{-\alpha(\rho-\gamma)}\mathbf{E}^\alpha\left[\left(\frac{Y_T}{e^{S_T}u^\gamma}\right)^{\alpha}\mathbb{1}_{\{T<\tau\}}\,\middle|\, Y_0 = u^\gamma\right].\end{aligned}$$

Now it is sufficient to show that

$$\limsup_{u\to\infty}\mathbf{E}^\alpha\left[\left(\frac{Y_T}{e^{S_T}u^\gamma}\right)^{\alpha}\mathbb{1}_{\{T<\tau\}}\,\middle|\, Y_0 = u^\gamma\right] < \infty, \tag{6.9}$$

to prove that the r.h.s. of (6.8) is $\mathcal{O}(u^{-\alpha(\rho-\gamma)})$, and hence, $\mathbf{P}(T(u^\beta) < \tau_d, \mathfrak{E}_1(u)) = o(u^{-\alpha})$. To show (6.9), note that

$$\frac{Y_T}{e^{S_T}u^\gamma} = e^{-S_T}u^{-\gamma}\left(e^{S_T}u^\gamma + e^{S_T}\sum_{k=1}^{T}|B_k|e^{-S_k}\right) = 1 + u^{-\gamma}\sum_{k=1}^{T}|B_k|e^{-S_k}.$$

Thus, we have that

$$\frac{Y_T}{e^{S_T}u^\gamma}\mathbb{1}_{\{T<\tau\}} \le 1 + u^{-\gamma}\sum_{k=1}^{T}|B_k|e^{-S_k}\mathbb{1}_{\{T<\tau\}} \le 1 + u^{-\gamma}\sum_{k=1}^{\infty}|B_k|e^{-S_k}\mathbb{1}_{\{k<\tau\}},$$

and hence,

$$\begin{aligned}
\mathbf{E}^\alpha\left[\left(\frac{Y_T}{e^{S_T}u^\gamma}\mathbb{1}_{\{T<\tau\}}\right)^\alpha \,\middle|\, Y_0 = u^\gamma\right]^{1/\alpha} &\le \mathbf{E}^\alpha\left[\left(1 + u^{-\gamma}\sum_{k=1}^{\infty}|B_k|e^{-S_k}\mathbb{1}_{\{k<\tau\}}\right)^\alpha \,\middle|\, Y_0 = u^\gamma\right]^{1/\alpha} \\
&\le 1 + \sum_{k=1}^{\infty}\mathbf{E}^\alpha\left[u^{-\alpha\gamma}|B_k|^\alpha e^{-\alpha S_k}\mathbb{1}_{\{k<\tau\}} \,\middle|\, Y_0 = u^\gamma\right]^{1/\alpha} \qquad (6.10)\\
&= 1 + u^{-\gamma}\sum_{k=1}^{\infty}\mathbf{E}^\alpha\left[e^{-\alpha S_k}|B_k|^\alpha\mathbb{1}_{\{k<\tau\}} \,\middle|\, Y_0 = u^\gamma\right]^{1/\alpha} \\
&= 1 + u^{-\gamma}\sum_{k=1}^{\infty}\mathbf{E}\left[|B_k|^\alpha\mathbb{1}_{\{k<\tau\}} \,\middle|\, Y_0 = u^\gamma\right]^{1/\alpha} \qquad (6.11)\\
&\le 1 + u^{-\gamma}\sum_{k=1}^{\infty}(\mathbf{E}|B_k|^\alpha)^{1/\alpha}\mathbf{P}(\tau \ge k \,|\, Y_0 = u^\gamma)^{1/\alpha} \qquad (6.12)\\
&\le 1 + u^{-\gamma}(\mathbf{E}|B_1|^\alpha)^{1/\alpha}\left(\mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = u^\gamma]\right)^{1/\alpha}\sum_{k=1}^{\infty}k^{-(\alpha+L)/\alpha}, \qquad (6.13)
\end{aligned}$$

for some $L > 0$, where (6.10) is from Fatou's lemma and Minkowski's inequality, (6.11) is from Remark 2.6 with $T = k$ and $R = |B_k|^\alpha\mathbb{1}_{\{k<\tau\}}$, (6.12) is from the fact that $\mathbb{1}_{\{k<\tau\}} \le \mathbb{1}_{\{k\le\tau\}}$ and $\mathbb{1}_{\{k\le\tau\}} \in m\mathcal{F}_{k-1}$ so that $\mathbb{1}_{\{k\le\tau\}}$ and $|B_k|^\alpha$ are independent, and (6.13) is from Markov's inequality. Using Lemma 6.5 above, we prove (6.9), which, in turn, proves (6.7). This concludes the proof of Proposition 6.1. □

Set

$$\mathscr{G}_+(u) = u^{(1-\beta)\alpha}\mathbf{P}^{\mathscr{D}^\alpha_{T(u^\beta)}}\left(\sum_{n=T(u^\beta)}^{K^\gamma_\beta(u)-1}X_n > u \,\middle|\, \mathcal{F}_{T(u^\beta)}\right)\left(\frac{X_{T(u^\beta)}}{u^\beta}\right)^{-\alpha}\mathbb{1}_{\{Z_{T(u^\beta)}>0\}}, \qquad (6.14)$$

and

$$\mathscr{G}_-(u) = u^{(1-\beta)\alpha}\mathbf{P}^{\mathscr{D}^\alpha_{T(u^\beta)}}\left(\sum_{n=T(u^\beta)}^{K^\gamma_\beta(u)-1}X_n > u \,\middle|\, \mathcal{F}_{T(u^\beta)}\right)\left|\frac{X_{T(u^\beta)}}{u^\beta}\right|^{-\alpha}\mathbb{1}_{\{Z_{T(u^\beta)}\le 0\}}. \qquad (6.15)$$

Recall C_∞ in (3.3). The following two lemmas are useful in proving Proposition 6.2.

Lemma 6.6. *Suppose that Assumptions 1 and 2 hold. Under the measure* $\mathbf{P}^\alpha$*,*

$$\mathscr{G}_+(u) \xrightarrow{a.s.} C_\infty\mathbb{1}_{\{Z>0\}} \quad \text{and} \quad \mathscr{G}_-(u) \xrightarrow{a.s.} 0, \quad \text{as } u \to \infty.$$

Moreover, $\mathscr{G}_+(u)$ *and* $\mathscr{G}_-(u)$ *are bounded in* u *by some constants almost surely.*

Recall that Z_n, τ_d, and $T(u)$ are defined in (2.6), (3.1), and (6.1), respectively.

Lemma 6.7. *Suppose that Assumptions 1 and 2 hold. The random variables* $Z^+_{T(u^\beta)}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}$ *and* $Z^-_{T(u^\beta)}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}$ *are bounded by*

$$\bar{Z} = |X_0| + \sum_{n=1}^{\infty}|B_n|e^{-S_n}\mathbb{1}_{\{n<\tau_d\}}.$$

In addition, $\mathbf{E}^\alpha[\bar{Z}^\alpha] < \infty$.

Proof of Proposition 6.2. We focus on deriving the first asymptotics, since the second one follows using similar arguments. Note that

$$\begin{aligned} u^\alpha \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > u, T(u^\beta) < \tau_d\right) &= u^\alpha \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > u, X_{T(u^\beta)} > 0, T(u^\beta) < \tau_d\right) \\ &\quad + u^\alpha \mathbf{P}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > u, X_{T(u^\beta)} < 0, T(u^\beta) < \tau_d\right) \\ &= \textbf{(I.1)} + \textbf{(I.2)}. \end{aligned} \tag{6.16}$$

We first consider **(I.1)**. Applying the dual change of measure $\mathscr{D}^\alpha_{T(u^\beta)}$ together with Result 2.5, we obtain that

$$\textbf{(I.1)} = \mathbf{E}^{\mathscr{D}^\alpha_{T(u^\beta)}}[g_{\tau_d-1}(X_0,\ldots,X_{\tau_d-1})e^{-\alpha S_{T(u^\beta)}}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}],$$

where

$$g_{\tau_d-1}(X_0,X_1,\ldots,X_{\tau_d-1}) = \begin{cases} 1 & \text{if } \sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_n > u \text{ and } X_{T(u^\beta)} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Recall the expression for Z_n given in (2.6). Note that

$$\begin{aligned} \textbf{(I.1)} &= u^\alpha \mathbf{E}^{\mathscr{D}^\alpha_{T(u^\beta)}}\left[g_{\tau_d-1}(X_0,\ldots,X_{\tau_d-1})e^{-\alpha S_{T(u^\beta)}}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}\right] \\ &= u^\alpha \mathbf{E}^{\mathscr{D}^\alpha_{T(u^\beta)}}\left[g_{\tau_d-1}(X_0,\ldots,X_{\tau_d-1})\cdot |X_{T(u^\beta)}|^{-\alpha}\cdot |X_{T(u^\beta)}|^{\alpha}\cdot e^{-\alpha S_{T(u^\beta)}}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}\right] \\ &= \mathbf{E}^{\mathscr{D}^\alpha_{T(u^\beta)}}\left[\mathscr{G}_+(u)(Z^+_{T(u^\beta)})^\alpha \mathbb{1}_{\{T(u^\beta)<\tau_d\}}\right], \end{aligned} \tag{6.17}$$

for all $n \geq 0$. Using Lemma 6.6, Lemma 6.7, the dominated convergence theorem and the fact that $T(u^\beta) \to \infty$ as $u \to \infty$, we obtain that

$$\begin{aligned} \lim_{u\to\infty} \textbf{(I.1)} &= \lim_{u\to\infty} \mathbf{E}^{\mathscr{D}^\alpha_{T(u^\beta)}}\left[(Z^+_{T(u^\beta)})^\alpha \mathbb{1}_{\{T(u^\beta)<\tau_d\}}\mathscr{G}_+(u)\right] = \lim_{u\to\infty} \mathbf{E}^\alpha\left[(Z^+_{T(u^\beta)})^\alpha \mathbb{1}_{\{T(u^\beta)<\tau_d\}}\mathscr{G}_+(u)\right] \\ &= \mathbf{E}^\alpha\left[\lim_{u\to\infty}(Z^+_{T(u^\beta)})^\alpha \mathbb{1}_{\{T(u^\beta)<\tau_d\}}\mathscr{G}_+(u)\right] = \mathbf{E}^\alpha\left[(Z^+)^\alpha \mathbb{1}_{\{\tau_d=\infty\}}C_\infty\right] \\ &= C_\infty \mathbf{E}^\alpha\left[(Z^+)^\alpha \mathbb{1}_{\{\tau_d=\infty\}}\right]. \end{aligned}$$

Analogously, we have that

$$\textbf{(I.2)} = \mathbf{E}^{\mathscr{D}^\alpha_{T(u^\beta)}}\left[(Z^-_{T(u^\beta)})^\alpha \mathbb{1}_{\{T(u^\beta)<\tau_d\}}\mathscr{G}_-(u)\right] \to 0, \quad \text{as } u \to \infty, \tag{6.18}$$

where $\mathscr{G}_-(u)$ was defined in (6.15). Using (6.16), (6.17), and (6.18), we prove the first asymptotics in Proposition 6.2. The second one can be shown analogously. ☐

We need the following lemmas to prove Proposition 6.3. Let $Y_{n+1} = A_{n+1}Y_n + |B_{n+1}|$ and let r be the first time that Y_n regenerates.

Lemma 6.8. *Suppose that Assumptions 1 and 2 hold. Let $\epsilon > 0$, and let $L > 0$ be such that $\lfloor \alpha - \epsilon \rfloor \geq 1$. Then there exists a positive constant c such that, for sufficiently large x,*

$$\mathbf{E}[r^{\alpha+L} \,|\, Y_0 = x] \leq c x^{\lfloor \alpha-\epsilon \rfloor}.$$

In particular, $\mathbf{E}[r^{\alpha+L} \,|\, Y_0 = x] = \mathcal{O}(x)$.

Lemma 6.9. *Suppose that Assumptions 1 and 2 hold. We have that*

$$\lim_{u\to\infty} u^\alpha \mathbf{P}(T(u) < r_1) = \mathbf{E}\left[e^{-\alpha\mathfrak{X}}\right]\mathbf{E}^\alpha\left[|Z|^\alpha \mathbb{1}_{\{r_1=\infty\}}\right],$$

where $\mathfrak{X}$ is the positive random variable such that $\log X_{T(u)} - \log u$ converges in distribution to $\mathfrak{X}$ as $u\to\infty$ under $\mathbf{P}^\alpha$.

Proof of Proposition 6.3. By replacing τ_d with r_1, the proposition can be shown using almost identical arguments as in the proof of Proposition 6.1. Nonetheless, we need to show that

- $\mathbf{P}(T(u^\beta) < r_1) \sim cu^{-\alpha\beta}$ for some constant c, and that
- $\mathbf{E}[r^{\alpha+\epsilon}|Y_0 = x] = \mathcal{O}(x)$, where $Y_{n+1} = A_{n+1}Y_n + |B_{n+1}|$ and $r-1$ is the first time that (Y_n, η_n) returns to the set $[-d, d]\times\{1\}$.

For this, we use Lemmas 6.8 and 6.9 above. □

Proof of Proposition 6.4. Using Lemma 6.6, Lemma 6.7, the dominated convergence theorem and the fact that $T(u^\beta)\to\infty$ as $u\to\infty$, one can prove the first asymptotics. The second one follows by a similar analysis. □

Next we provide proofs of all lemmas in this section. To show Lemma 6.5, we introduce a result on bounding functionals of passage times for Markov chains. Let $\{V_n\}_{n\ge0}$ be an $\{\mathcal{F}_n\}$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}_+$. Let $\{U_n\}_{n\ge0}$ be another $\{\mathcal{F}_n\}$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}_+$ such that U_n is integrable for all $n\ge0$. Let $\tau_b^V = \inf\{n\ge 0\colon V_n\le b\}$ be the first time V_n returning to the set $[0,b]$.

Result 6.10 (Theorem 2′ of [1]). *Suppose there exists a positive number d and functions g and h that are positive on (b,∞) and*

$$U_n \le h(V_n), \qquad \forall n\ge 0$$
$$\mathbf{E}[U_{n+1} - U_n \,|\, \mathcal{F}_n] \le -g(V_n) \qquad \text{on} \qquad \{\tau_b^V > n\}.$$

Suppose that f is a function on $[0,\infty]$ such that

- $f\in C^2$, $f>0$, $f'>0$, $\lim_{x\to\infty} f(x) = \infty$
- *f is convex for sufficiently large x*
- *$\log f'$ is concave for sufficiently large x*
- *f satisfies*
$$\liminf_{y\to\infty} \frac{g(y)}{f'\circ f^{-1}\circ h(y)} > 0.$$
- *there exisits a positive constant c_f such that*
$$\limsup_{y\to\infty}\frac{f(2y)}{f(y)} \le c_f.$$

Then there exists a positive constant c such that, for all $x\ge b$

$$\mathbf{E}[f(\tau_b^V)\,|\,V_0 = x] \le ch(x).$$

Proof of Lemma 6.5. We first apply Result 6.10 with $f(y) = y^{\alpha+L}$, $g(y) = c_2 y^{\underline{\alpha}}$, $h(y) = y^{\underline{\alpha}}$ where $\underline{\alpha} = \lfloor \alpha - \epsilon \rfloor$, and c_2 is a constant that we construct below, $U_n = h(Y_n)$, $V_n = Y_n$, and $\mathcal{F}_n = \sigma(Y_i : i \le n)$. From the binomial formula, we see that there exist positive constants c_1 that depends on the first $(\underline{\alpha}-1)$-st moments of A_1 and B_1 such that, on $\{Y_n \ge 1\}$,

$$\mathbf{E}[U_{n+1} - U_n \,|\, \mathcal{F}_n] \le (\mathbf{E}[A_1^{\underline{\alpha}}] - 1) Y_n^{\underline{\alpha}} + c_1 Y_n^{\underline{\alpha}-1}.$$

Using the fact that $0 < \underline{\alpha} < \alpha$ and the moment generating function of $\log A_1$ is strictly convex on $[0, \alpha]$, we have $\mathbf{E}[A_1^{\underline{\alpha}}] < 1$. Thus, there exists a sufficiently large constant d' and sufficiently small constant c_2 such that, on $\{Y_n > d'\}$,

$$\mathbf{E}[U_{n+1} - U_n \,|\, \mathcal{F}_n] \le (\mathbf{E}[A_1^{\underline{\alpha}}] - 1) Y_n^{\underline{\alpha}} + c_1 Y_n^{\underline{\alpha}-1} \le -c_2 Y_n^{\underline{\alpha}}.$$

As mentioned at the beginning of the proof, we set $g(y) = c_2 y^{\underline{\alpha}} = c_2 h(y)$ so that

$$\mathbf{E}[U_{n+1} - U_n \,|\, \mathcal{F}_n] \le -g(Y_n).$$

It is now straightforward to check that f, g, and h satisfy all the conditions in Result 6.10. If we set $\tilde{\tau} = \inf\{n \ge 1 \colon Y_n \le d'\}$, Result 6.10 implies that there exists a positive constant c_3 such that

$$\mathbf{E}[\tilde{\tau}^{\alpha+L} \,|\, Y_0 = x] \le c_3 x^{\lfloor \alpha - \epsilon \rfloor} \tag{6.19}$$

for all $x \ge d'$. We assume w.l.o.g. that $d' \ge d$. Note that Y_n satisfies the same set assumptions as X_n, and hence, Lemma 2.4 applies to Y_n as well, and τ is bounded by the regeneration time of Y_n. Therefore, we can choose a t so that

$$\sup_{y \in [0,d']} \mathbf{E}[t^{\tau} \,|\, Y_0 = y]^{1/(\alpha+L)} < \infty.$$

