informs
https://pubsonline.informs.org/journal/moor

MATHEMATICS OF OPERATIONS RESEARCH
Articles in Advance, pp. 1–32
ISSN 0364-765X (print), ISSN 1526-5471 (online)

Sample-Path Large Deviations for Unbounded Additive Functionals of the Reflected Random Walk

Mihail Bazhba,[a] Jose Blanchet,[b] Chang-Han Rhee,[c,*] Bert Zwart[d,e]

[a] Quantitative Economics, University of Amsterdam, 1012 WP Amsterdam, Netherlands; [b] Management Science and Engineering, Stanford University, Stanford, California 94305; [c] Industrial Engineering and Management Sciences, Northwestern University, Evanston, Illinois 60208; [d] Stochastics Group, Centrum Wiskunde & Informatica, 1098 XG Amsterdam, Netherlands; [e] Eindhoven University of Technology, 5612 AZ Eindhoven, Netherlands

*Corresponding author

Contact: M.Bazhba@uva.nl (MB); jose.blanchet@stanford.edu (JB); chang-han.rhee@northwestern.edu, https://orcid.org/0000-0002-1651-4677 (C-HR); Bert.Zwart@cwi.nl, https://orcid.org/0000-0001-9336-0096 (BZ)

Received: April 1, 2020
Revised: March 6, 2022; December 23, 2022
Accepted: August 27, 2023
Published Online in Articles in Advance: March 27, 2024

MSC2020 Subject Classification: Primary: 60F10, 60G50

https://doi.org/10.1287/moor.2020.0094

Abstract. We prove a sample-path large deviation principle (LDP) with sublinear speed for unbounded functionals of certain Markov chains induced by the Lindley recursion. The LDP holds in the Skorokhod space $\mathbb{D}[0,1]$ equipped with the M_1' topology. Our technique hinges on a suitable decomposition of the Markov chain in terms of regeneration cycles. Each regeneration cycle denotes the area accumulated during the busy period of the reflected random walk. We prove a large deviation principle for the area under the busy period of the Markov random walk, and we show that it exhibits a heavy-tailed behavior.

Funding: The research of B. Zwart and M. Bazhba is supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek [Grant 639.033.413]. The research of J. Blanchet is supported by the National Science Foundation (NSF) [Grants 1915967, 1820942, and 1838576] as well as the Defense Advanced Research Projects Agency [Grant N660011824028]. The research of C.-H. Rhee is supported by the NSF [Grant CMMI-2146530].

Keywords: Lindley recursion • busy period asymptotics • sample-path large deviations • heavy tails

1. Introduction

In this paper, we develop sample-path large deviation principles (LDPs) for additive functionals of a Markov chain, which is important in operations research (OR), namely, Lindley's recursion. This Markov chain describes the waiting time sequence in a single-server queue under a first in, first out discipline and under independent and identically distributed (i.i.d.) interarrival times and service times. We focus on the case in which the input is light-tailed; that is the service times and interarrival times have a finite moment-generating function in a neighborhood of the origin.

Whereas the model that we consider is vital to many OR applications and, therefore, important in its own right, our main contributions are also fundamental from a methodological standpoint. We contribute, as we explain, to the development of key tools in the study of sample-path large deviations for additive functionals of light-tailed and geometrically ergodic Markov chains.

A rich body of theory, pioneered by Donsker and Varadhan in classic work that goes back more than 40 years (see, for example, Donsker and Varadhan [9]), provides powerful tools designed to study large deviations for additive functionals of light-tailed and geometrically ergodic Markov chains. Roughly speaking, these are chains that converge exponentially fast to stationarity and whose stationary distribution is light-tailed.

Unfortunately, despite remarkable developments in the area, including the more recent contributions in Kontoyiannis and Meyn [13], the prevailing assumptions in the literature are often not applicable to natural functionals of well-behaved geometrically ergodic models, such as Lindley's recursion with light-tailed input.

In particular, every existing general result describing sample-path large deviations of functionals of a process such as Lindley's recursion must assume the function of interest to be bounded. Hence, the current state of the art rules out very important cases, such as the sample-path behavior of the empirical average of the waiting time sequence in a single-server queue over large time scales. Our development allows one to study sample-path large deviations for the cumulative waiting time sequence of a single-server queue. In particular, we provide methodological ideas that, we believe, will be useful in further development of the general theory of sample-path large deviations for additive functionals of geometrically ergodic Markov processes. More precisely, our contributions are summarized as follows:

A. Let $\{X_k, k \geq 0\}$ follow Lindley's recursion. Assume that the associated increments have a finite moment-generating function in a neighborhood of the origin and the traffic intensity is less than one, and let $f(x) = x^p$ for any $p > 0$. We establish a sample-path large deviation principle for $\overline{Y}_n(\cdot) = \sum_{k=1}^{\lfloor n \cdot \rfloor} f(X_k)/n$ as $n \to \infty$ with respect to (w.r.t.) the M_1' topology on $\mathbb{D}[0, T]$ with a good rate function and a sublinear speed function, all of which are fully characterized in Theorem 2.1. Though our result only pertains to a specific Markov chain, it can be extended to more general stochastic recursions and diffusions; this will be pursued in future work. Related work, covering the case of one-dimensional Langevin diffusions can be seen in Bazhba et al. [1].

B. We believe that our overall strategy for establishing Theorem 2.1 can be applied generally to the sample-path large deviation analysis of additive functionals of geometrically ergodic Markov chains. Our strategy involves splitting the sample path into cycles, roughly corresponding to returns to a compact set (in the case of the Lindley recursion, the origin). Then, we show that the additive functional in a cycle has a Weibullian tail. Finally, we use ideas similar to those developed in Bazhba et al. [3], involving sample-path large deviations for random walks with Weibullian increments for the analysis. The result in Bazhba et al. [3], however, cannot be applied directly to our setting here because of two reasons. First, the cycle in progress at the end of the time interval is different from the rest. Second, the number of cycles (and, thus, the number of terms in the decomposition) is random.

The sublinear speed of convergence highlighted in (A) underscores the main qualitative difference between our result and those traditionally obtained in the Donsker–Varadhan setting. In our setting, as hinted in (B), the large deviations behavior of $\overline{Y}_n$ is characterized by heavy-tailed phenomena (in the form of Weibullian tails), which arise when studying the tails of the additive functional over a given busy period. Our choice of $f(\cdot)$ underscores the frailty of the boundedness assumptions required to apply the Donsker–Varadhan type theory. Note that, although $f(\cdot)$ grows slowly when $p \approx 0$, just a small amount of growth derails the application of the standard theory.

The choice of topology is an important aspect of our result. In Bazhba et al. [3], it is argued that M_1' is a natural topology to consider for developing a full sample-path large deviation principle for random walks with Weibullian increments. It is explained that such a result is impossible in the context of the J_1 topology in $\mathbb{D}[0, T]$. Actually, to be precise, the topology that we consider is a stronger variation of the one considered by Puhalskii and Whitt [17, 18], who introduced the M_1' topology in $\mathbb{D}[0, \infty)$ but in such a way that its direct projection onto $\mathbb{D}[0, T]$ loses important continuous functions (such as the maximum of the path in the interval). The key aspect in our variation is the evaluation of the metric at the right endpoint. The version that we consider merges the jumps in the same way in which it is done at the left endpoint in the standard M_1' description. This variation results in a stronger topology when restricted to functions on compact intervals, and it includes the maximum as a continuous function. An important reason for using the M_1' topology is that it allows merging jumps. This seems to be particularly relevant given that, in our setting, the large deviations behavior eventually merges the increments within the busy periods.

In addition to the two elements mentioned in (B), which make the result in Bazhba et al. [3] not directly applicable, our choice of a strong topology also makes the approach in Bazhba et al. [3] difficult to use. In fact, in contrast to Bazhba et al. [3], in this paper, we use a projective limit strategy to obtain our large deviation principle. A direct approach we explored, using the result in Bazhba et al. [3], consisted of replacing the random number of busy periods by its fluid limits (for which there is a large deviations companion with a linear speed rate). Then, we tried to verify that this replacement results in an exponentially good approximation. This would have been a successful strategy if we had used the version of the M_1' topology considered by Puhalskii and Whitt [17], but unfortunately, such exponential approximation does not hold in the presence of our stronger topology.

The development of Theorem 2.1 highlights interesting and somewhat surprising qualitative insights. For example, consider the case $f(x) = x$, corresponding to the area drawn under the waiting time as a curve. As we show, deviations of order $O(1)$ upward from the typical behavior of the process $\overline{Y}_n(\cdot)$ occur because of extreme behavior in a single busy period of duration $O(n^{1/2})$. A somewhat surprising insight involves the busy period in process at time n, which is split into two parts of size $O(n^{1/2})$ involving the age and forward lifetime of the cycle (the former contributes to the area calculations, whereas the latter does not). This asymmetry, relative to the other busy periods during the time horizon $[0, n]$, which are completely accounted for inside the area calculation, raises the question of whether a correction in the LDP is needed, because of this effect, at the end of the time horizon. The answer is no; the contribution to the current busy period and the ones inside the time horizon are symmetric. This result is highlighted in Theorems 2.2 and 2.3, which characterize the variational problem governing extreme busy periods.

There are several related works that deal with large deviations for the area under the waiting time sequence in a busy period. But they focus on queue length as in Blanchet et al. [4] or assume that the moment-generating function of the increment is finite everywhere as in Duffy and Meyn [10]. None of these works obtains sample-path results. Instead, we do not assume that the moment-generating function of the service times or interarrival times is finite everywhere. To handle this level of generality, we employ recently developed sample-path LDPs (Borovkov and

Mogulskii [5, 6], Vysotsky [21]). This level of generality requires us to put in a substantial amount of work to rule out discontinuous solutions of the functional optimization problems that appear in the large deviations analysis.