Using Minkowski's inequality we obtain that

$$\begin{aligned}
\mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} &= \mathbf{E}[(\tilde{\tau} + \tau - \tilde{\tau})^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} \\
&\le \mathbf{E}[\tilde{\tau}^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} + \mathbf{E}[(\tau - \tilde{\tau})^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} \\
&\le \mathbf{E}[\tilde{\tau}^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} + \sup_{y \in [0,d']} \mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = y]^{1/(\alpha+L)} \\
&\le \mathbf{E}[\tilde{\tau}^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} + \sup_{y \in [0,d']} \mathbf{E}[t^{\tau} \,|\, Y_0 = y]^{1/(\alpha+L)} + c_4, \\
&\le \mathbf{E}[\tilde{\tau}^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} + c_5,
\end{aligned}$$

for some $c_4, c_5 > 0$. Along with (6.19), this implies that there exists a $c > 0$ such that $\mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = x] \le c x^{\lfloor \alpha - \epsilon \rfloor}$ for sufficiently large x. ☐

The following lemma is useful in proving Lemma 6.6. Define

$$\mathfrak{E}_2(u) = \{|B_n| \le u^{\gamma}, \forall n \in \{1, \dots, K_{\beta}^{\gamma}(u)\}\}. \tag{6.20}$$

Lemma 6.11. *Suppose that Assumptions 1 and 2 hold. Fix an arbitrary constant v such that $|v| > 1$. For any $\beta + \gamma > 1$ and any $\epsilon > 0$ there exists a u_0 sufficiently large so that, for all $u \ge u_0$,*

$$\mathbf{P}((\mathfrak{E}_2(u))^c \,|\, X_0 = vu^{\beta}) \le \epsilon |v| u^{-(1-\beta)\alpha}.$$

Proof of Lemma 6.6. We first prove the statements associated with $\mathscr{G}_+(u)$. As $\mathbb{1}_{\{Z_{T(u^{\beta})} > 0\}} \xrightarrow{\text{a.s.}} \mathbb{1}_{\{Z > 0\}}$ under $\mathbf{P}^{\alpha}$, it is sufficient to show that

$$\lim_{u \to \infty} u^{(1-\beta)\alpha} \mathbf{P}\left(\sum_{k=T(u^{\beta})}^{K_{\beta}^{\gamma}(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^{\beta})}}{u^{\beta}} = v \right) = C_{\infty} v^{\alpha}, \qquad \text{for } v > 1.$$

Set $S_n^{(u)} \triangleq \sum_{i=1}^n \log(A_i + u^{-\gamma}\cdot|B_i|)$ and fix $v \geq 1$. Note that since

$$\frac{X_n}{X_{n-1}} \leq A_n + \frac{|B_n|}{|X_{n-1}|} < A_n + |B_n|u^{-\gamma} \qquad \text{on} \qquad T(u^\beta) < n < K_\beta^\gamma(u),$$

we have that, conditional on $X_0 = vu^\beta$, $|X_k| \leq vu^\beta e^{S_k^{(u)}}$ for all $k < K_\beta^\gamma(u)$. Therefore, from the Markov property,

$$\begin{aligned}
&\mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta} = v\right) = \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_0}{u^\beta} = v\right) \leq \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} |X_k| > u \,\middle|\, \frac{X_0}{u^\beta} = v\right) \\
&\leq \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} e^{S_k^{(u)}} > \frac{u^{1-\beta}}{v}\right) \leq \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k^{(u)}} > \frac{u^{1-\beta}}{v}\right) = \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} + \sum_{k=0}^{\infty} e^{S_k^{(u)}} - \sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v}\right) \\
&\leq \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v} - \delta\right) + \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k^{(u)}} - \sum_{k=0}^{\infty} e^{S_k} > \delta\right) \qquad (6.21)
\end{aligned}$$

for any $\delta > 0$. Note that from (3.3),

$$\mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v} - \delta\right) \sim C_\infty \left(\frac{u^{1-\beta}}{v}\right)^{-\alpha}. \qquad (6.22)$$

Moreover, using the Markov's inequality and the fact that $S_n^{(u)} \geq S_n$ we obtain that

$$\begin{aligned}
u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_n^{(u)}} - \sum_{k=0}^{\infty} e^{S_n} > \delta\right) &\leq \delta^{-1}u^{(1-\beta)\alpha}\mathbf{E}\left[\sum_{k=0}^{\infty} e^{S_n^{(u)}} - \sum_{k=0}^{\infty} e^{S_n}\right] \\
&= \delta^{-1}u^{(1-\beta)\alpha}\left(\sum_{k=0}^{\infty}\left(\mathbf{E}[A_1 + |B_1|u^{-\gamma}]\right)^k - \sum_{k=0}^{\infty}\left(\mathbf{E}[A_1]\right)^k\right) \\
&= \delta^{-1}u^{(1-\beta)\alpha}\left(\frac{1}{1 - \mathbf{E}A_1 - u^{-\gamma}\mathbf{E}|B_1|} - \frac{1}{1 - \mathbf{E}A_1}\right) \\
&= \delta^{-1}u^{(1-\beta)\alpha}\left(\frac{u^{-\gamma}\mathbf{E}|B_1|}{(1 - \mathbf{E}A_1 - u^{-\gamma}\mathbf{E}|B_1|)(1 - \mathbf{E}A_1)}\right) \\
&= \mathcal{O}(u^{(1-\beta)\alpha-\gamma}),
\end{aligned}$$

for sufficiently large u's. In the second equality, we used $\mathbf{E}A_1 < 1$, (which follows from $\alpha > 1$) and $\mathbf{E}(A_1 + |B_1|u^{-\gamma}) < 1$ for sufficiently large u's. By choosing β sufficiently close to 1 so that $(1-\beta)\alpha < \gamma$, we have that

$$u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k^{(u)}} - \sum_{k=0}^{\infty} e^{S_k} > \delta\right) = o(1). \qquad (6.23)$$

Using (6.21)–(6.23), an upper bound is given by, for $v > 1$

$$\limsup_{u\to\infty} u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta} = v\right) \leq C_\infty v^\alpha. \qquad (6.24)$$

Next, we show the corresponding lower bound. By the Markov property we obtain that

$$\mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta} = v\right) = \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_0}{u^\beta} = v\right). \qquad (6.25)$$

Set $\underline{S}_n^{(u)} = \sum_{i=1}^n \log(A_i - u^{-\gamma} \cdot |B_i|)^+$. Note that, conditional on $X_0 = vu^\beta$

$$\left|\frac{X_n}{X_{n-1}}\right| \geq \left(A_n - \frac{|B_n|}{|X_{n-1}|}\right)^+ > (A_n - u^{-\gamma}|B_n|)^+, \qquad \forall n \leq K_\beta^\gamma(u),$$

which, in turn, implies that

$$vu^\beta e^{\underline{S}_k^{(u)}} \leq |X_k|, \qquad \forall k \leq K_\beta^\gamma(u). \tag{6.26}$$

In particular,

$$vu^\beta e^{\underline{S}^{(u)}_{K_\beta^\gamma(u)}} \leq \left|X_{K_\beta^\gamma(u)}\right| \leq u^\gamma.$$

Therefore,

$$K_\beta^\gamma(u) \geq \inf\{n \geq 1 \colon \underline{S}_n^{(u)} \leq -\log v - (\beta - \gamma)\log u\} \triangleq K'(u) \tag{6.27}$$

conditional on $X_0 = vu^\beta$. Recall that $\mathfrak{E}_2(u) = \{|B_n| \leq u^\gamma, \forall n \in \{1, \dots, K_\beta^\gamma(u)\}\}$, and note that on $\mathfrak{E}_2(u)$ and $X_0 = vu^\beta$, $X_k \geq 0$ for all $k < K_\beta^\gamma(u)$. To see this, let $\kappa \triangleq \inf\{k \geq 0 : X_k < 0\}$ and observe that $\kappa < K_\beta^\gamma(u)$ implies that $A_{\kappa-1}X_{\kappa-1} \geq 0$ and $X_\kappa < -u^\gamma$, and hence, $B_\kappa < -u^\gamma$, leading to a contradiction. In view of this, (6.25), and (6.26), we have

$$\begin{aligned}
\mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta} = v\right) &= \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_0}{u^\beta} = v\right) \\
&\geq \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} |X_k| > u,\, \mathfrak{E}_2(u) \,\middle|\, \frac{X_0}{u^\beta} = v\right) \\
&\geq \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} e^{\underline{S}_k^{(u)}} > \frac{u^{1-\beta}}{v},\, \mathfrak{E}_2(u) \,\middle|\, \frac{X_0}{u^\beta} = v\right) \\
&\geq \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} e^{\underline{S}_k^{(u)}} > \frac{u^{1-\beta}}{v} \,\middle|\, \frac{X_0}{u^\beta} = v\right) - \mathbf{P}\left(\mathfrak{E}_2(u)^c \,\middle|\, \frac{X_0}{u^\beta} = v\right) \\
&= \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} e^{\underline{S}_k^{(u)}} > \frac{u^{1-\beta}}{v} \,\middle|\, \frac{X_0}{u^\beta} = v\right) + o(u^{\alpha(1-\beta)})|v|,
\end{aligned}$$

where the last equality is from Lemma 6.11. On the other hand, from (6.27),

$$\begin{aligned}
\mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} e^{\underline{S}_k^{(u)}} > \frac{u^{1-\beta}}{v} \,\middle|\, \frac{X_0}{u^\beta} = v\right) &\geq \mathbf{P}\left(\sum_{k=0}^{K'(u)-1} e^{\underline{S}_k^{(u)}} > \frac{u^{1-\beta}}{v}\right) \\
&\geq \mathbf{P}\left(\sum_{k=0}^{\infty} e^{\underline{S}_k^{(u)}} > \frac{u^{1-\beta}}{v} + \delta\right) - \mathbf{P}\left(\sum_{k=K'(u)}^{\infty} e^{\underline{S}_k^{(u)}} > \delta\right)
\end{aligned} \tag{6.28}$$

for any $\delta > 0$. Note that

$$\begin{aligned}
\mathbf{P}\left(\sum_{k=K'(u)}^{\infty} e^{\underline{S}_k^{(u)}} > \delta\right) &\leq \delta^{-1}\mathbf{E}\left[\sum_{k=K'(u)}^{\infty} e^{\underline{S}_k^{(u)}}\right] = \delta^{-1}\mathbf{E}\left[e^{\underline{S}^{(u)}_{K'(u)}} \sum_{k=0}^{\infty} e^{\underline{S}^{(u)}_{k+K'(u)} - \underline{S}^{(u)}_{K'(u)}}\right] \\
&\leq \frac{u^{\gamma-\beta}}{\delta v}\mathbf{E}\left[\sum_{k=0}^{\infty} e^{\underline{S}_k^{(u)}}\right] \leq \frac{u^{\gamma-\beta}}{\delta}\mathbf{E}\left[\sum_{k=0}^{\infty} e^{\underline{S}_k^{(u)}}\right] \leq \frac{u^{\gamma-\beta}}{\delta}\mathbf{E}\left[\sum_{k=0}^{\infty} e^{S_k}\right] = o(u^{\gamma-\beta}),
\end{aligned}$$

and hence,

$$u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=K'(u)}^{\infty} e^{\underline{S}_k^{(u)}} > \delta\right) = o(1), \qquad \text{for } \beta > (\alpha+\gamma)/(\alpha+1), \tag{6.29}$$

allowing us to choose a suitable value of β since $(\alpha+\gamma)/(\alpha+1) < 1$. Therefore, it remains to show that the first term in (6.28) is lower bounded by $C_\infty u^{-(1-\beta)\alpha} v^\alpha$. Note that

$$\begin{aligned}
&\mathbf{P}\left(\sum_{k=0}^{\infty} e^{\underline{S}_k^{(u)}} > \frac{u^{1-\beta}}{v} + \delta\right) \\
&\geq \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v} + 2\delta\right) - \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} - \sum_{k=0}^{\infty} e^{\underline{S}_k^{(u)}} > \delta\right) \\
&\geq \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v} + 2\delta\right) - \delta^{-1}\mathbf{E}\left[\sum_{k=0}^{\infty} e^{S_n} - \sum_{k=0}^{\infty} e^{\underline{S}_n^{(u)}}\right] \\
&= \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v} + 2\delta\right) - \delta^{-1}\left(\sum_{k=0}^{\infty}(\mathbf{E}A_1)^k - \sum_{k=0}^{\infty}(\mathbf{E}(A_1 - u^{-\gamma}|B_1^*|)^+)^k\right) \\
&= \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v} + 2\delta\right) - \delta^{-1}\left(\frac{1}{1-\mathbf{E}A_1} - \frac{1}{1-\mathbf{E}(A_1 - u^{-\gamma}|B_1|)^+}\right) \\
&\geq \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v} + 2\delta\right) - \delta^{-1}\frac{\mathbf{E}A_1 - \mathbf{E}(A_1 - u^{-\gamma}|B_1|)^+}{(1-\mathbf{E}A_1)(1-\mathbf{E}(A_1 - u^{-\gamma}|B_1|)^+)} \\
&\geq \mathbf{P}\left(\sum_{k=0}^{\infty} e^{S_k} > \frac{u^{1-\beta}}{v} + 2\delta\right) - \delta^{-1}\frac{u^{-\gamma}\mathbf{E}|B_1|}{(1-\mathbf{E}A_1)(1-\mathbf{E}(A_1 - u^{-\gamma}|B_1|)^+)}.
\end{aligned}$$

Again from (3.3) along with the assumption that $(1-\beta)\alpha < \gamma$, we get

$$\mathbf{P}\left(\sum_{k=0}^{\infty} e^{\underline{S}_k^{(u)}} > \frac{u^{1-\beta}}{v} + \delta\right) \geq C_\infty u^{-(1-\beta)\alpha} v^\alpha + o(u^{-(1-\beta)\alpha}). \tag{6.30}$$

In view of (6.28)–(6.30), we have that

$$\liminf_{u\to\infty} u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta} = v\right) \geq C_\infty v^\alpha.$$

Combining this with (6.24) we have that

$$\lim_{u\to\infty} u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta}\right)\left(\frac{X_{T(u^\beta)}}{u^\beta}\right)^{-\alpha} = C_\infty,$$

$\mathbf{P}^\alpha$-almost surely.

Next we show boundedness of $\mathscr{G}_+(u)$. Using (6.24), for $\epsilon > 0$, and by separately considering $v \leq 1+\epsilon$ and $v \geq 1+\epsilon$, there exists $U(\epsilon)$ (independent of v) such that

$$\left(\frac{u^{(1-\beta)}}{v}\right)^\alpha \mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta} = v\right) \leq C_\infty + \epsilon,$$

for all $u^{(1-\beta)} \ge vU(\epsilon)$. Moreover, for all $0 < u^{(1-\beta)} < vU(\epsilon)$,

$$u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta} = v\right) \le u^{(1-\beta)\alpha} \le v^\alpha U(\epsilon)^\alpha. \tag{6.31}$$

Thus

$$u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=T(u^\beta)}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_{T(u^\beta)}}{u^\beta} = v\right) \le \max\{C_\infty + \epsilon, U(\epsilon)^\alpha\} v^\alpha,$$

for all $u > 0$. This implies that $\mathscr{G}_+(u) \le \max\{C_\infty + \epsilon, U(\epsilon)^\alpha\} < \infty$.

Finally, we show the statements involved with $\mathscr{G}_-$. By the Markov property, it is sufficient to show that, for any arbitrary $\epsilon > 0$ and $v < -1$

$$\limsup_{u\to\infty} u^{(1-\beta)\alpha}\mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_0}{u^\beta} = v\right) \le \epsilon|v|^\alpha.$$

Recall $\mathfrak{E}_2(u) = \{|B_n| \le u^\gamma, \forall 1 \le n < K_\beta^\gamma(u)\}$ defined in (6.20). We have that

$$\begin{aligned}\mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} X_k > u \,\middle|\, \frac{X_0}{u^\beta} = v\right) &\le \mathbf{P}\left(\sum_{k=0}^{K_\beta^\gamma(u)-1} X_k > u,\, \mathfrak{E}_2(u) \,\middle|\, \frac{X_0}{u^\beta} = v\right) + \mathbf{P}\left(\mathfrak{E}_2(u)^c \,\middle|\, X_0 = vu^\beta\right) \\ &= \mathbf{P}\left(\mathfrak{E}_2(u)^c \,\middle|\, X_0 = vu^\beta\right) = o(u^{-(1-\beta)\alpha})|v|,\end{aligned}$$

thanks to Lemma 6.11. The boundedness of $\mathscr{G}_u^-$ follows using similar arguments as in (6.31). □

Remark 6.12. Using similar arguments as in the proof of Lemma 6.6, one can show that

$$\lim_{u\to\infty} u^{(1-\beta)\alpha}\mathbf{P}^{\mathscr{D}}\left(\sum_{n=T(u^\beta)}^{K_\beta^\gamma(u)-1} |X_n| > u \,\middle|\, \mathcal{F}_{T(u^\beta)}\right)\left|\frac{X_{T(u^\beta)}}{u^\beta}\right|^{-\alpha} = C_\infty.$$

As a consequence of this result, we have that

$$u^\alpha \mathbf{P}(\sum_{n=0}^{r_1-1} |X_n| > u) \to C_\infty \mathbf{E}^\alpha[|Z|^\alpha \mathbb{1}_{\{r_1=\infty\}}], \qquad u \to \infty.$$

Proof of Lemma 6.7. Note that $Z^+_{T(u^\beta)}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}$ and $Z^-_{T(u^\beta)}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}$ are bounded by $|Z_{T(u^\beta)}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}|$, for which we have that

$$|Z_{T(u^\beta)}\mathbb{1}_{\{T(u^\beta)<\tau_d\}}| \le |X_0| + \sum_{n=1}^{T(u^\beta)} |B_n|e^{-S_n}\mathbb{1}_{\{T(u^\beta)<\tau_d\}} \le |X_0| + \sum_{n=1}^{\infty} |B_n|e^{-S_n}\mathbb{1}_{\{n<\tau_d\}} = \bar{Z}.$$

Moreover, using Minkowski's inequality we have that

$$\begin{aligned}\left(\mathbf{E}^\alpha[\bar{Z}^\alpha]\right)^{1/\alpha} &\le \left(\mathbf{E}|X_0|^\alpha\right)^{1/\alpha} + \sum_{n=1}^{\infty}\left(\mathbf{E}^\alpha\left[|B_n|^\alpha e^{-\alpha S_n}\mathbb{1}_{\{n<\tau_d\}}\right]\right)^{1/\alpha} \\ &= \left(\mathbf{E}|X_0|^\alpha\right)^{1/\alpha} + \sum_{n=1}^{\infty}\left(\mathbf{E}\left[|B_n|^\alpha \mathbb{1}_{\{n<\tau_d\}}\right]\right)^{1/\alpha} \\ &\le \left(\mathbf{E}|X_0|^\alpha\right)^{1/\alpha} + (\mathbf{E}|B_1|^{\alpha+\epsilon})^{1/(\alpha+\epsilon)}\sum_{n=1}^{\infty}\mathbf{P}(\tau_d > n)^{\epsilon/(\alpha(\alpha+\epsilon))} < \infty,\end{aligned}$$

where in the second last inequality we used Hölder's inequality, and the finiteness follows from the fact that $\mathbf{P}(\tau_d > n)$ decays exponentially in n uniformly in $|X_0| \le d$, as established in Lemma 2.4. □

Proof of Lemma 6.8. Recall that $\tau = \inf\{n \geq 1 \colon Y_n \leq d\}$. Using Minkowski's inequality we obtain that

$$\begin{aligned}
&\mathbf{E}[r^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} = \mathbf{E}[(\tau + r - \tau)^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} \\
&\leq \mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} + \mathbf{E}[(r-\tau)^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} \\
&\leq \mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} + \sup_{y \in [0,d]} \mathbf{E}[r^{\alpha+L} \,|\, Y_0 = y]^{1/(\alpha+L)} \\
&\leq \mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} + \sup_{y \in [0,d]} \mathbf{E}[t^{r} \,|\, Y_0 = y]^{1/(\alpha+L)} + \mathcal{O}(1),
\end{aligned}$$

as $x \to \infty$, where, by following the arguments as in the proof of Lemma 2.4, t can be chosen such that $\sup_{y\in[0,d]} \mathbf{E}[t^r \,|\, Y_0 = y] < \infty$. For this choice of t, we have that

$$\mathbf{E}[r^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} \leq \mathbf{E}[\tau^{\alpha+L} \,|\, Y_0 = x]^{1/(\alpha+L)} + \mathcal{O}(1), \qquad \text{as } x \to \infty.$$

Finally, using Lemma 6.5 above we have $\mathbf{E}[r^{\alpha+L} \,|\, Y_0 = x] \leq c x^{\lfloor \alpha - \epsilon \rfloor}$ for sufficiently large x. □

Proof of Lemma 6.9. Note that both $|Z^\alpha_{T(u)}| \mathbb{1}_{\{T(u) < r_1\}}$ and $|Z^\alpha_n| \mathbb{1}_{\{n \leq r_1\}}$ are bounded by

$$\bar{Z} = |X_0| + \sum_{n=1}^{\infty} |B_n| e^{-S_n} \mathbb{1}_{\{n < r_1\}},$$

whose α-th moment is finite thanks to Lemma 6.7. Moreover, note that $\{X_n\}_{n\geq 0}$ is transient in the α-shifted measure (cf. Lemma 2.7 above), and hence, $T(u) < \infty$ a.s. Applying a change of measure argument, we obtain that

$$\begin{aligned}
u^\alpha \mathbf{P}(T(u) < r_1) &= u^\alpha \mathbf{E}^\alpha[e^{-\alpha S_{T(u)}} \mathbb{1}_{\{T(u)<r_1\}}] = \mathbf{E}^\alpha\left[|Z_{T(u)}|^\alpha \left| \frac{X_{T(u)}}{u} \right|^{-\alpha} \mathbb{1}_{\{T(u)<r_1\}} \right] \\
&= \mathbf{E}^\alpha\left[|Z_n|^\alpha \mathbb{1}_{\{n \leq T(u)\}} \mathbb{1}_{\{n \leq r_1\}} \mathbf{E}^\alpha\left[\left| \frac{X_{T(u)}}{u} \right|^{-\alpha} \middle| \mathcal{F}_n \right] \right] \\
&\quad + \mathbf{E}^\alpha\left[\left(|Z_{T(u)}|^\alpha \mathbb{1}_{\{T(u)<r_1\}} - |Z_n|^\alpha \mathbb{1}_{\{n \leq T(u)\}} \mathbb{1}_{\{n \leq r_1\}} \right) \left| \frac{X_{T(u)}}{u} \right|^{-\alpha} \right] \\
&= \textbf{(III.1)} + \textbf{(III.2)},
\end{aligned}$$

where $\{\mathcal{F}_n\}_{n\geq 0}$ is the natural filtration. Since $(X_{T(u)}/u)^{-\alpha} \leq 1$ and $T(u) \to \infty$ a.s. as $u \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} \lim_{u\to\infty} \textbf{(III.2)} &\leq \lim_{n\to\infty} \lim_{u\to\infty} \mathbf{E}^\alpha \left[|Z_{T(u)}|^\alpha \mathbb{1}_{\{T(u)<r_1\}} - |Z_n|^\alpha \mathbb{1}_{\{n \leq T(u)\}} \mathbb{1}_{\{n\leq r_1\}} \right] \\
&= \lim_{n\to\infty} \mathbf{E}^\alpha \left[\lim_{u\to\infty} \left(|Z_{T(u)}|^\alpha \mathbb{1}_{\{T(u)<r_1\}} - |Z_n|^\alpha \mathbb{1}_{\{n \leq T(u)\}} \mathbb{1}_{\{n\leq r_1\}} \right) \right] \\
&= \lim_{n\to\infty} \mathbf{E}^\alpha \left[|Z|^\alpha \mathbb{1}_{\{r_1 = \infty\}} - |Z_n|^\alpha \mathbb{1}_{\{n \leq r_1\}} \right] = 0.
\end{aligned}$$

It remains to consider **(III.1)**. Note that, given $\mathcal{F}_n$, $n \leq T(u)$, the random variable $\log|X_{T(u)}| - \log u$ converges in distribution to some positive random variable $\mathfrak{X}$—which is independent of $\mathcal{F}_n$, $n \leq T(u)$—as $u \to \infty$, under the α-shifted measure (cf. e.g. Theorem 3.8 of [11]). Hence we have that

$$\lim_{u\to\infty} \mathbf{E}^\alpha \left[\left| \frac{X_{T(u)}}{u} \right|^{-\alpha} \middle| \mathcal{F}_n \right] \mathbb{1}_{\{n \leq T(u)\}} = \mathbf{E}\left[e^{-\alpha \mathfrak{X}} \right].$$

Moreover, using dominated convergence and the fact $\mathbb{1}_{\{n\leq T(u)\}} \mathbf{E}^\alpha[|X_{T(u)}/u|^{-\alpha} \,|\, \mathcal{F}_n] \leq 1$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \lim_{u\to\infty} \textbf{(III.1)} &= \lim_{n\to\infty} \mathbf{E}^\alpha \left[|Z_n|^\alpha \mathbb{1}_{\{n\leq r_1\}} \lim_{u\to\infty} \mathbf{E}^\alpha \left[\left| \frac{X_{T(u)}}{u} \right|^{-\alpha} \middle| \mathcal{F}_n \right] \mathbb{1}_{\{n\leq T(u)\}} \right] \\
&= \mathbf{E}\left[e^{-\alpha\mathfrak{X}} \right] \lim_{n\to\infty} \mathbf{E}^\alpha \left[|Z_n|^\alpha \mathbb{1}_{\{n\leq r_1\}} \right] = \mathbf{E}\left[e^{-\alpha\mathfrak{X}} \right] \mathbf{E}^\alpha \left[|Z|^\alpha \mathbb{1}_{\{r_1 = \infty\}} \right],
\end{aligned}$$

completing the proof □

Proof of Lemma 6.11. To begin with, we write, for some $\delta > 0$,

$$\begin{aligned}\mathbf{P}((\mathfrak{E}_2(u))^c|X_0 = vu^\beta) &= \mathbf{P}(\exists n < K_\beta^\gamma(u)\colon |B_n| > u^\gamma|X_0 = vu^\beta)\\ &\le \mathbf{P}(\exists n < \tau_d\colon |B_n| > u^\gamma|X_0 = vu^\beta)\\ &\le \mathbf{P}(\exists n \le u^\delta\colon |B_n| > u^\gamma) + \mathbf{P}(\tau_d \ge u^\delta|X_0 = vu^\beta)\\ &= \textbf{(II.1)} + \textbf{(II.2)}.\end{aligned}$$

To bound **(II.1)**, we have that

$$\textbf{(II.1)} \le u^\delta \mathbf{P}(|B_1| > u^\gamma) \le u^{\delta-\alpha\gamma}\mathbf{E}|B_1|^\alpha = o(u^{-(1-\beta)\alpha}),$$

for $(1-\beta)\alpha + \delta - \alpha\gamma < 0$. Since $\textbf{(II.2)} \le u^{-(\alpha+L)\delta}\mathbf{E}[\tau_d^{\alpha+L}|X_0 = vu^\beta]$, it is sufficient to bound $\mathbf{E}[\tau_d^{\alpha+L}|X_0 = vu^\beta]$. Recall $\{Y_n\}_{n\ge 0}$ is the $\mathbb{R}_+$-valued Markov chain defined by $Y_{n+1} = A_{n+1}Y_n + |B_{n+1}|$, for $n \ge 0$, and $\tau = \inf\{n \ge 1\colon Y_n \le d\}$. Note that $\mathbf{E}[\tau_d^{\alpha+L}|X_0 = vu^\beta] \le \mathbf{E}[\tau^{\alpha+L}|Y_0 = |v|u^\beta]$. Combining this with Lemma 6.5, we conclude that there exist c and u_0 such that

$$\textbf{(II.2)} \le u^{-(\alpha+L)\delta}\mathbf{E}[\tau^{\alpha+L}|Y_0 = |v|u^\beta] \le c|v|u^\beta u^{-(\alpha+L)\delta}, \qquad \forall u \ge u_0.$$

Note that we can set $L = L(\delta, \alpha, \beta)$ to be arbitrarily large. Combining the estimates above for **(II.2)** and **(II.2)**, we conclude the proof. □

7 Proofs of Sections 3.2 and 3.3

Again, we briefly describe our strategy of proof before diving into the technicalities. Define $\bar{X}'_n = \{\bar{X}'_n(t), t \in [0,1]\}$, where

$$\bar{X}'_n(t) = \frac{1}{n}\sum_{i=1}^{N(nt)} X'_i \quad \text{and} \quad X'_i = \sum_{j=r_{i-1}}^{r_i-1} X_j, \tag{7.1}$$

where $\{r_i\}_{i\ge 0}$ is the sequence of regeneration times as in Remark 2.1, and

$$N(s) = \sup\{j \ge 0\colon r_j - 1 \le s\}. \tag{7.2}$$

Thanks to Theorem 4.1 of [33] and Theorem 3.1 above, we are able to establish an asymptotic equivalence between $\bar{X}'_n$ and some random walk $\bar{W}_n$ that will be specified below. This allows us to provide a large deviations result for $\bar{X}'_n$, using Lemma 2.11. In both the one-sided and the two-sided case, we will show that the residual process $\bar{X}_n - \bar{X}'_n$ is negligible in an asymptotic sense.

We state here three lemmata that will play key roles in the proofs of Theorems 3.2 and 3.3. Let $\bar{W}_n = \{\bar{W}_n(t), t \in [0,1]\}$ be such that

$$\bar{W}_n(t) = \frac{1}{n}\sum_{i=1}^{\lfloor nt/\mathbf{E}r_1\rfloor} X'_i, \tag{7.3}$$

where X'_i is as in (7.1). We begin with stating an asymptotic equivalence between $\bar{X}'_n$ and $\bar{W}_n$, however, w.r.t. the J_1-topology, which is stronger than the M'_1-topology introduced in the beginning of Section 3.2. Let d_{J_1} denote the Skorokhod J_1 metric on $\mathbb{D}$, which is defined by

$$d_{J_1}(\xi_1, \xi_2) = \inf_{\lambda\in\Lambda} ||\lambda - id||_\infty \vee ||\xi_1 \circ \lambda - \xi_2||_\infty, \qquad \xi_1, \xi_2 \in \mathbb{D},$$

where id denotes the identity mapping, $||\cdot||_\infty$ denotes the uniform metric, that is, $\|x\|_\infty = \sup_{t\in[0,1]}|x(t)|$, and Λ denotes the set of all strictly increasing, continuous bijections from $[0,1]$ to itself. Moreover, for $j \ge 0$, define

$$\mathbb{D}^\mu_{\leqslant j} = \{\xi \in \underline{\mathbb{D}}^\mu_{\leqslant j}\colon \xi(0) = 0\} \quad \text{and} \quad \mathbb{D}^\mu_{\ll j} = \{\xi \in \underline{\mathbb{D}}^\mu_{\ll j}\colon \xi(0) = 0\}.$$

Lemma 7.1. *Consider the metric space $(\mathbb{D}, d_{J_1})$. Suppose that Assumptions 1 and 2 hold. For any $j \geq 0$, the following holds.*

1. *If $B_1 \geq 0$ and C_+ as in Theorem 3.1 is strictly positive, then the stochastic process $\bar{X}'_n$ is asymptotically equivalent to $\bar{W}_n$ w.r.t. $n^{-j(\alpha-1)}$ and $\mathbb{D}^{\mu}_{\ll j-1}$.*

2. *If C_+ and C_- as in Theorem 3.1 satisfy $C_+C_- > 0$, then the stochastic process $\bar{X}'_n$ is asymptotically equivalent to $\bar{W}_n$ w.r.t. $n^{-j(\alpha-1)}$ and $\mathbb{D}^{\mu}_{\ll j}$.*

Proof. We only show *part 2)*, since *part 1)* can be proved by a similar argument. By Lemma 2.11, it is sufficient to show, for any $\delta > 0$ and $\gamma > 0$,

$$\begin{aligned}&\limsup_{n\to\infty} n^{j(\alpha-1)}\mathbf{P}(\bar{X}'_n \in (\mathbb{D}\setminus\mathbb{D}^{\mu}_{\ll j})^{-\gamma}, d_{J_1}(\bar{X}'_n, \bar{W}_n) \geq \delta)\\ &= \limsup_{n\to\infty} n^{j(\alpha-1)}\mathbf{P}(\bar{W}_n \in (\mathbb{D}\setminus\mathbb{D}^{\mu}_{\ll j})^{-\gamma}, d_{J_1}(\bar{X}'_n, \bar{W}_n) \geq \delta) = 0. \end{aligned} \tag{7.4}$$

To prove (7.4), it is convenient to consider the space of paths on a longer time horizon $[0,2]$. Let $\bar{W}_n$ denote the stochastic process $\{\bar{W}_n(t), t \in [0,2]\}$ over the time horizon $[0,2]$, and $\mathbb{D}^{\mu;[0,2]}_{\ll j}$ denote the space of step functions on $[0,2]$ that corresponds to $\mathbb{D}^{\mu}_{\ll j}$. Let $d^{[0,2]}_{J_1}$ denote the Skorokhod J_1 metric on $\mathbb{D}^{[0,2]} = \mathbb{D}([0,2],\mathbb{R})$. Note that $d_{J_1}(\bar{W}_n, \mathbb{D}^{\mu}_{\ll j}) \geq \gamma$ implies that $d^{[0,2]}_{J_1}(\bar{W}^{[0,2]}_n, \mathbb{D}^{\mu;[0,2]}_{\ll j}) \geq \gamma$, and $d_{J_1}(\bar{X}'_n, \mathbb{D}^{\mu}_{\ll j}) \geq \gamma$ implies that either $d^{[0,2]}_{J_1}(\bar{W}^{[0,2]}_n, \mathbb{D}^{\mu;[0,2]}_{\ll j}) \geq \gamma$ or $2n/\mathbf{E}r_1 \leq N(n)$. Therefore, (7.4) is implied by

$$\limsup_{n\to\infty} n^{j(\alpha-1)}\mathbf{P}(d^{[0,2]}_{J_1}(\bar{W}^{[0,2]}_n, \mathbb{D}^{\mu;[0,2]}_{\ll j}) \geq \gamma, d_{J_1}(\bar{X}'_n, \bar{W}_n) \geq \delta) = 0. \tag{7.5}$$