Another hurdle in developing tail asymptotics for the additive functional in a busy period (reported in Theorems 2.2 and 2.3) is the fact that the functional describing the area under the busy period is not continuous. To deal with this, we exploit path properties of the most probable—in an asymptotic sense—trajectories of the busy period along with the continuity of the area functional over a fixed time horizon. In particular, we rigorously show how to approximate the area over the busy period (which has a random endpoint) with the area over a large, fixed horizon. This is counterintuitive at first because the former approach allows one to remove the reflection operator. However, the latter approach does not have a first passage time (which is a discontinuous function) as a horizon, and this turns out to carry more weight. The proof of our sample-path LDP is provided in Section 3. Section 4 focuses on the technical details behind deriving the tail asymptotics for the area under a busy period. The paper is closed with three appendices covering auxiliary duality results for Markov chains (Appendix A), large deviations results (Appendix B), and smoothness properties of our variational problem (Appendix C).

2. Model Description and Main Results

2.1. Preliminaries

We consider the time-homogeneous Markov chain $\{X_n, n \geq 0\}$ that is induced by the Lindley recursion, that is, $X_{n+1} \triangleq [X_n + U_{n+1}]^+$, $n \geq 0$, and $X_0 = 0$. Note that the random variables $\{U_i, i \geq 1\}$ are i.i.d. copies of a random variable U such that $\mu \triangleq \mathbf{E}(U) < 0$. The state space of the Markov chain $\{X_n, n \geq 0\}$ is the half-line of nonnegative real numbers. We make the following technical but necessary assumptions.

Assumption 2.1. *Let θ_+ and θ_- be the supremum and infimum of the set $\{\theta : \mathbf{E}(e^{\theta U}) < \infty\}$, respectively. We assume that $-\infty \leq \theta_- < 0 < \theta_+ \leq \infty$.*

Assumption 2.2. *For θ_+ and θ_- in Assumption 2.1,*

$$\lim_{n\to\infty} \frac{\log \mathbf{P}(U \geq n)}{n} = -\theta_+, \ \lim_{n\to\infty} \frac{\log \mathbf{P}(-U \geq n)}{n} = \theta_-.$$

Assumption 2.3. *We assume that $\mathbf{P}(U > 0) > 0$.*

The purpose of this paper is to prove a sample-path LDP for $\overline{Y}_n = \{\overline{Y}_n(t), t \in [0, T]\}$, where

$$\overline{Y}_n(t) \triangleq \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} X_i^p, \quad t \in [0, 1],$$

and $p > 0$ is a fixed constant. We introduce basic notions that are used in the statement of one of our main results (Theorem 2.1). First, we set $\alpha \triangleq 1/(1+p)$. Let $\mathbb{D}[0, T]$ denote the Skorokhod space: the space of càdlàg paths from $[0, T]$ to $\mathbb{R}$. We sometimes also consider the space $\mathbb{D}[0, \infty)$ of càdlàg paths from $[0, \infty)$ to $\mathbb{R}$.

Let $\mathcal{T}_{M_1'}$ denote the M_1' Skorokhod topology, whose precise definition is provided subsequently. Unless specified otherwise, we assume that $\mathbb{D}[0, T]$ is equipped with $\mathcal{T}_{M_1'}$ throughout the rest of this paper.

Definition 2.1. For $\xi \in \mathbb{D}[0, T]$, define the extended completed graph $\Gamma'(\xi)$ of ξ as

$$\Gamma'(\xi) \triangleq \{(u, t) \in \mathbb{R} \times [0, T] : u \in [\xi(t-) \wedge \xi(t), \ \xi(t-) \vee \xi(t)]\},$$

where $\xi(0-) \triangleq 0$. Define an order on the graph $\Gamma'(\xi)$ by setting $(u_1, t_1) < (u_2, t_2)$ if either $t_1 < t_2$ or $t_1 = t_2$ and $|\xi(t_1-) - u_1| < |\xi(t_2-) - u_2|$. We call a continuous nondecreasing function $(u, t) = ((u(s), t(s)), s \in [0, T])$ from $[0, T]$ to $\Gamma'(\xi)$ a parameterization of $\Gamma'(\xi)$ if $\Gamma'(\xi) = \{(u(s), t(s)) : s \in [0, T]\}$. We also call such (u, t) a parameterization of ξ, and we denote the set of all parameterizations of ξ with $\Pi'(\xi)$.

Definition 2.2. Define the M_1' metric $d_{M_1'}$ on $\mathbb{D}$ as follows

$$d_{M_1'}(\xi, \zeta) \triangleq \inf_{\substack{(u,t)\in\Pi'(\xi)\\(v,r)\in\Pi'(\zeta)}} \{\|u - v\|_\infty + \|t - r\|_\infty\}.$$

We say that $\xi \in \mathbb{D}[0, T]$ is a pure jump path if $\xi = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i, T]}$ such that $x_i \in \mathbb{R}$ and $u_i \in [0, T]$ for each $i \geq 1$ and the u_i's are all distinct. Let $\mathbb{D}_{\leqslant\infty}^{\uparrow}[0, T]$ be the subspace of $\mathbb{D}[0, T]$ consisting of nondecreasing pure jump paths that assume nonnegative values at the origin. Let $\mathbb{BV}[0, T]$ be the subspace of $\mathbb{D}[0, T]$ consisting of paths with finite variation. Every $\xi \in \mathbb{BV}[0, T]$ has a Lebesgue decomposition with respect to the Lebesgue measure. That is,

$\xi = \xi^{(a)} + \xi^{(s)}$, where $\xi^{(a)}$ denotes the absolutely continuous part of ξ, and $\xi^{(s)}$ denotes the singular part of ξ. Subsequently, using Hahn's decomposition theorem, we can decompose $\xi^{(s)}$ into its nondecreasing singular part $\xi^{(u)}$ and nonincreasing singular part $\xi^{(d)}$ so that $\xi^{(s)} = \xi^{(u)} + \xi^{(d)}$. Without loss of generality (w.l.o.g.), we assume that $\xi^{(s)}(0) = \xi^{(u)}(0) = \xi^{(d)}(0) = 0$. We sometimes also consider $\mathbb{BV}[0,\infty)$: the subspace of $\mathbb{D}[0,\infty)$ consisting of paths that are of bounded variation on any compact interval.

2.2. Sample-Path Large Deviations

In this section, we present the sample-path large deviation principle for $\overline{Y}_n$ and the main ideas of its proof. We start with a few definitions. Let Ψ be the reflection map defined by $\Psi(\xi)(t) \triangleq \xi(t) - \inf_{s\in[0,t]}\{\xi(s) \wedge 0\}$, $\forall t \geq 0$. Define $\mathcal{T}(\xi) \triangleq \inf\{t > 0 : \Psi(\xi)(t) = 0\}$, $B_y \triangleq \{\xi \in \mathbb{BV}[0,\infty) : \xi(0) = y, \int_0^{\mathcal{T}(\xi)} \Psi(\xi)(s)^p ds \geq 1\}$, and $\Lambda^*(y) \triangleq \sup_{\theta\in\mathbb{R}}\{\theta y - \log \mathbf{E}(e^{\theta U})\}$. Set

$$I_y(\xi) \triangleq \begin{cases} \int_0^{\mathcal{T}(\xi)} \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+\xi^{(u)}(\mathcal{T}(\xi)) + \theta_-\xi^{(d)}(\mathcal{T}(\xi)) & \text{if } \xi(0) = y \text{ and } \xi \in \mathbb{BV}[0,\infty), \\ \infty & \text{otherwise,} \end{cases}$$

and denote with $\mathcal{B}_y^*$ the optimal value of the variational problem

$$\mathcal{B}_y^* \triangleq \inf_{\xi\in B_y} I_y(\xi). \tag{$\mathcal{B}_y$}$$

Similarly, denote with $\mathcal{B}_\pi^*$ the optimal value of the variational problem

$$\mathcal{B}_\pi^* \triangleq \inf_{y\in[0,\infty),\ \xi\in B_y} \{\beta y + I_y(\xi)\}, \tag{$\mathcal{B}_\pi$}$$

where $\beta \triangleq \sup\{\theta \geq 0 : \mathbf{E}(e^{\theta U}) \leq 1\}$ is the decay rate of the steady-state distribution π of the reflected random walk (see Result B.2). Note that $\beta \leq \theta_+$, and β is strictly positive in view of Assumption 2.1 and the assumption that $\mu < 0$. Note also that $\mathcal{B}_\pi^* = \inf_{y\in[0,\infty)}\{\beta y + \mathcal{B}_y^*\}$.

Let $T_0 \triangleq 0$ and $T_i \triangleq \inf\{k > T_{i-1} : X_k = 0\}$ for $i \geq 1$, and subsequently, define $\lambda \triangleq \mathbf{E}(\sum_{i=1}^{T_1} X_i^p)/\mathbf{E}(T_1)$. Define $\mathbb{D}^{(\lambda)}[0,T] \triangleq \{\xi \in \mathbb{D}[0,T] : \xi(t) = \lambda t + \zeta(t),\ \forall t \in [0,T],\ \zeta \in \mathbb{D}^{\uparrow}_{\leqslant\infty}[0,T]\}$, that is, the subspace of increasing paths with slope λ and countable upward jumps. Recall that $\alpha = 1/(1+p)$.

Theorem 2.1. *The stochastic process $\overline{Y}_n$ satisfies a large deviation principle in $(\mathbb{D}[0,T], \mathcal{T}_{M_1'})$ with the speed n^α and the rate function $I_{\mathbf{Y}} : \mathbb{D}[0,T] \to \mathbb{R}_+$ defined as*

$$I_{\mathbf{Y}}(\zeta) \triangleq \begin{cases} \mathcal{B}_0^* \sum_{t:\zeta(t)\neq\zeta(t-)} (\zeta(t) - \zeta(t-))^\alpha & \text{if } \zeta \in \mathbb{D}^{(\lambda)}[0,T], \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

That is, for any measurable set A,

$$-\inf_{A^\circ} I_{\mathbf{Y}}(\xi) \leq \liminf_{n\to\infty} \frac{\log \mathbf{P}(\overline{Y}_n \in A)}{n^\alpha} \leq \limsup_{n\to\infty} \frac{\log \mathbf{P}(\overline{Y}_n \in A)}{n^\alpha} \leq -\inf_{\overline{A}} I_{\mathbf{Y}}(\xi). \tag{2.2}$$

The full proof of Theorem 2.1 is deferred to Section 3. The strategy relies on a suitable representation for $\overline{Y}_n$ using renewal theory, which is presented next. The sequence $\{T_j, j \geq 1\}$ induces a renewal process $\{N(t), t \geq 0\}$ defined by $N(t) \triangleq \max\{k \geq 0 : T_k \leq t\}$, $t \geq 0$. We decompose the process $\overline{Y}_n$ as follows. For fixed $t \geq 0$,

$$\overline{Y}_n(t) = \frac{1}{n}\sum_{j=1}^{N(nt)} \sum_{i=T_{j-1}+1}^{T_j} X_i^p + \frac{1}{n}\sum_{i=T_{N(nt)}+1}^{\lfloor nt\rfloor} X_i^p, \tag{2.3}$$

with the convention that $\sum_{i=T_{N(nt)}}^{\lfloor nt\rfloor} X_i^p$ is zero in case the superscript $\lfloor nt\rfloor$ is strictly smaller than the subscript $T_{N(nt)}$. We introduce some notation for the analysis of $\overline{Y}_n$. Define

- $\tau_j \triangleq T_j - T_{j-1}, j \geq 1$, the interarrival times of the renewal process N.
- $W_j \triangleq \sum_{i=T_{j-1}+1}^{T_j} X_i^p, j \geq 1$, the area under X_i^p during the jth busy period of X_n.

- $\overline{Z}_n(t) \triangleq \frac{1}{n}\sum_{j=1}^{N(nt)} W_j, t \in [0,1]$, the aggregate process (excluding the last regeneration cycle).
- $\overline{R}_n(t) \triangleq \frac{1}{n}\sum_{i=T_{N(n)}+1}^{\lfloor nt \rfloor} X_i^p, t \in [0,1]$, the process that equals $\overline{Y}_n$ during the last regeneration cycle.
- $V_n \triangleq \sum_{i=T_{N(n)}+1}^{n} X_i^p$, the area under X_i^p during the last regeneration cycle.
- $\overline{S}_n(t) \triangleq \frac{1}{n} V_n \mathbb{1}_{\{1\}}(t)$, the process with one jump, which aggregates the area under X_i^p during the last regeneration cycle.

We often refer to the functions associated with these quantities by dropping the argument; for example, by writing $\overline{R}_n$, we refer to $\{\overline{R}_n(t), t \in [0,T]\}$. The strategy to prove our main result, Theorem 2.1, builds on tail estimates for W_1 and V_n, which are presented in Theorems 2.2 and 2.3. Using Theorem 2.3, we derive an LDP for $\overline{S}_n$ in Lemma 3.1. For the LDP of $\overline{Z}_n$ in $\mathbb{D}[0,T]$, we start by obtaining an LDP for the finite projections of $\overline{Z}_n$ in Lemmas 3.2–3.4. Then, the finite-dimensional LDP is lifted in the standard projective limit to the pointwise convergence topology in Lemma 3.6 and, finally, extended to the M_1' topology using the continuity of the identity map in the subspace of increasing cádlág paths in Lemma 3.7. In the last step, we infer an LDP for $\overline{Z}_n + \overline{S}_n$ through the use of a continuous mapping approach, and hence, we obtain the LDP for $\overline{Y}_n$.

For the sample-path LDP of $\overline{Y}_n$, we prove the exponential equivalence of $\overline{Y}_n$ and $\overline{Z}_n + \overline{S}_n$ in Lemma 3.8 by considering the M_1' distance of $\overline{Y}_n$ with the aggregate process $\overline{Z}_n$ and the last regeneration cycle $\overline{R}_n$ pushed to the end of the time horizon. Consequently, the LDP of $\overline{Y}_n$ is deduced because of the LDPs of $\overline{S}_n$ and $\overline{Z}_n$ in $(\mathbb{D}[0,T], \mathcal{T}_{M_1'})$.

Before embarking on the execution of this technical program, it is worth commenting on the role of $\overline{R}_n$ because this element allows us to expose the importance of a careful analysis involving the area during a busy period. As mentioned in the introduction, one may wonder if the contribution of $\overline{R}_n$ may end up counting differently in the form of the LDP. The typical path for $\overline{Y}_n$ is a straight line with drift equal to the steady-state workload. Our developments indicate that most likely large deviations behavior away from the most likely path occur because of isolated busy periods that exhibit extreme behavior. For example, in the case $p = 1$, substantially extreme busy periods (leading to large deviations of order $O(n)$) have a duration of $O(n^{1/2})$ and exhibit excursions of order $O(n^{1/2})$, therefore accumulating an area of order $O(n)$.

Each busy period, including the one in progress at the end of the time horizon, contributes the same way in the rate function. This follows from Theorems 2.2 and 2.3, but may be somewhat remarkable. The reason is that, when the cycle in progress at the end of the time horizon is extreme, its duration is of order $O(n^{1/2})$. This suggests that the remainder of the cycle is also of order $O(n^{1/2})$. It turns out that this long time duration has no significant contribution to the total area: whereas the remainder of the cycle in progress may be large, the position of the chain is actually of order $o(n^{1/2})$ from the end of the time horizon, so the total contribution to the area of the remaining portion of the cycle is negligible. This calculation is exposed in Proposition 4.3, and a time-reversal argument is given in Appendix A.

2.3. Busy Period Asymptotics

It is clear that a large deviations analysis of the area under a busy period is indispensable for deriving the sample-path LDP of $\overline{Y}_n$ in Theorem 2.1. Our next two theorems provide the asymptotic estimation for the tails of W_1 and V_n, showing that they exhibit Weibull behavior. We discuss their statements and defer their proofs to Section 4.

Theorem 2.2. *Recall that $W_1 = \sum_{k=1}^{T_1} X_k^p$ and $\alpha = 1/(1+p)$. It holds that*

$$\lim_{t\to\infty} \frac{1}{t^\alpha} \log \mathbf{P}(W_1 \geq t) = -\mathcal{B}_0^*. \tag{2.4}$$

For V_n, our analysis points to Weibull-like asymptotic behavior similar to W_1 except that the prefactor associated with V_n is $\mathcal{B}_\pi^*$ (instead of $\mathcal{B}_0^*$). It turns out that (see Proposition 4.3) the prefactor $\mathcal{B}_\pi^*$ is equal to $\mathcal{B}_0^*$. This leads to the conclusion that every busy period, including the one in progress at the end of the time horizon, has the same tail asymptotics.

Theorem 2.3. *For the area of the last busy period, we have the following tail asymptotics: for any $b \geq 0$,*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(V_n \geq nb) = -\mathcal{B}_0^* \cdot b^\alpha. \tag{2.5}$$

The tail asymptotics for W_1 and V_n are derived using a recently developed LDP for random walks with light-tailed increments from Borovkov and Mogulskii [5, 6] and Vysotsky [21]; compare this with Result B.3. Specifically, the tail probability of W_1 can be written as the tail probability of the image $\Phi(\overline{K}_n^0)$ of the unrestricted scaled random walk $\{\overline{K}_n^0(t), t \in [0,T]\}$, where $\overline{K}_n^y(t) \triangleq y + \frac{1}{n}K_{\lfloor nt \rfloor}$, $K_n \triangleq \sum_{i=1}^n U_i$, and the functional Φ is defined as $\Phi(\xi) \triangleq \int_0^{\mathcal{T}(\xi)}(\Psi(\xi)(s))^p ds$. Note that $\Phi : \mathbb{D}[0,\infty) \to \mathbb{R}_+$ is not continuous, and hence, the proof for the tail asymptotics of W_1 gets more involved than simply applying the contraction principle. We derive large deviations upper and lower bounds and show that they coincide.

For the upper bound, we replace the hitting time T_1 with a sufficiently large value T. This enables us to study the area of X_n over the finite time horizon $[0,T]$. For T large enough, we show that the area of the reflected random walk over the whole time horizon $[0,T]$ serves as an asymptotic upper bound for W_1 (see the proof of Proposition 4.2), and it is expressed as a functional of $\overline{K}_n$ (see Lemma 4.6). This functional is shown to be uniformly continuous in the M_1' topology on level sets of the rate function associated with the LDP for $\overline{K}_n$ (cf. Lemma 5.2). Invoking Result B.3, recently established in Vysotsky [21], we get a large deviation upper bound.

For the lower bound, we confine the functional of the area under the busy period over a fixed time horizon by imposing an extra condition; see the proof of Proposition 4.2. Subsequently, we exploit some regularity properties of a variational problem associated with the lower bound to show that $\mathcal{B}_0$ has the same optimal value as the variational problem associated with the large deviation upper and lower bounds. We have organized the presentation in such a way that analytic details associated to variational problems are gathered in Section 5.

For V_n, we follow the same approach with some slight modifications. In order to carry out our analysis for V_n, we associate the tail of V_n with the tail of W_1 through Lemmas A.1 and A.2. In particular, we prove that

$$\lim_{n\to\infty} \frac{\log \mathbf{P}_0(V_n > nx)}{n^\alpha} = \lim_{n\to\infty} \frac{\log \mathbf{P}_\pi(W_1 > nx)}{n^\alpha}.$$

To show this, we also rely on Nuyens and Zwart [16, result B.2], describing the asymptotic behavior of the invariant measure π. Finally, we repeat similar steps as in the analysis of W_1.