To prove (7.5), we adopt the construction of a piecewise linear non-decreasing homeomorphism $\bar{\lambda}_n$ from [33, the proof of Theorem 4.1]. Let $t_0 = 0$ and t_i be the i-th jump time of $N(n\cdot)$ and t_L be the last jump time of $N(n\cdot)$. Let $L = (\lfloor n/\mathbf{E}r_1\rfloor - 1) \wedge N(n)$. Define $\bar{\lambda}_n$ in such a way that $\bar{\lambda}_n(t) = \mathbf{E}r_1 N(nt)/n$ on $t_0, \ldots, t_L$, $\bar{\lambda}_n(1) = 1$, and $\bar{\lambda}_n$ is a piecewise linear interpolation in between. For such $\bar{\lambda}_n$, $\bar{W}_n(\bar{\lambda}_n(t)) = \bar{X}'_n(t)$ for all $t \in [0, t_L]$, and hence, $\|\bar{W}_n \circ \bar{\lambda}_n - \bar{X}'_n\|_\infty = \sup_{t\in[t_L,1]} |\bar{W}_n \circ \bar{\lambda}_n(t) - \bar{X}'_n(t)|$. Therefore,

$$\begin{aligned} d_{J_1}(\bar{W}_n, \bar{X}'_n) &= \inf_{\lambda\in\Lambda} \|\lambda - id\|_\infty \vee \|\bar{W}_n \circ \lambda - \bar{X}'_n\|_\infty \leq \|\bar{\lambda}_n - id\|_\infty \vee \|\bar{W}_n \circ \bar{\lambda}_n - \bar{X}'_n\|_\infty\\ &= \|\bar{\lambda}_n - id\|_\infty \vee \sup_{t\in[t_L,1]} |\bar{W}_n \circ \bar{\lambda}_n(t) - \bar{X}'_n(t)|. \end{aligned} \tag{7.6}$$

The second term can be bounded (with high probability) as follows. For an arbitrary $\epsilon > 0$, consider two cases: $\lfloor n/\mathbf{E}r_1 - n\epsilon\rfloor < N(n) < \lfloor n/\mathbf{E}r_1\rfloor$ and $\lfloor n/\mathbf{E}r_1\rfloor \leq N(n) \leq \lfloor n/\mathbf{E}r_1 + n\epsilon\rfloor$. Set

$$W_n = \sum_{i=1}^{\lfloor n/\mathbf{E}r_1\rfloor} X'_i.$$

If $\lfloor n/\mathbf{E}r_1 - n\epsilon\rfloor < N(n) < \lfloor n/\mathbf{E}r_1\rfloor$, by the construction of $\bar{\lambda}_n$,

$$\sup_{t\in[t_L,1]} |\bar{W}_n \circ \bar{\lambda}_n(t) - \bar{X}'_n(t)| \leq \sup_{s,t\in[1-\epsilon,1]} |\bar{W}_n(s) - \bar{W}_n(t)|. \tag{7.7}$$

On the other hand, if $\lfloor n/\mathbf{E}r_1\rfloor \leq N(n) \leq \lfloor n/\mathbf{E}r_1 + n\epsilon\rfloor$,

$$\sup_{t\in[t_L,1]} |\bar{W}_n \circ \bar{\lambda}_n(t) - \bar{X}'_n(t)| \leq \sup_{s,t\in[1,1+\epsilon]} |\bar{W}_n(s) - \bar{W}_n(t)|. \tag{7.8}$$

From (7.7) and (7.8), we see that on the event $\{\lfloor n/\mathbf{E}r_1 - n\epsilon\rfloor < N(n) \leq \lfloor n/\mathbf{E}r_1 + n\epsilon\rfloor\}$,

$$\sup_{t\in[t_L,1]} |\bar{W}_n \circ \bar{\lambda}_n(t) - \bar{X}'_n(t)| \leq \sup_{s,t\in[1-\epsilon,1+\epsilon]} |\bar{W}_n(s) - \bar{W}_n(t)|. \tag{7.9}$$

Using (7.6) and (7.9), we obtain that

$$\begin{aligned}
&\mathbf{P}(d_{J_1}^{[0,2]}(\bar{W}_n^{[0,2]}, \mathbb{D}_{\ll j}^{\mu;[0,2]}) \geq \gamma, d_{J_1}(\bar{X}_n', \bar{W}_n) \geq \delta) \\
&\leq \mathbf{P}\left(d_{J_1}^{[0,2]}(\bar{W}_n^{[0,2]}, \mathbb{D}_{\ll j}^{\mu;[0,2]}) \geq \gamma, \sup_{s,t\in[1-\epsilon,1+\epsilon]} |\bar{W}_n(s) - \bar{W}_n(t)| \geq \delta \right) \\
&\quad + \mathbf{P}(\{\lfloor n/\mathbf{E}r_1 - n\epsilon \rfloor < N(n) \leq \lfloor n/\mathbf{E}r_1 + n\epsilon \rfloor\}^c) + \mathbf{P}(\|\bar{\lambda}_n - id\|_\infty \geq \delta). \qquad (7.10)
\end{aligned}$$

Thanks to Cramér's theorem, the second term in (7.10) decays geometrically. Moreover, for the last term in (7.10), we have that

$$\begin{aligned}
\mathbf{P}(\|N(n\,\cdot)/n - \cdot/\mathbf{E}r_1\|_\infty > \delta) &= \mathbf{P}\left(\sup_{t\in[0,1]} |N(nt)/n - t/\mathbf{E}r_1| > \delta\right) \\
&= 1 - \lim_{m\to\infty} \mathbf{P}\left(\sup_{0\leq l\leq 2^m} \left|\frac{N(nl/2^m)}{n} - \frac{l}{\mathbf{E}r_1 2^m}\right| \leq \delta\right) \\
&= 1 - \lim_{m\to\infty} \mathbf{P}\left(\left|\frac{N(nl/2^m)}{n} - \frac{l}{\mathbf{E}r_1 2^m}\right| \leq \delta, \forall 0 \leq l \leq 2^m\right) \\
&= 1 - \lim_{m\to\infty} \mathbf{P}\left(\frac{N(nl/2^m)}{n} \in \left[\frac{l}{\mathbf{E}r_1 2^m} - \delta, \frac{l}{\mathbf{E}r_1 2^m} + \delta\right], \forall l \leq 2^m\right).
\end{aligned}$$

Let $\Delta_i = r_i - r_{i-1}$. Using the fact that $N(n) < k \iff \sum_{i=1}^k \Delta_i > n$, we obtain that

$$\begin{aligned}
&\mathbf{P}(\|N(nt)/n - t/\mathbf{E}r_1\|_\infty > \delta) \\
&= 1 - \lim_{m\to\infty} \mathbf{P}\left(\sum_{i=1}^{\lfloor (l/(\mathbf{E}r_1 2^m) - \delta)n \rfloor + 1} \Delta_i \leq nl/2^m < \sum_{i=1}^{\lfloor (l/(\mathbf{E}r_1 2^m) + \delta)n \rfloor + 1} \Delta_i, \forall 0 \leq l \leq 2^m \right) \\
&= 1 - \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{\lfloor (t/\mathbf{E}r_1 - \delta)n \rfloor + 1} \Delta_i - nt \leq 0, \inf_{t\in[0,1]} \sum_{i=1}^{\lfloor (t/\mathbf{E}r_1 + \delta)n \rfloor + 1} \Delta_i - nt > 0 \right) \\
&\leq 1 - \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{\lfloor (t/\mathbf{E}r_1 - \delta/2)n \rfloor} \Delta_i - nt < 0, \inf_{t\in[0,1]} \sum_{i=1}^{\lfloor (t/\mathbf{E}r_1 + 3\delta/2)n \rfloor} \Delta_i - nt > 0 \right) \\
&= 1 - \mathbf{P}\Bigg(\sup_{t\in[0,1/\mathbf{E}r_1 - \delta/2]} \frac{1}{n}\left(\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i - n\mathbf{E}r_1 t - n\mathbf{E}r_1\delta\right) < 0, \\
&\qquad\qquad \inf_{t\in[\delta,1/\mathbf{E}r_1 + 3\delta/2]} \frac{1}{n}\left(\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i - n\mathbf{E}r_1 t + n\mathbf{E}r_1\delta\right) > 0 \Bigg) \\
&= 1 - \mathbf{P}\left(\sup_{t\in[0,1/\mathbf{E}r_1 - \delta/2]} \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i - \mathbf{E}r_1 t < \mathbf{E}r_1\delta, \inf_{t\in[\delta,1/\mathbf{E}r_1 + 3\delta/2]} \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i - \mathbf{E}r_1 t > -\mathbf{E}r_1\delta \right) \\
&\to 0,
\end{aligned}$$

as $n \to \infty$, at an exponential rate, by Etemadi's inequality, Lemma 2.4, and Cramér's theorem.

For the first term in (7.10), we have that (see [33, page 21])

$$\limsup_{n\to\infty} n^{j(\alpha-1)} \mathbf{P}\left(d_{J_1}^{[0,2]}(\bar{W}_n^{[0,2]}, \mathbb{D}_{\ll j}^{\mu;[0,2]}) \geq \gamma, \sup_{s,t\in[1-\epsilon,1+\epsilon]} |\bar{W}_n(s) - \bar{W}_n(t)| \geq \delta \right) \leq c\epsilon$$

for some $c > 0$, where the intuition behind the asymptotics above is that, given the rare event takes place, the random walk $\bar{W}_n^{[0,2]}$ must have j big jumps and one of them has to occur in the time interval $[1-\epsilon, 1+\epsilon]$. Since the choice of $\epsilon > 0$ was arbitrary, (7.4) is proved by letting $\epsilon \to 0$. ☐

The next two lemmata are useful for future purposes.

Lemma 7.2. *For $\xi, \zeta \in \mathbb{D}$, we have that $d_{M_1'}(\xi, \zeta) \leq d_{J_1}(\xi, \zeta)$.*

Proof. As explained in Section 2.1 of [35], $d_{M_1'}(\xi, \zeta) \leq d_{M_1}(\xi, \zeta)$. Furthermore, by Theorem 12.3.2 of [36], $d_{M_1}(\xi, \zeta) \leq d_{J_1}(\xi, \zeta)$. ☐

Recall that $\mathrm{Disc}(\xi)$ is the set of discontinuities of $\xi \in \mathbb{D}$ and was defined in (3.4).

Lemma 7.3. *If $d_{M_1'}(\xi_n, \xi) \to 0$ as $n \to \infty$, then, for each $t \in Disc(\xi)^c$*

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sup_{t_1 \in \mathscr{B}_\delta(t) \cap [0,1]} |\xi_n(t_1) - \xi(t_1)| = 0.$$

Proof. Let $t \in \mathrm{Disc}(\xi)^c$. We first prove the statement for the case where $t \in (0,1)$. Let $\epsilon > 0$ be fixed. Choose $\delta = \delta(\epsilon) > 0$ such that

$$|\xi(t_1) - \xi(t)| < \epsilon, \qquad \text{for } t_1 \in \mathscr{B}_\delta(t) \subseteq (0,1). \tag{7.11}$$

By the definition of the M_1' convergence, for the given ϵ, there exists n_0, such that $d_{M_1'}(\xi_n, \xi) < (\delta \wedge \epsilon)/8$ for all $n \geq n_0$. Moreover, for each fixed $n \geq n_0$, one can find $(u_n, v_n) \in \Gamma'(\xi_n)$ and $(u, v) \in \Gamma'(\xi)$ such that

$$\|u_n - u\|_\infty \vee \|v_n - v\|_\infty < (\delta \wedge \epsilon)/4. \tag{7.12}$$

Let $\underline{s}$, s, $\overline{s}$ be such that $v(\underline{s}) = t - \delta/2$, $v(s) = t$ and $v(\overline{s}) = t + \delta/2$. Moreover, by (7.12) we have that $v_n(\underline{s}) < t - \delta/4$ and $v_n(\overline{s}) > t + \delta/4$. Thus, for all $t_1 \in (t - \delta/4, t + \delta/4)$ there exists $s_n \in (\underline{s}, \overline{s})$ such that $(u_n(s_n), v_n(s_n)) = (\xi_n(t_1), t_1)$. Combining this with (7.11) and (7.12), we obtain that

$$\begin{aligned} |\xi_n(t_1) - \xi(t_1)| &\leq |\xi_n(t_1) - \xi(t)| + |\xi(t_1) - \xi(t)| = |u_n(s_n) - u(s)| + |\xi(t_1) - \xi(t)| \\ &\leq |u_n(s_n) - u(s_n)| + |u(s_n) - u(s)| + \epsilon \\ &\leq (\delta \wedge \epsilon)/2 + \epsilon + \epsilon < 3\epsilon. \end{aligned}$$

Finally, the case where $t \in \{0, 1\}$ can be dealt with similarly. ☐

The remainder of this section is split into two parts that deal with Theorems 3.2 and 3.3.

7.1 Proof of Theorem 3.2

We consider the case where B_1 is nonnegative. Let us give the "roadmap" of proving Theorem 3.2.

- In Corollary 7.5 below we establish a sample-path large deviations result for the aggregated process $\bar{X}_n'$ (see (7.1) above) by considering a suitably defined random walk together with utilizing Theorem 4.1 of [33]. For the $\mathbb{M}$-convergence in Corollary 7.5 we need Lemma 7.4 below.
- In Proposition 7.6 we show the asymptotic equivalence between the aggregated process $\bar{X}_n'$ and the original process $\bar{X}_n$. Again, one technical lemma, see Lemma 7.7 below, is needed.
- Part 1) of Theorem 3.2 follows by combining Corollary 7.5 with Proposition 7.6. Part 2) is a direct consequence of part 1).

Lemma 7.4. *For all $j \geq 0$ and all $z \in \mathbb{R}$, the set $\underline{\mathbb{D}}^z_{\leqslant j}$ is closed w.r.t. $(\mathbb{D}, d_{M_1'})$.*

Recall that C_j^z was defined in (3.6) for $z \in \mathbb{R}$.

Corollary 7.5. *Suppose that Assumptions 1 and 2 hold. Moreover, let $B_1 \geq 0$ and C_+ as in Theorem 3.1 be strictly positive. For any $j \geq 0$,*

$$n^{j(\alpha-1)}\mathbf{P}(\bar{X}'_n \in \cdot\,) \to (C_+\mathbf{E}r_1)^j C^\mu_j(\,\cdot\,),$$

in $\mathbb{M}(\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})$ as $n \to \infty$.

Proposition 7.6. *Suppose that Assumptions 1 and 2 hold. If $B_1 \geq 0$ and C_+ as in Theorem 3.1 is strictly positive, then $\bar{X}_n$ is asymptotically equivalent to $\bar{X}'_n$ w.r.t. $(n\mathbf{P}(X'_1 \geq n))^j$ and $\underline{\mathbb{D}}^\mu_{\leqslant j-1}$.*

Proof of Theorem 3.2. Part 1) follows by combining Corollary 7.5 with Proposition 7.6. Part 2) is a direct consequence of part 1). ☐

Proof of Lemma 7.4. We give the proof for the case where $z = 0$, while the proof for $z \neq 0$ follows using the same arguments. The statement is trivial for $\underline{\mathbb{D}}_{\leqslant 0} = \{0\}$; we focus on the case where $j \geq 1$. Let $\xi_n, n \geq 1$, be a sequence such that $\xi_n \in \underline{\mathbb{D}}_{\leqslant j}$, for all $n \geq 1$, and $\lim_{n\to\infty} d_{M'_1}(\xi_n, \xi) = 0$ for some $\xi \in \mathbb{D}$. Our goal is to prove that $\xi \in \underline{\mathbb{D}}_{\leqslant j}$. Note that by Lemma 7.3 above, for every $t \in \mathrm{Disc}(\xi)^c \cup \{1\}$,

$$\lim_{n\to\infty} \xi_n(t) = \xi(t). \tag{7.13}$$

We first show that ξ has at most j discontinuity points. Assume that $|\mathrm{Disc}(\xi)| \geq j+1$. Then there exists $0 \leq t_{1,-} < t_{1,+} < \cdots < t_{j+1,-} < t_{j+1,+} \leq 1$ such that $t_{i,-}, t_{i,+} \in \mathrm{Disc}(\xi)^c \cup \{1\}$, and $|\xi(t_{i,-}) - \xi(t_{i,+})| > 0$, for all $i \in \{1, \ldots, j+1\}$. By (7.13), there exists N' such that $|\xi_{N'}(t_{i,-}) - \xi_{N'}(t_{i,+})| > 0$ for all $i \in \{1, \ldots, j+1\}$. This leads to the contradiction that $|\mathrm{Disc}(\xi_{N'})| \leq j$. Now let $\underline{t} < \bar{t}$ be two neighbouring discontinuity points of ξ. We claim that ξ is constant on $(\underline{t}, \bar{t})$. To see this, assume that the opposite statement holds. Then there exists $t_1 < t_{j+2}$ such that $\underline{t} < t_1 < t_{j+2} < \bar{t}$ and $\xi(t_1) \neq \xi(t_{j+2})$. W.l.o.g. we assume that $\xi(t_1) < \xi(t_{j+2})$. Since ξ is continuous on $(\underline{t}, \bar{t})$, there exists $t_1 < t_2 < \cdots < t_{j+2}$ such that

$$\xi(t_1) < \xi(t_2) < \cdots < \xi(t_{j+2}) \qquad \text{with} \quad \epsilon' = \min_{i\in\{1,\ldots,j+1\}} \xi(t_{i+1}) - \xi(t_i). \tag{7.14}$$