The constant $\mathcal{B}_0^*$ appears in all our theorems and is the solution of a variational problem. We show in Proposition 4.1 that $\mathcal{B}_0^* \in (0,\infty)$. This property is all that is needed for our main sample-path large deviations results. Nevertheless, it is of interest to compute $\mathcal{B}_0^*$. This can be done by solving a suitable variational problem, which, in turn, is typically done using the associated Euler–Lagrange equations. However, for the Euler–Lagrange equations to characterize the solution, it must be shown that an optimizing path ξ^* exists that is sufficiently smooth, that is, not just absolutely continuous, but differentiable with continuous derivative. In general, showing a priori sufficient smoothness of an optimizer is a nontrivial task, but we explain how to execute this for the case $p = 1$, using a framework presented in Cesari [7]. The details can be found in Appendix C.

3. Proof of the Sample-Path LDP

In this section, we prove Theorem 2.1. For notational convenience, we take $T = 1$ throughout this section. We begin our analysis with a lemma that establishes the large deviations behavior of the area under the busy period active at time n. To this end, define $\mathbb{D}[0,1]^{\leqslant 1} \triangleq \{\xi \in \mathbb{D}[0,1] : \xi = x\mathbb{1}_{\{1\}} \text{ for some } x \geq 0\}$. Recall that $\overline{S}_n = \frac{1}{n}V_n\mathbb{1}_{\{1\}}$ and $V_n = \sum_{i=T_{N(n)}+1}^n X_i^p$.

Lemma 3.1. *$\overline{S}_n$ satisfies the LDP in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$ with the speed n^α and the rate function $I_S : \mathbb{D}[0,1] \to \mathbb{R}_+$, where*

$$I_S(\zeta) \triangleq \begin{cases} \mathcal{B}_0^*(\zeta(1) - \zeta(1-))^\alpha & \text{if } \zeta \in \mathbb{D}[0,1]^{\leqslant 1}, \\ \infty & \text{otherwise}. \end{cases} \tag{3.1}$$

Proof. Define a function $\Upsilon : \mathbb{R}_+ \to \mathbb{D}[0,1]^{\leqslant 1}$ as $\Upsilon(x) \triangleq x \cdot \mathbb{1}_{\{1\}}$. Then, $\overline{S}_n = \Upsilon\left(\frac{1}{n}V_n\right)$, and it is straightforward to see that Υ is a continuous function w.r.t. the M_1' topology. Therefore, the desired LDP follows from the contraction principle if we prove that $\frac{1}{n}V_n$ satisfies an LDP in $\mathbb{R}_+$ with the sublinear speed n^α and the good rate function $I_V : \mathbb{R}_+ \to \mathbb{R}_+$, where $I_V(x) \triangleq \mathcal{B}_0^* \cdot x^\alpha$. To prove the LDP for $\frac{1}{n}V_n$, note first that $\mathbf{P}\left(\frac{1}{n}V_n \in \cdot\right)$ is exponentially tight (w.r.t. the speed n^α) from Theorem 2.3. Therefore, it is enough to establish the weak LDP. For the weak LDP, we start with showing that, for any $a, b \in \mathbb{R}$, $B \triangleq (a,b) \cap \mathbb{R}_+$ satisfies $\limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\frac{1}{n}V_n \in B\right)}{n^\alpha} = \liminf_{n\to\infty} \frac{\log \mathbf{P}\left(\frac{1}{n}V_n \in B\right)}{n^\alpha}$. Because this holds trivially with value $-\infty$ if $0 \geq b$ or $a \geq b$, we assume that $0 \vee a < b$. Note that, from Theorem 2.3,

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\frac{1}{n}V_n \in B\right)}{n^\alpha} \leq \limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\frac{1}{n}V_n \geq 0 \vee a\right)}{n^\alpha} \leq -\mathcal{B}_0^* \cdot (0 \vee a)^\alpha.$$

For the lower bound, note that Theorem 2.3 implies that $\mathbf{P}\big(\frac{1}{n}V_n \geq b\big)/\mathbf{P}\big(\frac{1}{n}V_n \geq 0 \vee a+\epsilon\big) \to 0$ for any $\epsilon > 0$. Therefore, we obtain

$$\liminf_{n\to\infty}\frac{\log \mathbf{P}\big(\frac{1}{n}V_n \in B\big)}{n^\alpha} \geq \liminf_{n\to\infty}\frac{\log\big(\mathbf{P}\big(\frac{1}{n}V_n \geq 0\vee a+\epsilon\big)-\mathbf{P}\big(\frac{1}{n}V_n\geq b\big)\big)}{n^\alpha}$$
$$\geq \liminf_{n\to\infty}\frac{\log\Big\{\mathbf{P}\big(\frac{1}{n}V_n \geq 0\vee a+\epsilon\big)\Big(1-\frac{\mathbf{P}(\frac{1}{n}V_n\geq b)}{\mathbf{P}(\frac{1}{n}V_n\geq 0\vee a+\epsilon)}\Big)\Big\}}{n^\alpha}$$
$$= \liminf_{n\to\infty}\frac{\log \mathbf{P}\big(\frac{1}{n}V_n \geq 0\vee a+\epsilon\big)}{n^\alpha} = -\mathcal{B}_0^*\cdot(0\vee a+\epsilon)^\alpha.$$

Taking $\epsilon \to 0$, we see that the limit supremum and the limit infimum coincide. Because $\{(a,b)\cap\mathbb{R}_+ : a,b\in\mathbb{R}, a\leq b\}$ forms a base of the Euclidean topology on $\mathbb{R}_+$, Dembo and Zeitouni [8, theorem 4.1.11] applies and, hence, proves the desired weak LDP with the rate function I_V. This concludes the proof.

We next work toward a sample-path LDP for $\overline{Z}_n$. We employ a well-known technique based on the projective limit theorem by Dawson and Gärtner; see Dembo and Zeitouni [8, theorem 4.6.1]. The following three lemmas lead to the first key step in this approach, which consists of obtaining the finite-dimensional LDP for $\overline{Z}_n$.

Lemma 3.2. *For any given* $0 = t_0 < t_1 < t_2 < \cdots < t_k$, *let* $\Delta t_i \triangleq t_i - t_{i-1}$ *for* $i = 1,\ldots,k$. *Then,*

$$\limsup_{n\to\infty}\frac{1}{n^\alpha}\log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j \geq na_1,\ldots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\geq na_k\right) \leq -\mathcal{B}_0^*\left(\sum_{i=1}^k(a_i-\lambda\Delta t_i)_+^\alpha\right), \tag{3.2}$$

$$\liminf_{n\to\infty}\frac{1}{n^\alpha}\log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j \geq na_1,\ldots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\geq na_k\right) \geq -\mathcal{B}_0^*\left(\sum_{i=1}^k(a_i-\lambda\Delta t_i)_+^\alpha\right), \tag{3.3}$$

where $(x)_+ \triangleq x\vee 0$.

Proof. Recall that $\tau_j = T_j - T_{j-1}$, where T_j is the jth hitting time of zero. Fix an arbitrary $\epsilon > 0$ and let $E_i^{(n)}(\epsilon) \triangleq n[l_i, u_i]$, where $l_i \triangleq t_i/\mathbf{E}\tau_1 - \epsilon$ and $u_i \triangleq t_i/\mathbf{E}\tau_1+\epsilon$. We use this notation throughout the proof of this lemma. For the upper bound in Equation (3.2), note that

$$\mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j \geq na_1,\ldots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\geq na_k\right) \leq \underbrace{\sum_{i=1}^k \mathbf{P}(N(nt_i)\notin E_i^{(n)}(\epsilon))}_{=\text{(I)}}$$
$$+\underbrace{\mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j \geq na_1,\ldots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\geq na_k, N(nt_i)\in E_i^{(n)}(\epsilon)\text{ for } i=1,\ldots,k\right)}_{=\text{(II)}}.$$

Note also that τ_1 is light-tailed because $\mathbf{P}(\tau_1\geq k) = \mathbf{P}(\sum_{i=1}^j U_i \geq 0, j=1,\ldots,k) \leq \mathbf{P}\big(\frac{1}{k}\sum_{i=1}^k U_i\geq 0\big)$, and $\mathbf{P}\big(\frac{1}{k}\sum_{i=1}^k U_i \geq 0\big)$ decays at a geometric rate as $k\to\infty$ because of our assumptions that $\mathbf{E}U_1 < 0$ and $\theta_+ > 0$ along with Cramér's theorem. Moreover,

$$\{N(nt_i)\notin E_i^{(n)}\} = \{N(n(l_i+\epsilon)\mathbf{E}\tau_1) < \lceil nl_i\rceil\}\cup\{N(n(u_i-\epsilon)\mathbf{E}\tau_1) > \lfloor nu_i\rfloor\}$$
$$\subseteq\left\{\frac{\sum_{i=1}^{\lceil nl_i\rceil}\tau_i}{\lceil nl_i\rceil} > \frac{n(l_i+\epsilon)}{\lceil nl_i\rceil}\mathbf{E}\tau_1\right\}\cup\left\{\frac{\sum_{i=1}^{\lceil nu_i\rceil}\tau_i}{\lceil nu_i\rceil} < \frac{n(u_i-\epsilon)}{\lceil nu_i\rceil}\mathbf{E}\tau_1\right\}.$$

Therefore, again, by Cramér's theorem,

$$\limsup_{n\to\infty} \frac{\log(\mathrm{I})}{n^{\alpha}} = -\infty. \tag{3.4}$$