On the other hand, for any $\epsilon > 0$, by (7.13) there exists $N = N(\epsilon)$ such that

$$\xi_N(t_i) \in (\xi(t_i) - \epsilon, \xi(t_i) + \epsilon), \qquad \text{for all } i \in \{1, \ldots, j+2\}. \tag{7.15}$$

In view of (7.14) and (7.15), by choosing $\epsilon < \epsilon'$ we conclude that ξ_N has at least $j+1$ discontinuity points, which leads to the contradiction that $|\mathrm{Disc}(\xi_N)| \leq j$. Thus we conclude that ξ is constant between any two neighbouring discontinuity points. Similarly one can show that $\xi(t^+) - \xi(t^-) > 0$ for every $t \in \mathrm{Disc}(\xi)$. ☐

Proof of Corollary 7.5. Note that $\underline{\mathbb{D}}^\mu_{\leqslant j} = \mathbb{D}^\mu_{\leqslant j} \cup \{\xi \in \mathbb{D} \colon \xi(0) > 0,\ \xi - \xi(0) \in \mathbb{D}^\mu_{\leqslant j-1}\}$. In particular, $\mathbb{D}^\mu_{\leqslant j} \subseteq \underline{\mathbb{D}}^\mu_{\leqslant j}$. Using Lemma 7.2, Corollary 7.5 is a consequence of Lemma 7.1 and Theorem 4.1 in [33]. ☐

The following lemma is essential in the proof of Proposition 7.6. Recall $\bar{X}'_n$ was defined in (7.1). Define

$$R_n = \{R_n(t), t \in [0,1]\}, \quad \text{where } R_n(t) = \frac{1}{n}\sum_{i=r_{N(n)}}^{\lfloor nt \rfloor - 1} X_i. \tag{7.16}$$

Lemma 7.7. *Suppose that Assumptions 1 and 2 hold. Moreover, let $B_1 \geq 0$ and C_+ as in Theorem 3.1 be strictly positive. The following holds for any $\delta > 0$, $\gamma > 0$, and $j \geq 0$.*

1. *First we have that*

$$\mathbf{P}(\bar{X}'_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, R_n(1) \geq \delta) = o((n\mathbf{P}(X'_1 \geq n))^{j+1}), \quad \textit{as } n \to \infty.$$

2. *Moreover, we have that*

$$\mathbf{P}(R_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\leqslant 1})^{-\gamma}) = o((n\mathbf{P}(X'_1 \geq n))^j), \quad \textit{as } n \to \infty.$$

Proof of Proposition 7.6. To begin with, for $\epsilon > 0$, define

$$\mathfrak{E}_3^\epsilon(n) = \{N_\epsilon^-(n) < N(n) \le N_\epsilon^+(n)\}, \tag{7.17}$$

where $N_\epsilon^-(n) = \lfloor n/\mathbf{E}r_1 - n\epsilon \rfloor$ and $N_\epsilon^+(n) = \lfloor n/\mathbf{E}r_1 + n\epsilon \rfloor$. Using Cramér's theorem, it is easy to see that $\mathbf{P}(\mathfrak{E}_3^\epsilon(n)^c)$ decays exponentially to 0 as $n \to \infty$. Defining $\Delta_i = r_i - r_{i-1}$, we have that

$$\{d_{M_1'}(\bar{X}_n, \bar{X}_n') \ge 2\delta\} \subseteq \{\exists\, i \le N(n) \ s.t. \ \Delta_i \ge n\delta\} \cup \{R_n(1) \ge \delta\}\,. \tag{7.18}$$

First we show that for any $j \ge 0$, $\delta > 0$, and $\gamma > 0$,

$$\lim_{n\to\infty} (n\mathbf{P}(X_1' \ge n))^{-j}\mathbf{P}(\bar{X}_n' \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, d_{M_1'}(\bar{X}_n, \bar{X}_n') \ge 2\delta) = 0.$$

By (7.18) we have that

$$\begin{aligned}
&\mathbf{P}(\bar{X}_n' \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, d_{M_1'}(\bar{X}_n, \bar{X}_n') \ge 2\delta) \\
&\le \mathbf{P}(\exists\, i \le N(n) \ s.t. \ \Delta_i \ge n\delta) + \mathbf{P}(\bar{X}_n' \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, \ R_n(1) \ge \delta) \\
&= \mathbf{P}(\exists\, i \le N(n) \ s.t. \ \Delta_i \ge n\delta) + o((n\mathbf{P}(X_1' \ge n))^j),
\end{aligned} \tag{7.19}$$

where in (7.19) we used Lemma 7.7 (1) above. It remains to analyze the first term in (7.19). Note that

$$\begin{aligned}
\mathbf{P}(\exists\, i \le N(n) \ s.t. \ \Delta_i \ge n\delta) &\le \mathbf{P}(\exists\, i \le N(n) \ s.t. \ \Delta_i \ge n\delta, \mathfrak{E}_3^\epsilon(n)) + \mathbf{P}(\mathfrak{E}_3^\epsilon(n)^c) \\
&= \mathbf{P}(\exists\, i \le N(n) \ s.t. \ \Delta_i \ge n\delta, \mathfrak{E}_3^\epsilon(n)) + o((n\mathbf{P}(X_1' \ge n))^j) \\
&\le \mathbf{P}\left(\exists\, i \le \lfloor n/\mathbf{E}\tau_1 + n\epsilon \rfloor \ s.t. \ \Delta_i \ge n\delta\right) + o((n\mathbf{P}(X_1' \ge n))^j) \\
&\le \lfloor n/\mathbf{E}r_1 + n\epsilon \rfloor \mathbf{P}(r_1 \ge n\delta) + o((n\mathbf{P}(X_1' \ge n))^j) \\
&= o((n\mathbf{P}(X_1' \ge n))^j),
\end{aligned}$$

for any $j \ge 0$. Next we show that

$$\lim_{n\to\infty} (n\mathbf{P}(X_1' \ge n))^{-j}\mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, d_{M_1'}(\bar{X}_n, \bar{X}_n') \ge 2\delta) = 0.$$

In view of the estimation right above, it is sufficient to show that

$$\lim_{n\to\infty} (n\mathbf{P}(X_1' \ge n))^{-j}\mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, \ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\leqslant j-1})_\rho, \ R_n(1) \ge \delta) = 0,$$

for some $\rho > 0$. Note that

$$\begin{aligned}
&\mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, \ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\leqslant j-1})_\rho, \ R_n(1) \ge \delta) \\
&= \mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, \ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\leqslant j-1})_\rho \cap (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-2})^{-\rho}, \ R_n(1) \ge \delta) \\
&\quad + \mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, \ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\leqslant j-1})_\rho \cap (\underline{\mathbb{D}}^\mu_{\leqslant j-2})_\rho, \ R_n(1) \ge \delta) \\
&\le \mathbf{P}(\bar{X}_n' \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-2})^{-\rho}, \ R_n(1) \ge \delta) \\
&\quad + \mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, \ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\leqslant j-2})_\rho) \\
&= \mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, \ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\leqslant j-2})_\rho) + o(n^{-j(\alpha-1)}).
\end{aligned}$$

Thus, it remains to consider the first term in the last equation. Combining Lemma 7.7 (2) above with the fact that

$$\mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\leqslant j-1})^{-\gamma}, \ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\leqslant j-2})_\rho) \le \mathbf{P}(R_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\leqslant 1})^{-\rho}) + o(n^{-j(\alpha-1)}),$$

for ρ small enough, we conclude the proof. ☐

Proof of Lemma 7.7. Part 1): We start showing the first equivalence. Defining $\bar{X}'_{\leqslant k,n} = \{\bar{X}'_{\leqslant k,n}(t), t \in [0,1]\}$ by $\bar{X}'_{\leqslant k,n}(t) = 1/n \sum_{i=1}^{N(nt)\wedge k} X'_i$, we have that

$$
\begin{aligned}
&\mathbf{P}(\bar{X}'_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma}, R_n(1) \geq \delta) \\
&\leq \mathbf{P}\left(\bar{X}'_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma},\ \sum_{i=r_{N(n)}}^{r_{N(n)+1}-1} X_i \geq n\delta,\ \mathfrak{E}_3^{\epsilon}(n)\right) + \mathbf{P}(\mathfrak{E}_3^{\epsilon}(n)^c) \\
&= \sum_{k=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(\bar{X}'_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma},\ X'_{N(n)+1} \geq n\delta,\ N(n) = k) + o((n\mathbf{P}(X'_1 \geq n))^{j+1}) \\
&= \sum_{k=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(\bar{X}'_{\leqslant k,n} \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma},\ X'_{k+1} \geq n\delta,\ N(n) = k) + o((n\mathbf{P}(X'_1 \geq n))^{j+1}) \\
&\leq \sum_{k=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(\bar{X}'_{\leqslant k,n} \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma},\ X'_{k+1} \geq n\delta) + o((n\mathbf{P}(X'_1 \geq n))^{j+1}) \\
&= \sum_{k=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(\bar{X}'_{\leqslant k,n} \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma})\mathbf{P}\left(X'_{k+1} \geq n\delta\right) + o((n\mathbf{P}(X'_1 \geq n))^{j+1}) \\
&\leq \mathbf{P}(X'_1 \geq n\delta) \sum_{k=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(\bar{X}'_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma/2}) + o((n\mathbf{P}(X'_1 \geq n))^{j+1}) \\
&\leq 2\epsilon n\mathbf{P}(X'_1 \geq n\delta)\mathbf{P}(\bar{X}'_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma/2}) + o((n\mathbf{P}(X'_1 \geq n))^{j+1}).
\end{aligned} \tag{7.20}
$$

It remains to consider the first term in (7.20). Using Corollary 7.5, we have that

$$
\limsup_{n\to\infty} (n\mathbf{P}(X'_1 \geq n))^{-(j+1)}\, 2\epsilon n\mathbf{P}(X'_1 \geq n\delta)\mathbf{P}(\bar{X}'_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\leqslant j-1})^{-\gamma/2}) \leq c\epsilon, \tag{7.21}
$$

for some $c > 0$ independent of ϵ. Part (1) is proved using (7.20) and (7.21), and letting $\epsilon \to 0$.

Part 2): Note that

$$
\mathbf{P}\left(R_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\leqslant 1})^{-\gamma}\right) = \mathbf{P}\left(R_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\leqslant 1})^{-\gamma},\ \frac{r_{N(n)}+1}{n} > \rho\right) \mathbf{P}\left(R_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\leqslant 1})^{-\gamma},\ \frac{r_{N(n)}+1}{n} \leq \rho\right)
$$

where the first term equals zero for sufficiently large $\rho \in (0,1)$. Hence, it is sufficient to consider the second term which is bounded by

$$
\begin{aligned}
\mathbf{P}\left(\frac{r_{N(n)}+1}{n} \leq \rho\right) &\leq \mathbf{P}\left(r_{N(n)} \leq n\rho\right) \leq \mathbf{P}\left(r_{N(n)} \leq n\rho,\ \mathfrak{E}_3^{\epsilon}(n)\right) + \mathbf{P}\left(\mathfrak{E}_3^{\epsilon}(n)^c\right) \\
&= \mathbf{P}\left(\sum_{i=1}^{N(n)} \Delta_i \leq n\rho,\ \mathfrak{E}_3^{\epsilon}(n)\right) + o((n\mathbf{P}(X'_1 \geq n))^j) \\
&\leq \mathbf{P}\left(\sum_{i=1}^{N_\epsilon^-(n)} \Delta_i \leq n\rho\right) + o((n\mathbf{P}(X'_1 \geq n))^j) \\
&\leq \mathbf{P}\left(\sum_{i=1}^{N_\epsilon^-(n)} \frac{\Delta_i}{N_\epsilon^-(n)} \leq \frac{\rho}{1/\mathbf{E}r_1 - \epsilon}\right) + o((n\mathbf{P}(X'_1 \geq n))^j).
\end{aligned} \tag{7.22}
$$

Note that, for every $\rho \in (0,1)$ there exists a sufficiently small $\epsilon > 0$ such that $\rho/(1/\mathbf{E}r_1 - \epsilon) < \mathbf{E}r_1$. For this choice of ϵ, the first term in (7.22) decays exponentially thanks to Cramér's theorem. $\square$

7.2 Proof of Theorem 3.3

We consider the case where B_1 is a general random variable taking values in $\mathbb{R}$. The idea behind the proof of Theorem 3.3 is similar to the one in the one-sided case.

- In Corollary 7.9 below we establish a sample-path large deviations result for the aggregated process $\bar{X}'_n$ (see (7.1) above).
- In Proposition 7.10 we show the asymptotic equivalence between the aggregated process $\bar{X}'_n$ and the original process $\bar{X}_n$. In Lemma 7.11 we deal with the technical issues appearing in Proposition 7.10.
- Part 1) of Theorem 3.3 follows by combining Corollary 7.9 with Proposition 7.10. Part 2) is a direct consequence of part 1).

Lemma 7.8. *For all $j \geq 0$ and all $z \in \mathbb{R}$, the set $\underline{\mathbb{D}}^z_{\ll j}$ is closed w.r.t. $(\mathbb{D}, d_{M_1'})$.*

The proof of Lemma 7.8 is similar to the proof of Lemma 7.4 and therefore omitted.
Recall $C^z_{j,k}$ was defined in (3.9). Let C_+, C_- be as in Theorem 3.1.

Corollary 7.9. *Suppose that Assumptions 1 and 2 hold. If $C_+C_- > 0$, then for any $j \geq 1$*

$$n^{j(\alpha-1)}\mathbf{P}(\bar{X}'_n \in \cdot\,) \to (\mathbf{E}r_1)^j \sum\nolimits_{(l,m)\in I_{=j}} (C_+)^l (C_-)^m C^\mu_{l,m}(\,\cdot\,),$$

in $\mathbb{M}(\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\ll j})$ as $n \to \infty$, where $I_{=j} = \{(l,m) \in \mathbb{Z}^2_+ \colon l + m = j\}$.

Proposition 7.10. *Suppose that Assumptions 1 and 2 hold. If $C_+C_- > 0$, then the following hold for all $j \geq 0$:*

1. *First*
$$\lim_{n\to\infty} n^{j(\alpha-1)}\mathbf{P}(\bar{X}'_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\ll j})^{-\gamma},\, d_{M_1'}(\bar{X}_n, \bar{X}'_n) > \delta) = 0.$$
2. *Assume additionally that $\mathbf{E}|B_1|^m < \infty$ for every $m \in \mathbb{Z}_+$. Then*
$$\lim_{n\to\infty} n^{j(\alpha-1)}\mathbf{P}(\bar{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\ll j})^{-\gamma},\, d_{M_1'}(\bar{X}_n, \bar{X}'_n) > \delta) = 0.$$

In particular, $\bar{X}_n$ is asymptotically equivalent to $\bar{X}'_n$ w.r.t. $n^{-j(\alpha-1)}$ and $\underline{\mathbb{D}}^\mu_{\ll j}$.

We need the following lemma to prove Proposition 7.10. Set

$$R_{p,n}(t) = \frac{1}{n}\sum_{i=r_p}^{\lfloor r_{p+1}t\rfloor - 1} X_i.$$

Let $T_1(u) = T(u) = \inf\{n \geq 0 \colon |X_n| > u\}$ and

$$T_{i+1}(u) = \inf\{n \geq T_i(u) \colon -\operatorname{sign}(X_{T_i}(u))X_n > u\}, \quad i \geq 1.$$

Define $\bar{X}_{i,n} = \{\bar{X}_{i,n}(t), t \in [0,1]\}$ and $\bar{X}'_{i,n} = \{\bar{X}'_{i,n}(t), t \in [0,1]\}$ by

$$\bar{X}_{i,n}(t) = \frac{1}{n}\sum_{l=r_{i-1}}^{\lfloor nt\rfloor \wedge r_i - 1} X_l, \quad \text{and} \quad \bar{X}'_{i,n}(t) = \frac{X'_i}{n}\mathbb{1}_{[r_i/n,1]}(t). \tag{7.23}$$

respectively.