Shifting our attention to (II),

$$\begin{aligned}
&\mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \ge na_1, \dots, \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \ge na_k, N(nt_i) \in E_i^{(n)}(\epsilon) \text{ for } i = 1, \dots, k\right) \\
&\le \sum_{i_1=\lceil n(t_1/\mathbf{E}\tau_1-\epsilon)\rceil}^{\lfloor n(t_1/\mathbf{E}\tau_1+\epsilon)\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}\tau_1-\epsilon)\rceil}^{\lfloor n(t_k/\mathbf{E}\tau_1+\epsilon)\rfloor} \mathbf{P}\left(\sum_{j=1}^{i_1} W_j \ge na_1, \dots, \sum_{j=i_{k-1}+1}^{i_k} W_j \ge na_k, N(nt_l) = i_l \text{ for } l = 1, \dots, k\right) \\
&\le \sum_{i_1=\lceil n(t_1/\mathbf{E}\tau_1-\epsilon)\rceil}^{\lfloor n(t_1/\mathbf{E}\tau_1+\epsilon)\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}\tau_1-\epsilon)\rceil}^{\lfloor n(t_k/\mathbf{E}\tau_1+\epsilon)\rfloor} \mathbf{P}\left(\sum_{j=1}^{i_1} W_j \ge na_1, \dots, \sum_{j=i_{k-1}+1}^{i_k} W_j \ge na_k\right) I(i_1 \le \cdots \le i_k) \\
&= \sum_{i_1=\lceil n(t_1/\mathbf{E}\tau_1-\epsilon)\rceil}^{\lfloor n(t_1/\mathbf{E}\tau_1+\epsilon)\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}\tau_1-\epsilon)\rceil}^{\lfloor n(t_k/\mathbf{E}\tau_1+\epsilon)\rfloor} \mathbf{P}\left(\sum_{j=1}^{i_1} W_j \ge na_1\right) \cdots \mathbf{P}\left(\sum_{j=i_{k-1}+1}^{i_k} W_j \ge na_k\right) \\
&\le (2\epsilon n)^k\, \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/\mathbf{E}\tau_1+\epsilon)\rfloor} W_j \ge na_1\right) \cdots \mathbf{P}\left(\sum_{j=\lceil n(t_{k-1}/\mathbf{E}\tau_1-\epsilon)\rceil}^{\lfloor n(t_k/\mathbf{E}\tau_1+\epsilon)\rfloor} W_j \ge na_k\right).
\end{aligned}$$

Now, we have that, from Result B.1 and Theorem 2.2,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n^{\alpha}} \log(\mathrm{II}) &\le \sum_{i=1}^{k} \limsup_{n\to\infty} \frac{1}{n^{\alpha}} \log \mathbf{P}\left(\sum_{j=\lceil n(t_{i-1}/\mathbf{E}\tau_1-\epsilon)\rceil}^{\lfloor n(t_i/\mathbf{E}\tau_1+\epsilon)\rfloor} W_j \ge na_i\right) + \limsup_{n\to\infty} \frac{\log(2\epsilon n)^k}{n^{\alpha}} \\
&\le -\mathcal{B}_0^* \sum_{i=1}^{k} (a_i - \lambda(\Delta t_i + 2\epsilon \mathbf{E}\tau_1))_+^{\alpha}.
\end{aligned}$$

Taking $\epsilon \to 0$, we arrive at

$$\limsup_{n\to\infty} \frac{1}{n^{\alpha}} \log(\mathrm{II}) \le -\mathcal{B}_0^* \sum_{i=1}^{k} (a_i - \lambda \Delta t_i)_+^{\alpha}. \tag{3.5}$$

In view of (3.4) and (3.5),

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^{\alpha}} \log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j \ge na_1, \dots, \sum_{j=N(nt_{i-1})}^{N(nt_i)-1} W_j \ge na_i, \dots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j \ge na_k\right) \\
&\le \max\left\{\limsup_{n\to\infty} \frac{\log(\mathrm{I})}{n^{\alpha}}, \limsup_{n\to\infty} \frac{\log(\mathrm{II})}{n^{\alpha}}\right\} \le -\mathcal{B}_0^* \sum_{i=1}^{k} (a_i - \lambda \Delta t_i)_+^{\alpha}.
\end{aligned}$$

For the lower bound in Equation (3.3), notice that

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \dots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k\right) \\
&\geq \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \dots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k, N(nt_i) \in E_i^{(n)}(\epsilon) \text{ for } i = 1, \dots, k\right) \\
&\geq \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E\tau_1-\epsilon)\rfloor-1} W_j > na_1, \dots, \sum_{j=\lceil n(t_{k-1}/E\tau_1+\epsilon)\rceil}^{\lfloor n(t_k/E\tau_1-\epsilon)\rfloor-1} W_j > na_k, N(nt_i) \in E_i^{(n)}(\epsilon) \text{ for } i = 1, \dots, k\right) \\
&\geq \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E\tau_1-\epsilon)\rfloor-1} W_j > na_1, \dots, \sum_{j=\lceil n(t_{k-1}/E\tau_1+\epsilon)\rceil}^{\lfloor n(t_k/E\tau_1-\epsilon)\rfloor-1} W_j > na_i\right) - (\mathrm{I}) \\
&= \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E\tau_1-\epsilon)\rfloor-1} W_j > na_1\right)\prod_{i=2}^{k}\mathbf{P}\left(\sum_{j=\lceil n(t_{i-1}/E\tau_1+\epsilon)\rceil}^{\lfloor n(t_i/E\tau_1-\epsilon)\rfloor-1} W_j > na_i\right) - (\mathrm{I}) \\
&= \underbrace{\mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E\tau_1-\epsilon)\rfloor-1} W_j > na_1\right)\prod_{i=2}^{k}\mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_i/E\tau_1-\epsilon)\rfloor-\lceil n(t_{i-1}/E\tau_1+\epsilon)\rceil} W_j > na_i\right)}_{=(\mathrm{III})} - (\mathrm{I}).
\end{aligned}
\tag{3.6}
$$

From Theorem 2.2, Result B.1, and (3.4), we get $\frac{(\mathrm{I})}{(\mathrm{III})} \to 0$ as $n \to \infty$. Therefore, (3.6) leads to

$$
\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n^\alpha}\log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \dots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k\right) \\
&\geq \liminf_{n\to\infty}\frac{1}{n^\alpha}\log\left\{(\mathrm{III})\left(1-\frac{(\mathrm{I})}{(\mathrm{III})}\right)\right\} = \liminf_{n\to\infty}\frac{1}{n^\alpha}\log(\mathrm{III}) \\
&= -\mathcal{B}_0^*\sum_{i=1}^{k}(a_i - \lambda(\Delta t_i - 2\epsilon\mathbf{E}\tau_1))_+^\alpha.
\end{aligned}
$$

Taking $\epsilon \to 0$, we arrive at (3.3) concluding the proof.

Our next lemma establishes the LDP for $\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)} W_j, \dots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$.

Lemma 3.3. *For any given $\mathbf{t} = (t_1, \dots, t_k)$ such that $0 = t_0 \leq t_1 < \cdots < t_k \leq 1$, the probability measures μ_n of $\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)} W_j, \dots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ satisfy the LDP in $\mathbb{R}_+^k$ w.r.t. Euclidean topology with the speed n^α and the good rate function $I_{\mathbf{t}}: \mathbb{R}_+^k \to \mathbb{R}_+$:*

$$
I_{\mathbf{t}}(x_1, \dots, x_k) \triangleq \begin{cases} \mathcal{B}_0^*\sum_{i=1}^{k}(x_i - \lambda\Delta t_i)^\alpha & \text{if } x_i \geq \lambda\Delta t_i,\ \forall i = 1, \dots, k, \\ \infty & \text{otherwise.} \end{cases}
\tag{3.7}
$$

Proof. Note that it is straightforward from (3.2) of Lemma 3.2 to see that $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \dots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ is exponentially tight by considering compact sets $\prod_{i=1}^{k}[0, a_i]$ for sufficiently large a_i's. Also, we claim that $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \dots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ satisfies a weak LDP. Once this claim is established, Dembo and Zeitouni [8, lemma 1.2.18] applies, showing that the full LDP is satisfied.

Now, to prove the claimed weak LDP, we start by showing that

$$\underbrace{\limsup_{n\to\infty}\frac{\log\mu_n(A)}{n^\alpha}}_{\triangleq\,\overline{\mathcal{L}}_A}=\underbrace{\liminf_{n\to\infty}\frac{\log\mu_n(A)}{n^\alpha}}_{\triangleq\,\underline{\mathcal{L}}_A} \tag{3.8}$$

for every $A\in\mathcal{A}\triangleq\{\prod_{i=1}^k((a_i,b_i)\cap\mathbb{R}_+): a_i<b_i\}$. Let

$$\mathcal{L}_A\triangleq\begin{cases}-\mathcal{B}_0^*\sum_{i=1}^k(a_i-\lambda\Delta t_i)_+^\alpha & \text{if } b_i\ge\lambda\Delta t_i \text{ for } i=1,\dots,k,\\ -\infty & \text{otherwise.}\end{cases}$$

We prove (3.8) by showing that $\overline{\mathcal{L}}_A\le\mathcal{L}_A\le\underline{\mathcal{L}}_A$. We consider two cases:

Case 1: $b_i\ge\lambda\Delta t_i$ for $i=1,\dots,k$.

Case 2: $b_i<\lambda\Delta t_i$ for some $i\in\{1,\dots,k\}$.

Let $A\triangleq\prod_{i=1}^k((a_i,b_i)\cap\mathbb{R}_+)$ and $a_i<b_i$ for $i=1,\dots,k$. We start with case 1. Because $A\subseteq\prod_{i=1}^k[a_i,b_i)$,

$$\overline{\mathcal{L}}_A\le\limsup_{n\to\infty}\frac{1}{n^\alpha}\log\mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j\ge na_1,\dots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\ge na_k\right)\le-\mathcal{B}_0^*\sum_{i=1}^k(a_i-\lambda\Delta t_i)^\alpha=\mathcal{L}_A, \tag{3.9}$$

where the second inequality is from (3.2). Because $\prod_{i=1}^k[a_i+\epsilon,b_i)\subseteq A$ for small enough $\epsilon>0$,

$$\begin{aligned}\underline{\mathcal{L}}_A&\ge\liminf_{n\to\infty}\frac{1}{n^\alpha}\log\mathbf{P}\left(\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)}W_j,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\right)\in\prod_{i=1}^k[a_i+\epsilon,b_i)\right)\\&\ge\liminf_{n\to\infty}\frac{1}{n^\alpha}\log\left\{\mathbf{P}\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)}W_j>a_1+\epsilon,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j>a_k+\epsilon\right)\right.\\&\qquad\left.-\sum_{l=1}^k\mathbf{P}\left(\sum_{j=N(nt_{i-1})+1}^{N(nt_i)}W_j\ge na_i\ \ \forall i\ne l,\sum_{j=N(nt_{l-1})+1}^{N(nt_l)}W_j\ge nb_l\right)\right\}\\&\ge\liminf_{n\to\infty}\frac{1}{n^\alpha}\log\mathbf{P}\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)}W_j>a_1+\epsilon,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j>a_k+\epsilon\right)\\&\quad+\liminf_{n\to\infty}\frac{1}{n^\alpha}\log\left(1-\frac{\sum_{l=1}^k\mathbf{P}\left(\frac{1}{n}\sum_{j=N(nt_{i-1})+1}^{N(nt_i)}W_j\ge a_i+\epsilon\ \ \forall i\ne l,\ \frac{1}{n}\sum_{j=N(nt_{l-1})+1}^{N(nt_l)}W_j\ge b_l\right)}{\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)}W_j>a_1+\epsilon,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j>a_k+\epsilon\right)}\right).\end{aligned} \tag{3.10}$$

Note that, because of the logarithmic asymptotics of Lemma 3.2, for every $l\in\{1,\dots,k\}$,

$$\frac{\mathbf{P}\left(\frac{1}{n}\sum_{j=N(nt_{i-1})+1}^{N(nt_i)}W_j\ge a_i+\epsilon \text{ for } i\in\{1,\dots,k\}\setminus l,\ \frac{1}{n}\sum_{j=N(nt_{j-1})+1}^{N(nt_l)}W_j\ge b_l\right)}{\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)}W_j>a_1+\epsilon,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j>a_k+\epsilon\right)}\to0,$$

and hence, the second term of (3.10) disappears. Therefore,

$$\begin{aligned}\underline{\mathcal{L}}_A&\ge\liminf_{n\to\infty}\frac{\log\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)}W_j>a_1+\epsilon,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j>a_k+\epsilon\right)}{n^\alpha}\\&\ge-\mathcal{B}_0^*\sum_{i=1}^k(a_i+\epsilon-\lambda\Delta t_i)^\alpha.\end{aligned}$$

Taking $\epsilon\to0$, we arrive at $\underline{\mathcal{L}}_A\ge\mathcal{L}_A$, which, together with (3.9), proves (3.8) for case 1.

For Case 2, note that, by Result B.1,

$$\overline{\mathcal{L}}_A \leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\sum_{j=N(nt_{i-1})+1}^{N(nt_i)} W_j < nb_i\right) = -\infty,$$

and hence, $\overline{\mathcal{L}}_A = \underline{\mathcal{L}}_A = \mathcal{L}_A = -\infty$.

Now, note also that

$$I_{\mathbf{t}}(x_1,\ldots,x_k) = \sup\{-\mathcal{L}_A : A \in \mathcal{A}, (x_1,\ldots,x_k) \in A\}. \tag{3.11}$$

Because $\mathcal{A}$ is a base of the Euclidean topology, the desired weak LDP follows from (3.8), (3.11), and Dembo and Zeitouni [8, theorem 4.1.11]. □

The following is an immediate corollary of Lemma 3.3.

Lemma 3.4. *For any given* $\mathbf{t} = (t_1,\ldots,t_k)$ *such that* $0 = t_0 \leq t_1 < \cdots < t_k \leq 1$, *the probability measures* (μ_n) *of* $\big(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j,\ldots,\frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\big)$ *satisfy an LDP in* $\mathbb{R}_+^k$ *with the speed* n^α *and with the good rate function* $\tilde{I}_{\mathbf{t}} : \mathbb{R}_+^k \to \mathbb{R}_+$,

$$\tilde{I}_{\mathbf{t}}(x_1,\ldots,x_k) \triangleq \begin{cases} \mathcal{B}_0^* \sum_{i=1}^{k} (x_i - x_{i-1} - \lambda\Delta t_i)^\alpha & \text{if } x_i - x_{i-1} \geq \lambda\Delta t_i, \text{ for } i = 1,\ldots,k, \\ \infty & \text{otherwise}. \end{cases} \tag{3.12}$$

Proof. The proof is an application of the contraction principle (Dembo and Zeitouni [8]). To this end, consider the function $f : \mathbb{R}_+^k \to \mathbb{R}_+^k, f(x_1,x_2,\ldots,x_k) \triangleq (x_1, x_1 + x_2,\ldots,x_1 + \cdots + x_k)$. Notice that

$$\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j,\ldots,\frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\right) = f\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j,\ldots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right),$$

where f is a continuous function. That is, $\big(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j,\ldots,\frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\big)$ satisfies a large deviation principle with the rate function $\tilde{I}_{\mathbf{t}}(y_1,\ldots,y_k) = \inf\{I_{\mathbf{t}}(x_1,\ldots,x_k) : y = f(x_1,\ldots,x_k)\}$. However, because $(y_1,\ldots,y_k) = f(x_1,\ldots,x_k)$, it is immediate that $y_1 \leq y_2 \leq \cdots \leq y_k$. Therefore,

$$\tilde{I}_{\mathbf{t}}(y_1,\ldots,y_k) = \begin{cases} \mathcal{B}_0^* \sum_{i=1}^{k} (y_i - y_{i-1} - \lambda\Delta t_i)^\alpha & \text{if } y_{i+1} - y_i \geq \lambda\Delta t_i \text{ for } i = 1,\ldots,k, \\ \infty & \text{otherwise}. \quad \square \end{cases}$$

Now, for a path $\xi \in \mathbb{D}[0,1]$, let

$$I_Z(\xi) \triangleq \begin{cases} \mathcal{B}_0^* \sum_{t:\xi(t)\neq\xi(t-)} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}^{(\lambda)}[0,1], \\ \infty & \text{otherwise}. \end{cases} \tag{3.13}$$

Because $\overline{Z}_n$ satisfies a finite-dimensional LDP, we can show that the Dawson–Gärtner projective limit theorem implies that $\overline{Z}_n$ satisfies an LDP in $\mathbb{D}[0,1]$ endowed with the pointwise convergence topology. The next lemma verifies that the rate function associated with the LDP of $\overline{Z}_n$ is indeed I_Z.

Lemma 3.5. *Let* $\mathbf{T} \triangleq \cup_{d=1}^{\infty} \{(t_1,\ldots,t_k) : 0 \leq t_1 < t_2 < \cdots < t_d \leq 1\}$ *be the collection of all ordered (in the increasing order) finite subsets of* $[0,1]$. *Then,*

$$\sup_{\mathbf{t}\in\mathbf{T}} \tilde{I}_{\mathbf{t}}(\xi(t_1),\ldots,\xi(t_k)) = I_Z(\xi).$$

Proof. The proof is essentially identical to the proof of Gantert [12, lemma 4] and, hence, omitted. □

We derive the sample-path LDP for the stochastic process $\overline{Z}_n$ w.r.t. the pointwise convergence topology, which we denote with $\mathcal{W}$. Recall that $\mathbb{D}^{(\lambda)}[0,1]$ denotes the subspace of increasing paths with slope λ.

Lemma 3.6. *The stochastic process* $\overline{Z}_n$ *satisfies a large deviation principle in* $(\mathbb{D}[0,1],\mathcal{W})$ *with the speed* n^α *and the good rate function* I_Z.

Proof. This is an immediate consequence of the Dawnson and Gärtner projective limit theorem, (Dembo and Zeitouni [8, theorem 4.6.1]) and Lemmas 3.4 and 3.5. □

Next, we establish the sample-path LDP for the stochastic process $\overline{Z}_n$ in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$.

Lemma 3.7. *The stochastic process $\overline{Z}_n$ satisfies a large deviation principle in $\mathbb{D}[0,1]$ w.r.t. the M_1' topology with the speed n^α and the good rate function I_Z.*

Proof. For the upper bound, consider a set $K_M \triangleq \{\xi \in \mathbb{D}[0,1] : \xi \text{ is nondecreasing}, \ \xi(0) \geq 0, \|\xi\|_\infty \leq M\}$. Let F be a closed set in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$. Then,

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Z}_n \in F) \leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log\{\mathbf{P}(\overline{Z}_n \in F \cap K_M) + \mathbf{P}(\overline{Z}_n \in K_M^c)\}$$

$$\leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log\left\{ \mathbf{P}(\overline{Z}_n \in F \cap K_M) + \mathbf{P}\left(\sum_{j=1}^{N(nt)} W_j \geq M\right)\right\}.$$

From Bazhba et al. [3, proposition A.2], one can check that pointwise convergence in K_M implies the convergence w.r.t. the M_1' topology, and K_M (and, hence, $F \cap K_M$ as well) is closed w.r.t. $\mathcal{T}_{M_1'}$. Suppose that ξ is in the closure of $F \cap K_M$ w.r.t. $\mathcal{W}$. Then, because of the aforementioned properties of K_M, there exists a sequence of paths $\{\xi_n\}$ in $F \cap K_M$ such that $\xi_n \to \xi$ w.r.t. $\mathcal{T}_{M_1'}$, which, in turn, implies that $\xi \in F \cap K_M$. That is, $F \cap K_M$ is closed in $\mathcal{W}$ as well. Now, applying the sample-path LDP w.r.t. $\mathcal{W}$ we have proved in Lemma 3.6, and then picking M large enough,

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Z}_n \in F) \leq \max\left\{ -\inf_{\xi \in F\cap K_M} I_Z(\xi), -\mathcal{B}_0^* M\right\} = -\inf_{\xi \in F \cap K_M} I_Z(\xi) \leq -\inf_{\xi\in F} I_Z(\xi).$$