Lemma 7.11. *Suppose that Assumptions 1 and 2 hold. Moreover, assume that $\mathbf{E}|B_1|^m < \infty$ for every $m \in \mathbb{Z}_+$. Let C_+, C_- be as in Theorem 3.1 such that $C_+C_- > 0$.*

1. *For any $i \geq 1$, $j \geq 2$, $\epsilon > 0$, and $\delta > 0$, there exists c_1, c_2 and n_1, n_2 (independent of i) respectively such that*

$$\mathbf{P}(d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta) \leq c_1 n^{-(2-\epsilon)\alpha}, \qquad \textit{for all } n \geq n_1,$$
$$\textit{and} \quad \mathbf{P}(\bar{X}_{i,n} \in (\mathbb{D} \setminus \mathbb{D}_{\ll j})^{-\delta}) \leq c_2 n^{-(j-\epsilon)\alpha}, \qquad \textit{for all } n \geq n_2.$$

2. *For any $j \geq 1$, $\hat{X}_n$ is asymptotically equivalent to $\bar{X}'_n$ w.r.t. $n^{-j(\alpha-1)}$ and $\mathbb{D}^{\mu}_{\ll j}$.*

3. *For any $i \in \{N^-_\epsilon(n), \dots, N^+_\epsilon(n)\}$, $j \geq 1$, $\delta > 0$, and $\epsilon > 0$, there exists c and n_0 (independent of i) such that*
$$\mathbf{P}(R_{i,n} \in (\mathbb{D} \setminus \mathbb{D}_{\ll j})^{-\delta}) \leq c n^{-(j-\epsilon)\alpha}, \qquad \textit{for all } n \geq n_0.$$

Remark 7.12. Without the additional assumption $\mathbf{E}|B_1|^m < \infty$ for every $m \in \mathbb{Z}_+$, one can still show that $\mathbf{P}(T_2(n^\beta) < r_1) = o(n^{-\alpha})$, by following the arguments as in the proof of Lemma 7.11. Hence, under Assumptions 1 and 2, uniformly in i,

$$\lim_{n\to\infty} n^\alpha \mathbf{P}(d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta) = 0.$$

Proof of Proposition 7.10. To begin with, recall that, for $\epsilon > 0$

$$\mathfrak{E}^\epsilon_3(n) = \{N^-_\epsilon(n) \leq N(n) \leq N^+_\epsilon(n)\},$$

where $N^-_\epsilon(n) = n\lfloor 1/\mathbf{E}r_1 - \epsilon \rfloor$ and $N^+_\epsilon(n) = n\lfloor 1/\mathbf{E}r_1 + \epsilon \rfloor$. Moreover, $\mathbf{P}((\mathfrak{E}^\epsilon_3(n))^c)$ decays exponentially to 0 as $n \to \infty$. Let R_n be as in (7.16). Recalling $\Delta_i = r_i - r_{i-1}$, we have that

$$\{d_{M_1'}(\bar{X}_n, \bar{X}'_n) > \delta\} \subseteq \left\{\exists\, i \leq N(n) \text{ s.t. } d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta\right\} \cup \{\|R_n\|_\infty \geq \delta\}. \tag{7.24}$$

To see (7.24), we assume that the opposite statement holds. Given that the event $\{d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) < \delta\}$ takes place, there exist $(u^i_1, v^i_1) \in \Gamma'(\bar{X}_{i,n})$ and $(u^i_2, v^i_2) \in \Gamma'(\bar{X}'_{i,n})$ such that $\|u^i_1 - u^i_2\|_\infty \vee \|v^i_1 - v^i_2\|_\infty < \delta + \eta$. W.l.o.g. we assume that

$$\{s\colon v^i_1(s) = r_{i-1}/n,\, u^i_1(s) = 0\} \cap \{s\colon v^i_2(s) = r_{i-1}/n,\, u^i_2(s) = 0\} \neq \emptyset, \tag{7.25}$$

as well as
$$\{s\colon v^i_1(s) = r_i/n,\, u^i_1(s) = X'_i/n\} \cap \{s\colon v^i_2(s) = r_i/n,\, u^i_2(s) = X'_i/n\} \neq \emptyset.$$

We give here the reasoning for (7.25), where the second equation can be obtained by following same arguments. Let $s_1 \in \{s\colon v^i_1(s) = r_{i-1}/n,\, u^i_1(s) = 0\}$ and $s_2 \in \{s\colon v^i_2(s) = r_{i-1}/n,\, u^i_2(s) = 0\}$. When $s_1 = s_2$, we are done. We assume $s_1 < s_2$, otherwise one can change the role of s_1 and s_2. Define a new parametric representation $(\bar{u}^i_2, \bar{v}^i_2) \in \Gamma'(\bar{X}'_{i,n})$ by

$$\bar{v}^i_2(s) = \begin{cases} v_1(s), & \text{for } s \in [0, s_1], \\ v_1(s_1), & \text{for } s \in (s_1, s_2), \\ v_2(s), & \text{for } s \in [s_2, 1], \end{cases} \qquad \bar{u}^i_2(s) = \begin{cases} 0, & \text{for } s \in [0, s_1], \\ 0, & \text{for } s \in (s_1, s_2), \\ u_2(s), & \text{for } s \in [s_2, 1]. \end{cases}$$

It is easy to check that indeed $(\bar{u}^i_2, \bar{v}^i_2)$ is a parametric representation of $\Gamma'(\bar{X}'_{i,n})$. Moreover, $\|u^i_1 - \bar{u}^i_2\|_\infty = \|u^i_1 - u^i_2\|_\infty < \delta + \eta$,

$$|v^i_1(s) - \bar{v}^i_2(s)| = |v^i_1(s) - v^i_1(s_1)| \leq v^i_1(s_2) - v^i_1(s_1) = v^i_1(s_2) - v^i_2(s_2) < \delta + \eta,$$

for $s \in (s_1, s_2)$, and hence, $\|v^i_1 - \bar{v}^i_2\|_\infty < \delta + \eta$. In view of the construction above, we can replace v^i_2 by $\bar{v}^i_2$, so that (7.25) holds. For the similar reasoning, on the event $\{\|R_n\|_\infty < \delta\} \subseteq \{d_{M_1'}(R_n, 0) < \delta\}$, there exist $(u^{N(n)+1}_1, v^{N(n)+1}_1) \in \Gamma'(R_n)$ and $(u^{N(n)+1}_2, v^{N(n)+1}_2) \in \Gamma'(0)$ such that

$$\|u^{N(n)+1}_1 - u^{N(n)+1}_2\|_\infty \vee \|v^{N(n)+1}_1 - v^{N(n)+1}_2\|_\infty < \delta + \eta,$$

and the intersection of

$$\{s\colon v_1^{N(n)+1}(s)=r_{N(n)}/n,\, u_1^{N(n)+1}(s)=0\}$$

and

$$\{s\colon v_2^{N(n)+1}(s)=r_{N(n)}/n,\, u_2^{N(n)+1}(s)=0\}$$

is an empty set. Now, we pick $s_-^1=0$, $s_+^{N(n)+1}=1$,

$$s_+^i\in\{s\colon v_1^i(s)=r_i/n,\, u_1^i(s)=X_i'/n\}\cap\{s\colon v_2^i(s)=r_i/n,\, u_2^i(s)=X_i'/n\},$$

for $i\in\{1,\dots,N(n)\}$, and

$$s_-^i\in\{s\colon v_1^i(s)=r_i/n,\, u_1^i(s)=0\}\cap\{s\colon v_2^i(s)=r_i/n,\, u_2^i(s)=0\},$$

for $i\in\{2,\dots,N(n)+1\}$. W.l.o.g. we assume that $s_+^i=s_-^{i+1}$, otherwise one can apply a strictly increasing, continuous bijection from $[0,1]$ to itself to the corresponding parametric representation, which preserves the uniform distance between parametric representations. Finally, we define parametric representations $(u_1,v_1)\in\Gamma'(\bar X_n)$ and $(u_2,v_2)\in\Gamma'(\bar X_n')$ by $v_i(s)=v_i^j(s)$, and $u_i(s)=u_i^j(s)+\sum_{k=1}^{j-1}X_k'$, for $s\in[s_-^j,s_+^j]$, $j\in\{1,\dots,N(n)+1\}$, and $i\in\{1,2\}$. It is easy to check that $\|u_1-u_2\|_\infty\vee\|v_1-v_2\|_\infty<\delta+\eta$, and hence, $d(\bar X_n,\bar X_n')\le\|u_1-u_2\|_\infty\vee\|v_1-v_2\|_\infty<\delta+\eta$. Letting $\eta\to0$ leads to the contradiction of $d_{M_1'}(\bar X_n,\bar X_n')>\delta$.

Part 1): For $\gamma>0$ and $j\ge1$, define

$$\mathcal{D}_{\geqslant j}^{\gamma}=\{\xi\in\mathbb{D}\colon |\mathrm{Disc}_\gamma(\xi)|\ge j\},\quad \mathrm{Disc}_\gamma(\xi)=\{t\in\mathrm{Disc}(\xi)\colon |\xi(t)-\xi(t^-)|\ge\gamma\}. \tag{7.26}$$

Note that (cf. the proof of Lemma 2 in [9]), for any $L>0$, there exists a $\bar\gamma=\bar\gamma(\gamma,L)>0$ sufficiently small such that

$$\mathbf{P}(\bar X_n'\in(\mathbb{D}\setminus\mathbb{D}_{\ll j}^{\mu})^{-\gamma}\cap(\mathcal{D}_{\geqslant j}^{\bar\gamma})^c)=o(n^{-L}). \tag{7.27}$$

Thus, it suffices to show that for any $j\ge1$ and any $\delta>0$

$$\lim_{n\to\infty}n^{j(\alpha-1)}\mathbf{P}(\bar X_n'\in\mathcal{D}_{\geqslant j}^{\bar\gamma},\, d_{M_1'}(\bar X_n,\bar X_n')\ge2\delta)=0.$$

By (7.24) we have that

$$\begin{aligned}\mathbf{P}(\bar X_n'\in\mathcal{D}_{\geqslant j}^{\bar\gamma},\, d_{M_1'}(\bar X_n,\bar X_n')\ge2\delta)&\le\mathbf{P}(\bar X_n'\in\mathcal{D}_{\geqslant j}^{\bar\gamma},\,\exists\, i\le N(n)\ s.t.\ d_{M_1'}(\bar X_{i,n},\bar X_{i,n}')\ge\delta)\\&\quad+\mathbf{P}(\bar X_n'\in\mathcal{D}_{\geqslant j}^{\bar\gamma},\,\|R_n\|_\infty\ge\delta)=\mathbf{(IV.1)}+\mathbf{(IV.2)},\end{aligned}$$

where

$$\mathbf{(IV.1)}=\mathbf{P}(\bar X_n'\in\mathcal{D}_{\geqslant j}^{\bar\gamma},\,\mathfrak{E}_3^\epsilon(n),\,\exists\, i\le N(n)\ s.t.\ d_{M_1'}(\bar X_{i,n},\bar X_{i,n}')\ge\delta)+o(n^{-j(\alpha-1)}).$$

For $p\in\mathbb{Z}_+$, let $\mathcal{P}(E,p)$ denote the set of all p-permutations of a discrete set E. Using Lemma 7.11 (1) and the fact that the blocks $\{X_{r_{i-1}},\dots,X_{r_i}\}$, $i\ge1$ are mutually independent, we obtain that

$$\begin{aligned}&\mathbf{P}(\bar X_n'\in\mathcal{D}_{\geqslant j}^{\bar\gamma},\,\mathfrak{E}_3^\epsilon(n),\,\exists\, i\le N(n)\text{ s.t.} d_{M_1'}(\bar X_{i,n},\bar X_{i,n}')\ge\delta)\\&\le\mathbf{P}(\exists(i_1,\dots,i_j)\in\mathcal{P}(\{1,\dots,N_\epsilon^+(n)\},j)\ \text{ s.t. }\ d_{M_1'}(\bar X_{i_1,n},\bar X_{i_1,n}')\ge\delta,\,|X_{i_p}'|\ge n\bar\gamma,\forall 2\le p\le j)\\&=\mathcal{O}(n^j)\mathbf{P}(d_{M_1'}(\bar X_{i_1,n},\bar X_{i_1,n}')\ge\delta)\mathbf{P}(|X_{i_p}'|\ge n\bar\gamma)^{j-1}=\mathcal{O}(n^j)o(n^{-\alpha})\mathcal{O}(n^{-(j-1)\alpha})=o(n^{-j(\alpha-1)}),\end{aligned}$$

where $\mathbf{P}(d_{M_1'}(\bar X_{i_1,n},\bar X_{i_1,n}')\ge\delta)$ is of order $o(n^{-\alpha})$ thanks to Remark 7.12. Recalling

$$\bar X_{\leqslant m,n}'=\left\{\frac1n\sum_{i=1}^{N(nt)\wedge m}X_i',\ t\in[0,1]\right\},$$

we have that

$$
\begin{aligned}
\textbf{(IV.2)} &\le \mathbf{P}\left(\bar{X}'_n \in \mathcal{D}^{\bar{\gamma}}_{\geqslant j},\ \sum_{i=r_{N(n)}}^{r_{N(n)+1}-1} |X_i| \ge n\delta,\ \mathfrak{E}^{\epsilon}_3(n)\right) + \mathbf{P}(\mathfrak{E}^{\epsilon}_3(n)^c) \\
&= \sum_{m=N^-_\epsilon(n)}^{N^+_\epsilon(n)} \mathbf{P}\left(\bar{X}'_n \in \mathcal{D}^{\bar{\gamma}}_{\geqslant j},\ \sum_{i=r_{N(n)}}^{r_{N(n)+1}-1} |X_i| \ge n\delta,\ N(n)=m\right) + o(n^{-j(\alpha-1)}) \\
&\le \sum_{m=N^-_\epsilon(n)}^{N^+_\epsilon(n)} \mathbf{P}\left(\bar{X}'_{\leqslant m,n} \in \mathcal{D}^{\bar{\gamma}}_{\geqslant j},\ \sum_{i=r_m}^{r_{m+1}-1} |X_i| \ge n\delta\right) + o(n^{-j(\alpha-1)}) \\
&= \mathbf{P}\left(\sum_{i=0}^{r_1-1} |X_i| \ge n\delta\right) \sum_{m=N^-_\epsilon(n)}^{N^+_\epsilon(n)} \mathbf{P}(\bar{X}'_{\leqslant m,n} \in \mathcal{D}^{\bar{\gamma}}_{\geqslant j}) + o(n^{-j(\alpha-1)}) \\
&\le \mathbf{P}\left(\sum_{i=0}^{r_1-1} |X_i| \ge n\delta\right) \sum_{m=N^-_\epsilon(n)}^{N^+_\epsilon(n)} \mathbf{P}(\bar{X}'_n \in \mathcal{D}^{\bar{\gamma}}_{\geqslant j}) + o(n^{-j(\alpha-1)}) \\
&\le \mathbf{P}\left(\sum_{i=0}^{r_1-1} |X_i| \ge n\delta\right) 2\epsilon n \mathbf{P}(\bar{X}'_n \in \mathcal{D}^{\bar{\gamma}}_{\geqslant j}) + o(n^{-j(\alpha-1)}) \\
&= 2\epsilon n \mathcal{O}(n^{-\alpha})\mathcal{O}(n^{-j(\alpha-1)}) = o(n^{-j(\alpha-1)}),
\end{aligned}
$$

where $\mathbf{P}(\sum_{i=0}^{r_1-1} |X_i| \ge n\delta)$ is of order $\mathcal{O}(n^{-\alpha})$ due to Remark 6.12.

Part 2): In view of part (1), it is sufficient to show that

$$\mathbf{P}(\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3}) = o(n^{-j(\alpha-1)}),$$

for some $\rho > 0$. Noting $\hat{X}_n(t) = (1/n)\sum_{i=0}^{(\lfloor nt\rfloor \wedge r_{N(n)})-1} X_i$ for $t \in [0,1]$, we have that

$$
\begin{aligned}
&\{\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3}\} \\
&\subseteq \{\bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3},\ \hat{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\rho}\} \cup \{\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \hat{X}_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho}\} \\
&\subseteq \{\bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3},\ \hat{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\rho}\} \cup \{\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \hat{X}_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j-1})_{\rho}\} \\
&\quad \cup \{\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \hat{X}_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho} \cap (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j-1})^{-\rho}\}.
\end{aligned}
$$

Iterating this procedure $j+k$ times, we obtain that

$$
\begin{aligned}
&\{\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3}\} \\
&\subseteq \{\bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3},\ \hat{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\rho}\} \cup \{\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \hat{X}_n \in (\underline{\mathbb{D}}^{\mu}_{0})_{\rho}\} \\
&\quad \cup \bigcup_{i=1}^{j+k-1} \{\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \hat{X}_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j+1-i})_{\rho} \cap (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho}\}.
\end{aligned}
\tag{7.28}
$$

Now, note that

$$\{\bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3},\ \hat{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\rho}\} \subseteq \{\hat{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\rho}, d_{M_1'}(\bar{X}'_n, \hat{X}_n) \ge \rho/3\}. \tag{7.29}$$

Moreover, for $\rho > 0$ sufficiently small, we have that

$$\{\bar{X}_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\ \hat{X}_n \in (\underline{\mathbb{D}}^{\mu}_{0})_{\rho}\} \subseteq \{R_n \in (\mathbb{D}\setminus\underline{\mathbb{D}}_{\ll j})^{-\rho}\}, \tag{7.30}$$

and that

$$\{\bar{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \hat{X}_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j+1-i})_{\rho} \cap (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho}\} \subseteq \{\hat{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho},\, R_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho}\},$$

for all $i \in \{1,\ldots,j+k-1\}$. In view of (7.28)–(7.31), we have that

$$\begin{aligned}
&\mathbf{P}(\bar{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3}) \\
&\le \mathbf{P}(\hat{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\rho}, d_{M_1'}(\bar{X}'_n,\hat{X}_n) \ge \rho/3) + \mathbf{P}(R_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll j})^{-\rho}) \\
&\quad + \sum_{i=1}^{j+k-1} \mathbf{P}(\hat{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho},\, R_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho}),
\end{aligned} \tag{7.31}$$

where the first term in the previous inequality is of order $o(n^{-j(\alpha-1)})$ due to Lemma 7.11 (2) above. Turning to estimating the summation in (7.31), we define $R_{p,n} = \{R_{p,n}(t), t \in [0,1]\}$ by

$$R_{p,n}(t) = \frac{1}{n}\sum_{i=r_p}^{\lfloor r_{p+1}t\rfloor -1} X_i.$$

Using the facts that $R_{N(n),n}(t) = R_n(r_{N(n)+1}t/n)$ and $r_{N(n)+1}/n > 1$ a.s., we have that

$$R_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho} \;\Rightarrow\; R_{N(n),n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho/2}. \tag{7.32}$$