Moving on to the lower bound, let G be an open set in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$. We assume that $I(G) < \infty$ because we have nothing to show otherwise. Fix an arbitrary $\xi \in G \cap \mathbb{D}^{(\lambda)}[0,1]$, and let k be such that an open ball of radius $\frac{1+\lambda}{k}$ around ξ is inside of G. That is, $B_{M_1'}\left(\xi; \frac{1+\lambda}{k}\right) \triangleq \left\{\zeta \in \mathbb{D}[0,1] : d_{M_1'}(\xi, \zeta) < \frac{1+\lambda}{k}\right\} \subseteq G$. Note that, because $\xi \in \mathbb{D}^{(\lambda)}[0,1]$ and $\overline{Z}_n$ is nondecreasing, $\left\{|\overline{Z}_n(i/k) - \xi(i/k)| < 1/k, \text{ for } i = 0, \ldots, k\right\} \subseteq \left\{\overline{Z}_n \in B_{M_1'}\left(\xi; \frac{1+\lambda}{k}\right)\right\}$. Therefore, in view of Lemma 3.4,

$$\liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Z}_n \in G) \geq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\overline{Z}_n \in B_{M_1'}\left(\xi, \frac{1+\lambda}{k}\right)\right)$$

$$\geq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(|\overline{Z}_n(i/k) - \xi(i/k)| < 1/k, \text{ for } i = 0, \ldots, k)$$

$$= -\inf_{(y_1,\ldots,y_k)\in\prod_{i=1}^k (\xi(i/k)-1/k,\, \xi(i/k)+1/k)} \tilde{I}_{\mathbf{t}}(y_1, \ldots, y_k)$$

$$\geq -\mathcal{B}_0^{*\,(p)} \sum_{i=1}^k (\xi(i/k) - \xi((i-1)/k) - \lambda/k)^\alpha$$

$$\geq -\mathcal{B}_0^{*\,(p)} \sum_{t:\xi(t)\neq\xi(t-)} (\xi(t) - \xi(t-))^\alpha = -I_Z(\xi).$$

Becaue ξ was an arbitrary element of $G \cap \mathbb{D}^{(\lambda)}[0,1]$, we arrive at the desired lower bound:

$$-\inf_{\xi\in G} I_Z(\xi) = -\inf_{\xi \in G\cap \mathbb{D}^{(\lambda)}[0,1]} I_Z(\xi) \leq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Z}_n \in G). \quad \square$$

Our next lemma shows that $\overline{Z}_n + \overline{S}_n$ is exponentially equivalent to $\overline{Y}_n$.

Lemma 3.8. *$\overline{Y}_n$ and $\overline{Z}_n + \overline{S}_n$ are exponentially equivalent in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$.*

Proof. Because of the construction of $\overline{Y}_n$, $\overline{Z}_n$, and $\overline{S}_n$, we have that, for any $\delta > 0$,

$$\{d_{M_1'}(\overline{Y}_n, \overline{Z}_n + \overline{S}_n) \geq \delta\} \subseteq \{(n - T_{N(n)})/n \geq \delta\} \cup \{\exists j \leq N(n) : \tau_j \geq n\delta\}. \tag{3.14}$$

To bound the probability of the first set, define $D_n(\epsilon) \triangleq \{N(n)/n \geq 1/\mathbf{E}\tau_1 - \epsilon\}$ for any $\epsilon > 0$ and notice that $\mathbf{P}((n - T_{N(n)})/n > \delta) = \mathbf{P}(T_{N(n)} \leq n(1-\delta)) = \mathbf{P}(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)) + \mathbf{P}(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)^c)$, and hence,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}((n - T_{N(n)})/n \geq \delta) \\
&\leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log\{\mathbf{P}(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)) + \mathbf{P}(D_n(\epsilon)^c)\} \\
&= \max\left\{\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)),\ \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(D_n(\epsilon)^c)\right\}.
\end{aligned} \tag{3.15}$$

Letting $\epsilon < \delta/(2\mathbf{E}\tau_1)$, we see that, from the same argument as the one preceding (3.4),

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)) &\leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(T_{\left\lfloor n\left(\frac{1}{\mathbf{E}\tau_1} - \epsilon\right)\right\rfloor} \leq n(1-\delta), D_n(\epsilon)\right) \\
&= \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(N(n(1-\delta)) \geq \left\lfloor n\left(\frac{1}{\mathbf{E}\tau_1} - \epsilon\right)\right\rfloor, D_n(\epsilon)\right) \\
&= -\infty.
\end{aligned}$$

Using the definition of a renewal process and Cramérs theorem, we obtain

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(D_n(\epsilon)^c) = -\infty. \tag{3.16}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}((n - T_{N(n)})/n > \delta) = -\infty. \tag{3.17}$$

Moving on to the bound for the probability of the second term in (3.14), for any $\epsilon > 0$,

$$\begin{aligned}
\mathbf{P}(\{\exists j \leq N(n) : \tau_j \geq n\delta\}) &= \mathbf{P}(\exists j \leq N(n) : \tau_j \geq n\delta, N(n)/n \leq 1/\mathbf{E}\tau_1 + \epsilon) + \mathbf{P}(N(n)/n > 1/\mathbf{E}\tau_1 + \epsilon) \\
&\leq \mathbf{P}(\exists j \leq \lceil n/\mathbf{E}(\tau_1) + n\epsilon\rceil : \tau_j \geq n\delta) + \mathbf{P}(N(n)/n > 1/\mathbf{E}\tau_1 + \epsilon) \\
&\leq \lceil n/\mathbf{E}(\tau_1) + n\epsilon\rceil \mathbf{P}(\tau_1 \geq n\delta) + \mathbf{P}(N(n)/n > 1/\mathbf{E}\tau_1 + \epsilon).
\end{aligned}$$

Because $\mathbf{P}(\tau_1 \geq n\delta)$ and $\mathbf{P}(N(n)/n > 1/\mathbf{E}\tau_1 + \epsilon)$ both decay at an exponential rate,

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\{\exists j \leq N(n) : \tau_j \geq n\delta\}) = -\infty.$$

This, along with (3.17) and (3.14), proves the desired exponential equivalence. □

Now, we have all the necessary components to prove Theorem 2.1.

Proof of Theorem 2.1. The preceding sequence of lemmas has resulted in LDPs of $\overline{Z}_n$ (Lemma 3.7) and $\overline{S}_n$ (Lemma 3.1). Because $\overline{Z}_n$ and $\overline{S}_n$ are independent, $(\overline{Z}_n, \overline{S}_n)$ satisfies an LDP in $\prod_{i=1}^2 \mathbb{D}[0,1]$ with the rate function $I_{Z,S}(\zeta, \xi) \triangleq I_Z(\zeta) + I_S(\xi)$; see, for example, Ganesh et al. [11, theorem 4.14].

Let $\phi : \prod_{i=1}^2 \mathbb{D}[0,1] \to \mathbb{D}[0,1]$ denote the addition function $\phi(\xi, \zeta) = \xi + \zeta$. Because ϕ is continuous on (ξ, ζ) as far as ξ and ζ do not share a jump time with opposite directions (which follows from a straightforward modification of Bazhba et al. [2, lemma B.1]), ϕ is continuous on the effective domain of $I_{Z,S}$. Let $I_{\mathbf{W}}(\zeta) \triangleq \inf\{I_{Z,S}(\xi_1, \xi_2) : \zeta = \xi_1 + \xi_2,\ \xi_1 \in \mathbb{D}^{(\lambda)}[0,1],\ \xi_2 \in \mathbb{D}^{\leqslant 1}[0,1]\}$, and note that it is straightforward to check that $I_W = I_Y$. By the extended contraction principle (see Puhalskii and Whitt [17]), we conclude that $\overline{Z}_n + \overline{S}_n$ satisfies the sample-path LDP with the rate function I_Y.

We now prove the large deviation upper bound. Let F be a closed set w.r.t. the M_1' topology, and let $F_\epsilon \triangleq \{\xi \in \mathbb{D}[0,1] : d_{M_1'}(\xi, F) \le \epsilon\}$. Then,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Y}_n \in F) \\
&= \limsup_{n\to\infty} \frac{1}{n^\alpha} \log\{\mathbf{P}(\overline{Y}_n \in F, d_{M_1'}(\overline{Y}_n, \overline{Z}_n + \overline{S}_n) \le \epsilon) + \mathbf{P}(d_{M_1'}(\overline{Y}_n, \overline{Z}_n + \overline{S}_n) > \epsilon))\} \\
&\le \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Z}_n + \overline{S}_n \in F_\epsilon) \le -\inf_{\xi \in F_\epsilon} I_Y(\zeta),
\end{aligned}$$

where the first inequality is due to Lemma 3.8. Note that $\lim_{\epsilon\to 0} \inf_{\xi\in F_\epsilon} I_Y(\zeta) = \inf_{\xi\in F} I_Y(\zeta)$, because I_W is good w.r.t. $\mathcal{T}_{M_1'}$ (see Bazhba et al. [3, proposition A.3]). The desired large deviation upper bound follows by taking $\epsilon \to 0$.

For the lower bound, let G be an open set in $\mathcal{T}_{M_1'}$. We assume that $\inf_{\xi\in G} I_Y(\xi) < \infty$ because the lower bound is trivial otherwise. For any given $\epsilon > 0$, pick $\zeta \in G$ such that $I(\zeta) \le \inf_{\xi\in G} I_Y(\xi) + \epsilon$. Let $\delta > 0$ be such that $B_{M_1'}(\zeta, 2\delta) \in G$. Then, we know from Lemma 3.8, $\mathbf{P}(d(\overline{Y}_n, \overline{Z}_n + \overline{S}_n) < \delta)/\mathbf{P}(\overline{Z}_n + \overline{S}_n \in B_{M_1'}(\zeta,\delta)) \to 0$, and hence,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Y}_n \in G) \\
&\ge \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Z}_n + \overline{S}_n \in B_{M_1'}(\zeta,\delta), d(\overline{Y}_n, \overline{Z}_n + \overline{S}_n) < \delta) \\
&\ge \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Z}_n + \overline{S}_n \in B_{M_1'}(\zeta,\delta)) \left\{1 - \frac{\mathbf{P}(d(\overline{Y}_n, \overline{Z}_n + \overline{S}_n) < \delta)}{\mathbf{P}(\overline{Z}_n + \overline{S}_n \in B_{M_1'}(\zeta,\delta))}\right\} \\
&= \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}(\overline{Z}_n + \overline{S}_n \in B_{M_1'}(\zeta,\delta)) \\
&\ge -\inf_{\xi \in B_{M_1'}(\zeta,\delta)} I_Y(\xi) \ge -I_Y(\zeta) \ge -\inf_{\xi\in G} I_Y(\xi) - \epsilon.
\end{aligned}$$

Taking $\epsilon \to 0$, we arrive at the desired lower bound. □

4. Tail Asymptotics for the Area of a Busy Period

Our focus in this section is on proving Theorems 2.2 and 2.3. In Section 4.1, we collect several analytic properties of key variational problems related to these theorems. The proofs of these analytic properties as well as some other analytic results are deferred to Section 5 to keep the focus of this section on probabilistic arguments as much as possible. In Section 4.2, we state two key propositions (Propositions 4.2 and 4.3), which are applied to providing the proofs of Theorems 2.2 and 2.3. The rest of this section is devoted to the proofs of Propositions 4.2 and 4.3. The focus is again on probabilistic ideas; the substantial number of additional analytical arguments that are directly needed are stated as lemmas of which the proofs can be found in Section 5.