Define $\bar{X}_{\leqslant p,n} = \{\bar{X}_{\leqslant p,n}(t), t\in[0,1]\}$ by $\bar{X}_{\leqslant p,n}(t) = (1/n)\sum_{i=0}^{(\lfloor nt\rfloor \wedge r_{N(n)\wedge p})-1} X_i$. In view of (7.32), we have that

$$\begin{aligned}
&\mathbf{P}(\hat{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho},\, R_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho}) \\
&\le \mathbf{P}(\hat{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho},\, R_{N(n),n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho/2}) \\
&\le \mathbf{P}(\hat{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho},\, R_{N(n),n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho/2},\, \mathfrak{E}_3^{\epsilon}(n)) + \mathbf{P}(\mathfrak{E}_3^{\epsilon}(n)^c) \\
&= \sum_{p=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(\hat{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho},\, R_{N(n),n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho/2},\, N(n)=p) + o(n^{-j(\alpha-1)}) \\
&= \sum_{p=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(\bar{X}_{\leqslant p,n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho},\, R_{p,n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho/2},\, N(n)=p) + o(n^{-j(\alpha-1)}) \\
&\le \sum_{p=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(\bar{X}_{\leqslant p,n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho})\mathbf{P}(R_{p,n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho/2}) + o(n^{-j(\alpha-1)}) \\
&\le \mathbf{P}(\hat{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j-i})^{-\rho/2}) \sum_{p=N_\epsilon^-(n)}^{N_\epsilon^+(n)} \mathbf{P}(R_{p,n} \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll i})^{-\rho/2}) + o(n^{-j(\alpha-1)}) \\
&= \mathcal{O}(n^{-(j-i)(\alpha-1)})2\epsilon\mathcal{O}(n^{-i(\alpha-1)}) + o(n^{-j(\alpha-1)}),
\end{aligned}$$

where in the final step we use Lemma 7.11 (2)–(3). Letting $\epsilon \to 0$, we prove that the summation in (7.31) is of order $o(n^{-j(\alpha-1)})$. Similarly, it can be shown that $\mathbf{P}(R_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}_{\ll j})^{-\rho})$, and hence, $\mathbf{P}(\bar{X}_n \in (\mathbb{D}\setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in (\underline{\mathbb{D}}^{\mu}_{\ll j})_{\rho/3})$ are of order $o(n^{-j(\alpha-1)})$. ☐

Proof of Lemma 7.11. Let $\mathbb{D}^s$ denote the set of all step functions in $\mathbb{D}$. Let $\mathbb{D}^{s,\uparrow}$ denote the set of all non-decreasing step functions in $\mathbb{D}$. Define the mapping $\Psi^{\uparrow}\colon \mathbb{D}^s \to \mathbb{D}^{s,\uparrow}$ by $\zeta = \Psi^{\uparrow}(\xi)$ and

$$\zeta(t) = \inf\{\zeta'(t) \in \mathbb{R}\colon \zeta' \in \mathbb{D}^{s,\uparrow},\, \zeta' \ge \xi\}, \qquad \text{for all } t \in [0,1]. \tag{7.33}$$

Basically, $\Psi^{\uparrow}(\xi)$ is the least possible nondecreasing step function such that $\Psi^{\uparrow}(\xi) \geq \xi$.

Part 1): First we show that $\mathbf{P}(d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta) \leq \mathbf{P}(T_2(n^\beta) < r_1) + o(n^{-(2-\epsilon)\alpha})$, for any $\beta \in (0,1)$. To begin with, setting $\beta^0 = (1-\beta)/2$ we have that

$$\begin{aligned}\mathbf{P}(d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta) &\leq \mathbf{P}(d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta,\, r_i - r_{i-1} \leq n^{\beta_0}) + \mathbf{P}(r_i - r_{i-1} > n^{\beta_0}) \\ &= \mathbf{P}(d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta,\, r_i - r_{i-1} \leq n^{\beta_0}) + o(n^{-(2-\epsilon)\alpha}).\end{aligned}$$

Hence, it is sufficient to show that

$$\mathbf{P}(d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta,\, r_i - r_{i-1} \leq n^{\beta_0}) \leq \mathbf{P}(T_2(n^\beta) < r_1). \tag{7.34}$$

Note that $d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta$ implies $\|\bar{X}_{i,n} - \bar{X}'_{i,n}\|_\infty \geq \delta$, and hence,

$$\delta \leq \sup_{k \leq r_i \wedge n} \left| \frac{1}{n} \sum_{j=r_{i-1}}^{k-1} X_j \right| \leq \sup_{k \leq r_i} \left| \frac{1}{n} \sum_{j=r_{i-1}}^{k-1} X_j \right|.$$

It is sufficient to show that

$$\left\{ \sup_{k \leq r_i} \left| \frac{1}{n} \sum_{j=r_{i-1}}^{k-1} X_j \right| \geq \delta,\, d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta,\, r_i - r_{i-1} \leq n^{\beta_0} \right\}$$

is a subset of $\{T_2(n^\beta) < r_1\}$. We distinguish between the cases 1) $\sup_{k \leq r_i} \frac{1}{n} \sum_{j=r_{i-1}}^{k-1} X_j \geq \delta$, and 2) $\inf_{k \leq r_i} \frac{1}{n} \sum_{j=r_{i-1}}^{k-1} X_j \leq -\delta$. We focus on 1), since 2) can be dealt with by replacing X_i by $-X_i$. Note that

$$\sup_{k \leq r_i \wedge n} \sum_{j=r_{i-1}}^{k-1} X_j \geq \delta n,\ r_i - r_{i-1} \leq n^{\beta_0}$$

implies the existence of $k_1 \in \{r_{i-1}, \ldots, r_i - 1\}$ such that $X_{k_1} > n^{1-\beta_0} > n^\beta$. Now, suppose that $X_k \geq -n^\beta$ for all $k \in \{r_{i-1}, \ldots, r_i - 1\}$. Then the following statements hold.

(i) For n sufficiently large, we have

$$\sup_{t \in [0,1]} \Psi^{\uparrow}(\bar{X}_{i,n})(t) - \sup_{t \in [0,1]} \bar{X}'_{i,n}(t) \leq n^{-1}(r_i - r_{i-1}) n^\beta \leq n^{\beta + \beta_0 - 1} \leq \delta/3,$$

and hence,

$$\sup_{t \in [0,1]} \bar{X}'_{i,n}(t) \geq \sup_{t \in [0,1]} \Psi^{\uparrow}(\bar{X}_{i,n})(t) - \delta/3 \geq 2/3\delta > 0.$$

Moreover, both $\Psi^{\uparrow}(\bar{X}_{i,n}) \in \mathbb{D}^{s,\uparrow}$ and $\bar{X}'_{i,n} \in \mathbb{D}^{s,\uparrow}$ are nonnegative functions in $\mathbb{D}$. Combining these with $r_i - r_{i-1} \leq n^{\beta_0}$, we have that, for sufficiently large n,

$$d_{M_1'}(\Psi^{\uparrow}(\bar{X}_{i,n}), \bar{X}'_{i,n}) \leq \left\{ \sup_{t \in [0,1]} \Psi^{\uparrow}(\bar{X}_{i,n})(t) - \sup_{t \in [0,1]} \bar{X}'_{i,n}(t) \right\} \vee (r_i - r_{i-1})/n \leq \delta/3.$$

(ii) For n sufficiently large,

$$d_{M_1'}(\Psi^{\uparrow}(\bar{X}_{i,n}), \bar{X}_{i,n}) \leq \|\Psi^{\uparrow}(\bar{X}_{i,n}) - \bar{X}_{i,n}\|_\infty \leq n^{-1}(r_i - r_{i-1}) n^\beta \leq n^{\beta + \beta_0 - 1} \leq \delta/3.$$

In view of (i) and (ii), we have that

$$d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \leq d_{M_1'}(\bar{X}_{i,n}, \Psi^{\uparrow}(\bar{X}_{i,n})) + d_{M_1'}(\Psi^{\uparrow}(\bar{X}_{i,n}), \bar{X}'_{i,n}) \leq 2\delta/3,$$

which leads to the contradiction of $d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \geq \delta$. Hence, we prove (7.34).

Next we show that $\mathbf{P}(\bar{X}_{i,n} \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\ll k})^{-\delta}) = \mathbf{P}(T_k(n^\beta) < r_1) + o(n^{-(k-\epsilon)\alpha})$ for any $\beta \in (0,1)$. First we claim that

$$d(\xi, \underline{\mathbb{D}}_{\ll k}) > \delta \;\Rightarrow \exists (t_0, \ldots, t_k) \text{ s.t. } 0 \leq t_0 < \cdots < t_k \leq 1,\, |\xi(t_i) - \xi(t_{i-1})| > \delta,\, i = 1, \ldots, k. \tag{7.35}$$

To see this, assume that the opposite holds. Set $s_0 = 0$ and

$$s_i = \sup\{t \in (s_{i-1}, 1]\colon |\xi(t) - \xi(s_{i-1})| \leq \delta\},$$

for $i = 1, \ldots, k$. Define $\zeta \in \mathbb{D}$ by $\zeta(t) = \xi(s_i)$ for $s_i \leq t < s_{i+1}$. Due to the assumption, we have $\zeta \in \underline{\mathbb{D}}_{\ll k}$, $d(\xi, \zeta) \leq \delta$, and hence, $d(\xi, \underline{\mathbb{D}}_{\ll k}) \leq \delta$. This leads to the contradiction of $d(\xi, \underline{\mathbb{D}}_{\ll k}) > \delta$. Thus, we proved (7.35). Using the fact that $\mathbf{P}(r_1 > n\delta/2)$ decays exponentially, we are able to restrict ourselves to the case where $r_1 \leq n\delta/2$. Let $(t_0, \ldots, t_k)$ be as in the r.h.s. of (7.35). Using the fact that, under the M_1' topology, jumps with the same sign "merge" into one jump in case they are "close", we conclude that $\text{sign}(\xi(t_i))\text{sign}(\xi(t_{i-1})) = -1$ for $i \in \{1, \ldots, k\}$. Combining this with the fact that $\mathbf{P}(r_1 > n^{(1-\beta)}) = o(n^{-(k-\epsilon)\alpha})$ we obtain that

$$\begin{aligned}\mathbf{P}(\bar{X}_{i,n} \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\ll k})^{-\delta}) &= \mathbf{P}(\bar{X}_{i,n} \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\ll k})^{-\delta},\, r_1 \leq n^{(1-\beta)}) + \mathbf{P}(r_1 > n^{(1-\beta)}) \\ &= \mathbf{P}(T_k(n^\beta) < r_1) + o(n^{-(k-\epsilon)\alpha})\end{aligned} \tag{7.36}$$

for any $\beta \in (0,1)$.

Now, it remains to show that $\mathbf{P}(T_k(u^\beta) < r_1) = \mathcal{O}(u^{-(k-\epsilon)\alpha})$ as $u \to \infty$. We prove this by induction in k. For the base case we need to show $\mathbf{P}(T_2(n^\beta) < r_1) = \mathcal{O}(n^{-(2-\epsilon)\alpha})$. Recalling $K_\beta^\gamma(u) = \inf\{n > T(u^\beta)\colon |X_n| \leq u^\gamma\}$, we have that

$$\begin{aligned}\mathbf{P}(T_2(u^\beta) < r_1) &= \mathbf{P}(T_2(u^\beta) < K_\beta^\gamma(u)) + \mathbf{P}(T_1(u^\beta) < K_\beta^\gamma(u) < T_2(u^\beta) < r_1) \\ &= \mathbf{P}(T_2(u^\beta) < K_\beta^\gamma(u)) + \mathcal{O}(u^{-(2\beta-\gamma)\alpha}),\end{aligned} \tag{7.37}$$

where $\mathbf{P}(T_1(u^\beta) < K_\beta^\gamma(u) < T_2(u^\beta) < r_1) = \mathcal{O}(u^{-(2\beta-\gamma)\alpha})$ can be deduced by following the arguments as in the proof of Proposition 6.1. Applying the dual change of measure $\mathscr{D}$ over the time interval $[0, T_1(u^\beta)]$, we obtain that

$$\begin{aligned}u^{(2\beta-\gamma)\alpha}\mathbf{P}(T_2(u^\beta) < K_\beta^\gamma(u)) &= u^{(2\beta-\gamma)\alpha}\mathbf{E}^{\mathscr{D}}\left[e^{-\alpha S_{T(u^\beta)}}\mathbb{1}_{\{T(u^\beta)<r_1\}}\mathbf{P}^{\mathscr{D}}(T_2(u^\beta) < K_\beta^\gamma(u)\,|\,\mathcal{F}_{T(u^\beta)})\right] \\ &= \mathbf{E}^{\mathscr{D}}\left[\mathbb{1}_{\{T(u^\beta)<r_1\}}u^{(\beta-\gamma)\alpha}\mathbf{P}^{\mathscr{D}}(T_2(u^\beta) < K_\beta^\gamma(u)\,|\,\mathcal{F}_{T(u^\beta)})\left|\frac{X_{T(u^\beta)}}{u^\beta Z_{T(u^\beta)}}\right|^{-\alpha}\right].\end{aligned} \tag{7.38}$$

Recalling $\mathfrak{E}_2(u) = \{|B_n| \leq u^\gamma, \forall 1 \leq n < K_\beta^\gamma(u)\}$, we have that, for $|v| \geq 1$

$$\begin{aligned}&\mathbf{P}^{\mathscr{D}}(T_2(u^\beta) < K_\beta^\gamma(u)\,|\,X_{T(u^\beta)} = vu^\beta) \\ &\leq \mathbf{P}^{\mathscr{D}}(|B_n| \leq u^\gamma, \forall T(u^\beta) < n < r_1,\, T_2(u^\beta) < K_\beta^\gamma(u)\,|\,X_{T(u^\beta)} = vu^\beta) \\ &\quad + \mathbf{P}^{\mathscr{D}}(\exists T(u^\beta) < n < r_1 \; s.t. \; |B_n| > u^\gamma\,|\,X_{T(u^\beta)} = vu^\beta) \\ &= \mathbf{P}((\mathfrak{E}_2(u))^c\,|\,X_0 = vu^\beta) = o(u^{-(\beta-\gamma)\alpha})v,\end{aligned} \tag{7.39}$$

where the tail estimate in (7.39) is obtained by following the arguments in the proof of Lemma 6.11 and taking advantage of the additional assumption that $\mathbf{E}|B_1|^m < \infty$ for every $m \in \mathbb{Z}_+$. Plugging (7.39) into (7.38) and using the dominated convergence theorem, we obtain that

$$u^{(2\beta-\gamma)\alpha}\mathbf{P}(T_2(u^\beta) < K_\beta^\gamma(u)) = o(1). \tag{7.40}$$

In view of (7.34), (7.37), and (7.40),

$$\mathbf{P}(T_2(n^\beta) < r_1) = \mathcal{O}(n^{-(2\beta-\gamma)\alpha}) = \mathcal{O}(n^{-(2-\epsilon)\alpha})$$

by choosing $\beta = 1-\epsilon/3$ and $\gamma = \epsilon/3$. Turning to the inductive step, suppose that $\mathbf{P}(T_k(u^\beta) < r_1) = \mathcal{O}(u^{-(k-\epsilon)\alpha})$. Note that

$$\mathbf{P}(T_{k+1}(u^\beta) < r_1) = \mathbf{P}(T_k(u^\beta) < K_\beta^\gamma(u) < T_{k+1}(u^\beta) < r_1) + \mathbf{P}(T_{k+1}(u^\beta) < K_\beta^\gamma(u)),$$

where for the first term in the previous sum we have that

$$\begin{aligned}\mathbf{P}(T_k(u^\beta) < K_\beta^\gamma(u) < T_{k+1}(u^\beta) < r_1) &\le \mathbf{P}(T_k(u^\beta) < r_1)\mathbf{P}(T(u^\beta) < r_1 | X_0 = u^\gamma) \\ &= \mathcal{O}(u^{-(k-\epsilon')\alpha})\mathcal{O}(u^{-(\beta-\gamma)\alpha}) = \mathcal{O}(u^{-(k+1-\epsilon)\alpha}),\end{aligned}$$

for suitable choice of β and γ. Hence, it remains to bound $\mathbf{P}(T_{k+1}(u^\beta) < K_\beta^\gamma(u))$. Applying the dual change of measure $\mathscr{D}$ over the time interval $[0, T_1(u^\beta)]$, we obtain that

$$\begin{aligned}&u^{((k+1)\beta-\gamma)\alpha}\mathbf{P}(T_{k+1}(u^\beta) < K_\beta^\gamma(u)) \\ &= \mathbf{E}^{\mathscr{D}}\left[\mathbb{1}_{\{T(u^\beta)<r_1\}}u^{(k\beta-\gamma)\alpha}\mathbf{P}^{\mathscr{D}}(T_{k+1}(u^\beta) < K_\beta^\gamma(u) \,|\, \mathcal{F}_{T(u^\beta)})\left|\frac{X_{T(u^\beta)}}{u^\beta Z_{T(u^\beta)}}\right|^{-\alpha}\right]. \qquad (7.41)\end{aligned}$$

Moreover, we have that, for $|v| \ge 1$,

$$\begin{aligned}&\mathbf{P}^{\mathscr{D}}(T_{k+1}(u^\beta) < K_\beta^\gamma(u) \,|\, X_{T(u^\beta)} = vu^\beta) \\ &\le \mathbf{P}^{\mathscr{D}}(\exists T(u^\beta) < n_1 < \cdots < n_k < r_1 \ s.t.\ |B_{n_i}| > u^\gamma, \forall i \le k \,|\, X_{T(u^\beta)} = vu^\beta) \\ &= \mathbf{P}(\exists 0 < n_1 < \cdots < n_k < r_1 \ s.t.\ |B_{n_i}| > u^\gamma, \forall i \le k \,|\, X_0 = vu^\beta) \\ &= \mathbf{P}(\exists 0 < n_1 < \cdots < n_k < r_1 \ s.t.\ |B_{n_i}| > u^\gamma, \forall i \le k) = o(u^{-(k\beta-\gamma)\alpha}), \qquad (7.42)\end{aligned}$$

where the tail estimate in (7.42) is obtained by following the arguments in the proof of Lemma 6.11 and taking advantage of the additional assumption that $\mathbf{E}|B_1|^m < \infty$ for every $m \in \mathbb{Z}_+$. Combining (7.41) and (7.42) with the fact that $|X_{T(u^\beta)}/u^\beta| \le 1$ we obtain that $\mathbf{P}(T_{k+1}(u^\beta) < K_\beta^\gamma(u))$, and hence, $\mathbf{P}(T_{k+1}(u^\beta) < r_1)$ are of order $\mathcal{O}(u^{-(k+1-\epsilon)\alpha})$.

Part 2): By a similar reasoning as in proving part (1) of Proposition 7.10, we have that

$$\begin{aligned}&\mathbf{P}(\bar{X}_n' \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\ll j})^{-\gamma},\ d_{M_1'}(\bar{X}_n', \hat{X}_n) \ge \delta) \\ &\le \mathbf{P}(\bar{X}_n' \in \mathcal{D}^{\bar{\gamma}}_{\geqslant j},\ \exists\, i \le N(n) \ s.t.\ d_{M_1'}(\bar{X}_{i,n}, \bar{X}_{i,n}') \ge \delta) + o(n^{-j(\alpha-1)}) \\ &= o(n^{-j(\alpha-1)}),\end{aligned}$$

where $\mathcal{D}^{\bar{\gamma}}_{\geqslant j}$ is defined as in (7.26). It remains to show that, for any $j \ge 1$, $\gamma > 0$, and $\delta > 0$, there exists some $\rho > 0$ so that

$$\mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\ll j})^{-\gamma},\ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\ll j})_\rho,\ d_{M_1'}(\bar{X}_n', \hat{X}_n) \ge \delta) = o(n^{-j(\alpha-1)}),$$

as $n \to \infty$. Recall, for $\gamma > 0$ and $j \ge 1$, $\mathcal{D}^\gamma_{\geqslant j} = \{\xi \in \mathbb{D}\colon |\mathrm{Disc}_\gamma(\xi)| \ge j\}$, where $\mathrm{Disc}_\gamma(\xi) = \{t \in \mathrm{Disc}(\xi)\colon |\xi(t) - \xi(t^-)| \ge \gamma\}$. Defining $\mathcal{D}^\rho_{=j} = \{\xi \in \mathbb{D}\colon |\mathrm{Disc}_\gamma(\xi)| = j\}$ for $j \in \mathbb{Z}$ and $\rho > 0$, we have

$$\begin{aligned}&\mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\ll j})^{-\gamma},\ \bar{X}_n' \in (\underline{\mathbb{D}}^\mu_{\ll j})_\rho,\ d_{M_1'}(\bar{X}_n', \hat{X}_n) \ge \delta) \\ &\le \mathbf{P}(\bar{X}_n' \in \mathcal{D}^{\rho_0}_{\geqslant j-1},\ d_{M_1'}(\bar{X}_n', \hat{X}_n) \ge \delta) + \mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\ll j})^{-\gamma},\ \bar{X}_n' \in (\mathcal{D}^{\rho_0}_{\geqslant j-1})^c) \\ &\le \mathbf{P}(\bar{X}_n' \in \mathcal{D}^{\rho_0}_{\geqslant j-1},\ d_{M_1'}(\bar{X}_n', \hat{X}_n) \ge \delta) + \sum_{i=1}^{j-1}\mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^\mu_{\ll j})^{-\gamma},\ \bar{X}_n' \in \mathcal{D}^{\rho_0}_{=j-i-1}) \\ &= \mathbf{P}(\bar{X}_n' \in \mathcal{D}^{\rho_0}_{\geqslant j-1},\ d_{M_1'}(\bar{X}_n', \hat{X}_n) \ge \delta) + \sum_{i=1}^{j-1}\mathbf{P}(E_j(i)). \qquad (7.43)\end{aligned}$$

Note that

$$
\begin{aligned}
&\mathbf{P}(\bar{X}'_n \in \mathcal{D}^{\rho_0}_{\geqslant j-1},\, d_{M_1'}(\bar{X}'_n, \hat{X}_n) \ge \delta) \\
&\le \mathbf{P}(\bar{X}'_n \in \mathcal{D}^{\rho_0}_{\geqslant j-1},\, \exists\, i \le N(n) \ s.t.\ d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \ge \delta) + o(n^{-j(\alpha-1)}) \\
&= \mathbf{P}(\bar{X}'_n \in \mathcal{D}^{\rho_0}_{\geqslant j-1},\, \mathfrak{E}_3^{\epsilon}(n),\, \exists\, i \le N(n) s.t.\ d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \ge \delta) + o(n^{-j(\alpha-1)}) \\
&\le \mathbf{P}(\exists (i_0, \dots, i_{j-2}) \in \mathcal{P}(\{1, \dots, N^+_\epsilon(n)\}, j-1) \text{ s.t.} \\
&\qquad d_{M_1'}(\bar{X}_{i_0,n}, \bar{X}'_{i_0,n}) \ge \delta,\, |X'_{i_p}| \ge n\rho_0, \forall 1 \le p \le j-2) \\
&= \mathcal{O}(n^{j-1} n^{-(2-\epsilon)\alpha} n^{-(j-2)\alpha}) + o(n^{-j(\alpha-1)}) = o(n^{-j(\alpha-1)}),
\end{aligned} \tag{7.44}
$$

where in (7.44) we use Lemma 7.11 (1) together with the fact that the blocks $\{X_{r_{i-1}}, \dots, X_{r_i}\}$, $i \ge 1$, are mutually independent, and the final equivalence is obtained by setting $\epsilon < 1/\alpha$. In view of the above computation, it remains to analyze $\mathbf{P}(E_j(k))$, $k \in \{1, \dots, j-1\}$ as in (7.43).

Let $I^* = \{i \le N(n) \colon d_{M_1'}(\bar{X}_{i,n}, \bar{X}'_{i,n}) \ge \rho_1\}$. Note that

$$
\begin{aligned}
&\mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2}) \\
&= \mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2},\, |I^*| \ge (k+2) \wedge (j-k-2)) \\
&\quad + \mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2},\, 1 \le |I^*| < (k+2) \wedge (j-k-2)) \\
&\quad + \mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2},\, |I^*| = 0) \\
&= \mathbf{(V.1)} + \mathbf{(V.2)} + \mathbf{(V.3)}.
\end{aligned}
$$

Suppose that $k \le j/2 - 2$, where the case $k > j/2 - 2$ can be dealt with similarly. Note that

$$
\begin{aligned}
\mathbf{(V.1)} &\le \mathbf{P}(\bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2},\, |I^*| \ge k+2,\, \mathfrak{E}_3^{\epsilon}(n)) + o(n^{-j(\alpha-1)}) \\
&\le \mathbf{P}(\exists (i_1, \dots, i_{j-k-2}) \in \mathcal{P}(\{1, \dots, N^+_\epsilon(n)\}, j-k-2) \text{ s.t.} \\
&\qquad d_{M_1'}(\bar{X}_{i_p,n}, \bar{X}'_{i_p,n}) \ge \rho, \forall 1 \le p \le k+2, \\
&\qquad |X'_{i_q}| \ge n\rho_0, \forall k+3 \le q \le j-k-2) \\
&\quad + o(n^{-j(\alpha-1)}) \\
&= \mathcal{O}(n^{j-k-2} n^{-(k+2)(2-\epsilon)\alpha} n^{-(j-2k-4)\alpha}) + o(n^{-j(\alpha-1)}) \\
&= \mathcal{O}(n^{-j(\alpha-1)} n^{-(k+2)+(k+2)\epsilon\alpha}) + o(n^{-j(\alpha-1)}) = o(n^{-j(\alpha-1)}),
\end{aligned}
$$

if $\epsilon < 1/\alpha$. Moreover, we have that $\mathbf{(V.3)} = o(n^{-j(\alpha-1)})$ for ρ_0 sufficiently small. Let $I' = \{i \le N(n) \colon \bar{X}'_{i,n} \ge \rho_0\}$. Turning to bounding $\mathbf{(V.2)}$ we have that

$$
\begin{aligned}
\mathbf{(V.2)} &= \mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2},\, 1 \le |I^*| \le k+1) \\
&= \sum_{k_1=1}^{k+1} \sum_{k_2=0}^{k_1} \mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2}, \\
&\qquad\qquad |I^*| = k_1,\, |I' \cap I^*| = k_2,\, \mathfrak{E}_3^{\epsilon}(n)) \\
&\quad + o(n^{-j(\alpha-1)}).
\end{aligned}
$$

Defining $J = \{(l'_1, \ldots, l'_{k_1}) \colon \mathbf{1}^T(l'_1, \ldots, l'_{k_1}) < k + 2 + k_2\}$, it is now sufficient to consider

$$
\begin{aligned}
&\mathbf{P}(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2},\, |I^*| = k_1,\, |I' \cap I^*| = k_2,\, \mathfrak{E}^{\epsilon}_3(n)) \\
&\leq \mathbf{P}\Bigg(\exists (i_1, \ldots, i_{j-k-2-k_2+k_1}) \in \mathcal{P}(\{1, \ldots, N^+_\epsilon(n)\}, j-k-2-k_2+k_1) \text{ s.t.} \\
&\qquad (\bar{X}_{i_1,n}, \ldots, \bar{X}_{i_{k_1},n}) \in \Big(\bigcup_{(l_1,\ldots,l_{k_1}) \in J} \prod_{p=1}^{k_1} \underline{\mathbb{D}}_{l_{i_p}} \Big)^{-\rho_2}, \\
&\qquad |X'_{i_q}| \geq n\rho_0,\, \forall k_1 + 1 \leq q \leq j-k-2-k_2+k_1 \Bigg) \\
&\quad + \mathbf{P}\Bigg(\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma},\, \bar{X}'_n \in \mathcal{D}^{\rho_0}_{=j-k-2},\, |I^*| = k_1,\, |I' \cap I^*| = k_2,\, \mathfrak{E}^{\epsilon}_3(n), \\
&\qquad \exists (i_1, \ldots, i_{j-k-2-k_2+k_1}) \in \mathcal{P}(\{1, \ldots, N^+_\epsilon(n)\}, j-k-2-k_2+k_1) \\
&\qquad \text{s.t. } (\bar{X}_{i_1,n}, \ldots, \bar{X}_{i_{k_1},n}) \in \Big(\bigcup_{(l_1,\ldots,l_{k_1}) \in J} \prod_{p=1}^{k_1} \underline{\mathbb{D}}_{l_{i_p}} \Big)_{\rho_2}, \\
&\qquad |X'_{i_q}| \geq n\rho_0,\, \forall k_1 + 1 \leq q \leq j-k-2-k_2+k_1 \Bigg) \\
&= \textbf{(V.2.a)} + \textbf{(V.2.b)}.
\end{aligned}
$$

Since $0 \leq k_2 \leq k_1 \leq k+1$ we have that

$$
\begin{aligned}
\textbf{(V.2.a)} &\leq \mathcal{O}(n^{j-1})\mathcal{O}(n^{-(k+2+k_2-k_1\delta)\alpha})\mathcal{O}(n^{-(j-k-2-k_2)\alpha}) \\
&= \mathcal{O}(n^{-j(\alpha-1)} n^{k_1\delta\alpha-1}) = o(n^{-j(\alpha-1)}),
\end{aligned}
$$

for $\delta < 1/((k+1)\alpha)$. It remains to show that $\textbf{(V.2.b)} = o(n^{-j(\alpha-1)})$. To see this, for $\epsilon > 0$ there exists

$$
(\zeta_1, \ldots, \zeta_{k_1}) \in \bigcup_{(l_1,\ldots,l_{k_1}) \in J} \prod_{p=1}^{k_1} \underline{\mathbb{D}}_{l_{i_p}} \tag{7.45}
$$

such that $d(\bar{X}_{i_p,n}, \zeta_{i_p}) \leq \rho_2 + \epsilon$, for all $1 \leq p \leq k_1$. Hence, we have that

$$
d\left(\hat{X}_n, \bar{X}'_n - \sum_{i \in I' \cap \{i_1,\ldots,i_{k_1}\}} \bar{X}'_{i,n} + \sum_{p=1}^{k_1} \zeta_{i_p} \right) \leq \rho_1 \vee (\rho_2 + \epsilon). \tag{7.46}
$$

For any $c > 0$, define $\Phi_c \colon \mathbb{D} \to \mathbb{D}$ by

$$
\Phi_c(\xi)(t) = \sum_{s \in [0,t] \cap \mathrm{Disc}(\xi,c)} (\xi(s) - \xi(s^-)), \quad \text{for } t \in [0,1], \tag{7.47}
$$

where $\mathrm{Disc}(\xi, c)\{t \in \mathrm{Disc}(\xi) \colon \xi(t) - \xi(t^-) \geq c\}$. Now we claim that

$$
\|\bar{X}'_n - \Phi_{\rho_0}(\bar{X}'_n) - \mu \cdot \mathrm{id}\|_\infty > \rho_3. \tag{7.48}
$$

To see this, suppose $\|\Phi_{\rho_0}(\bar{X}'_n) - \mu \cdot \mathrm{id}\|_\infty \le \rho_3$. Hence,

$$d\left(\bar{X}'_n - \sum_{i \in I' \cap \{i_1,\dots,i_{k_1}\}} \bar{X}'_{i,n} + \sum_{p=1}^{k_1} \zeta_{i_p}, \mu \cdot \mathrm{id} + \sum_{p=1}^{k_1} \zeta_{i_p} + \sum_{i \in I' \setminus \{i_1,\dots,i_{k_1}\}} \bar{X}'_{i,n}\right)$$
$$\le \left\| \bar{X}'_n - \sum_{i \in I'} \bar{X}'_{i,n} - \mu \cdot \mathrm{id} \right\|_\infty = \|\bar{X}'_n - \Phi_{\rho_0}(\bar{X}'_n) - \mu \cdot \mathrm{id}\|_\infty \le \rho_3. \tag{7.49}$$

In view of (7.46) and (7.49) we obtain that

$$d\left(\hat{X}_n, \mu \cdot \mathrm{id} + \sum_{p=1}^{k_1} \zeta_{i_p} + \sum_{i \in I' \setminus \{i_1,\dots,i_{k_1}\}} \bar{X}'_{i,n}\right) \le \rho_1 \vee (\rho_2 + \epsilon) + \rho_3,$$

where

$$\mu \cdot \mathrm{id} + \sum_{p=1}^{k_1} \zeta_{i_p} + \sum_{i \in I' \setminus \{i_1,\dots,i_{k_1}\}} \bar{X}'_{i,n} \in \underline{\mathbb{D}}^{\mu}_{\ll j}$$

due to (7.45). This leads to the contradiction of $\hat{X}_n \in (\mathbb{D} \setminus \underline{\mathbb{D}}^{\mu}_{\ll j})^{-\gamma}$ by choosing ρ_1, ρ_2 and ρ_3 small enough. In view of (7.48) we have that

$$\textbf{(V.2.b)} \le \mathbf{P}\left(\bar{X}'_n \in \left\{\xi \in \mathbb{D} \colon \xi(t) - \sup_{t \in [0,1]} \left| \Phi_{\rho_0}(\xi)(t) - \mu t \right| > \rho_3 \right\}\right)$$
$$= o(n^{-j(\alpha-1)}),$$

by choosing ρ_0 and ρ_3 such that $\rho_3/\rho_0 \notin \mathbb{Z}$ and $\lceil \rho_3/\rho_0 \rceil > j$.

Part 3): Since

$$\mathbf{P}(r_{i+1} - r_i > r_i \delta) \le \mathbf{P}(r_{i+1} - r_i > (n - \epsilon')\delta) + \mathbf{P}(r_i \ge n - \epsilon'),$$

$\mathbf{P}(r_{i+1} - r_i > r_i\delta)$ decays exponentially, for $i \in \{N^-_\epsilon(n), \dots, N^+_\epsilon(n)\}$. Combining this with (7.35), we are able to utilize the argument as in (7.36) and obtain that

$$\mathbf{P}(R_{i,n} \in (\mathbb{D} \setminus \underline{\mathbb{D}}_{\ll j})^{-\delta}) = \mathbf{P}(T_j(n^\beta) < r_1) + o(n^{-(j-\epsilon)\alpha})$$

for any $\beta \in (0,1)$. Since $\mathbf{P}(T_j(u^\beta) < r_1) = \mathcal{O}(u^{-(j-\epsilon)\alpha})$ for a suitable choice of β, the proof is completed. □

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