4.1. Key Auxiliary Variational Problems and Related Properties

Recall that $B_y^{\mathbb{AC}} = B_y \cap \mathbb{AC}[0,\infty)$. Our first lemma establishes that the infimum defining $\mathcal{B}_y^*$, taken over paths of bounded variation, can be confined to absolutely continuous paths. Its proof is given in Section 5.1.

Lemma 4.1. *Recall that $\mathcal{B}_y^*$ is the optimal value of the variational problem ($\mathcal{B}_y$).*

i. *Let $\overline{y} \triangleq (|\mu|(p+1))^\alpha$. For any $y \ge \overline{y}$, there exists a path $\xi^* \in B_y^{\mathbb{AC}}$ so that $I_y(\xi^*) = 0$ and $\mathcal{B}_y^* = 0$.*
ii. *For any $y \ge 0$, $\mathcal{B}_y^* = \inf_{\xi\in B_y^{\mathbb{AC}}} I_y(\xi)$.*

We provide the proof of the following proposition in Section 5.2. It facilitates the proof of Proposition 4.3, which is a key result for the tail asymptotics of W_1 and V_n.

Proposition 4.1. *The optimal value $\mathcal{B}_y^*$ of ($\mathcal{B}_y$) satisfies the following properties:*

i. *$y \longmapsto \mathcal{B}_y^*$ is nonincreasing in y, $y \in [0, \overline{y}]$.*
ii. *$y \longmapsto \mathcal{B}_y^*$ is Lipschitz continuous, $y \in [0, \overline{y}]$.*
iii. *For every $y \in [0, \overline{y})$, $\mathcal{B}_y^* \in (0, \infty)$.*

Fix $T > 0$ and consider a functional $\Phi_T : \mathbb{D}[0,T] \to \mathbb{R}_+$, where $\Phi_T(\xi) = \int_0^T (\Psi(\xi)(s))^p ds$. Now, let $\mathcal{V}_y^{T*}$ denote the optimal value of the optimization problem

$$\mathcal{V}_y^{T*} \triangleq \inf_{\xi \in V_y^T} I_y^{\mathbb{BV}[0,T]}(\xi), \tag{$\mathcal{V}_y^T$}$$

where

$$V_y^T \triangleq \{\xi \in \mathbb{D}[0,T] : \xi(0) = y, \Phi_T(\xi) \geq 1\},$$

and

$$I_y^{\mathbb{BV}[0,T]}(\xi) \triangleq \begin{cases} \int_0^T \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+ \xi^{(u)}(T) + \theta_- \xi^{(d)}(T) & \text{if } \xi(0) = y \text{ and } \xi \in \mathbb{BV}[0,T], \\ \infty & \text{otherwise.} \end{cases}$$

The variational problem ($\mathcal{V}_y^T$) naturally appears in large deviations estimates. The next lemma, proved in Section 5.3, summarizes several of its properties.

Lemma 4.2. *Consider an arbitrary path $\xi \in \mathbb{BV}[0,T]$ and set $y \triangleq \xi(0)$.*

i. *There exists a path $\zeta_1 \in \mathbb{BV}[0,T]$ such that*
   - i-1. $\zeta_1(0) = y$.
   - i-2. $\Phi_T(\zeta_1) \geq \Phi_T(\xi)$.
   - i-3. $I_y^{\mathbb{BV}[0,T]}(\zeta_1) \leq I_y^{\mathbb{BV}[0,T]}(\xi)$.
   - i-4. *For some $t \in [0,T]$, ζ_1 is nonnegative over $[0,t]$ and ζ_1 is linear with slope μ over $[t,T]$.*

ii. *There exists a path $\zeta_2 \in \mathbb{AC}[0,T]$ and a $z \in [0, \xi^{(u)}(T)]$ such that*
   - ii-1. $\zeta_2(0) = y + z$.
   - ii-2. $\Phi_T(\zeta_2) \geq \Phi_T(\xi)$.
   - ii-3. $\theta_+ \cdot z + I_{y+z}^{\mathbb{BV}[0,T]}(\zeta_2) \leq I_y^{\mathbb{BV}[0,T]}(\xi)$, *where we interpret $\theta_+ \cdot z$ as zero if $\theta_+ = \infty$ and $z = 0$.*
   - ii-4. *For some $t \in [0,T]$, ζ_2 is nonnegative over $[0,t]$ and ζ_2 is linear with slope μ over $[t,T]$.*

iii. *If, in addition, $\xi \in \mathbb{AC}[0,T]$, there exists a path $\zeta_3 \in \mathbb{AC}[0,T]$ such that*
   - iii-1. $\zeta_3(0) = y$.
   - iii-2. $\Phi_T(\zeta_3) \geq \Phi_T(\xi)$.
   - iii-3. $I_y^{\mathbb{BV}[0,T]}(\zeta_3) \leq I_y^{\mathbb{BV}[0,T]}(\xi)$.
   - iii-4. *ζ_3 is concave over $[0,T]$ and its derivative is bounded by μ from below.*

4.2. Proof of Theorems 2.2 and 2.3

The following propositions are instrumental.

Proposition 4.2.

i. *Recall that $T_1 = \inf\{k > 0 : X_k = 0\}$. Then,*

$$\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^{T_1/x} (X_{\lfloor ux \rfloor}/x)^p du \geq 1\right) \leq -\mathcal{B}_y^*.$$

ii. *Recall that $W_1 = \sum_{i=1}^{T_1} X_i^p$. Then,*

$$\liminf_{u\to\infty} \frac{1}{u^\alpha} \log \mathbf{P}_0(W_1 > u) \geq -\mathcal{B}_0^*.$$

We provide the proof of Proposition 4.2 in Section 4.3.

Proposition 4.3.

i. $\sum_{k=0}^{m-1} X_k^p > x^{1+p} \iff \int_0^{m/x} (\frac{X_{\lfloor ux \rfloor}}{x})^p du > 1$.

ii. *It holds that*

$$\mathcal{B}_0^* = \mathcal{B}_\pi^*,$$

iii. *Finally,*

$$\lim_{k\to\infty}\min_{i\geq 1}\left\{\frac{i-1}{k}\beta\overline{y}+\mathcal{B}^*_{\frac{i}{k}\overline{y}}\right\}=\inf_{y\in[0,\infty)}\{\beta y+\mathcal{B}^*_y\}=\mathcal{B}^*_\pi.$$

We provide the proof of Proposition 4.3 in Section 4.4.

With Propositions 4.2 and 4.3 in our hands, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. For the upper bound, setting $t=x^{p+1}=x^{1/\alpha}$,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t^\alpha}\log\mathbf{P}(W_1\geq t)&=\limsup_{x\to\infty}\frac{1}{x}\log\mathbf{P}_0\left(\sum_{k=0}^{T_1-1}X_k^p\geq x^{1+p}\right)\\&=\limsup_{x\to\infty}\frac{1}{x}\log\mathbf{P}_0\left(\int_0^{T_1/x}\left(\frac{X_{\lfloor ux\rfloor}}{x}\right)^p du\geq 1\right)\\&\leq-\mathcal{B}^*_0.\end{aligned}$$

We used $\sum_{i=1}^{T_1}X_i^p=\sum_{i=0}^{T_1-1}X_i^p$ to obtain the first equality, applied part (i) of Proposition 4.3 to derive the second equality, and invoked part (i) of Proposition 4.2 to establish the inequality. Together with the matching lower bound in part (ii) of Proposition 4.2, this yields the desired asymptotics (2.4). □

The proof of Theorem 2.3 is slightly more involved.

Proof of Theorem 2.3. We start by proving the large deviation upper bound for $\frac{1}{n}V_n$. Denote the time-reversed Markov process of $\{X_k,k=1,\dots,n\}$ with $\{X_k^*,k=0,\dots,n\}$, and let $T_1^*\triangleq\inf\{i>0:X_i^*=0\}$. Let $\overline{y}\triangleq(|\mu|(p+1))^\alpha$, and fix $b>0$. Setting $x^{p+1}\triangleq nb$, we obtain that

$$\begin{aligned}\mathbf{P}_0\left(\frac{1}{n}V_n\geq b\right)&=\mathbf{P}_0\left(\sum_{i=T_{N(n)}+1}^{n}X_i^p\geq nb\right)=\frac{1}{\pi(0)}\mathbf{P}_\pi\left(\sum_{i=0}^{T_1^*-1}(X_i^*)^p\geq nb,X_n^*=0\right)\\&\leq\frac{n+1}{\pi(0)}\mathbf{P}_\pi\left(\sum_{i=0}^{T_1-1}X_i^p\geq nb\right)=\frac{n}{\pi(0)}\mathbf{P}_\pi\left(\int_0^{T_1/x}\left(\frac{X_{\lfloor ux\rfloor}}{x}\right)^p du\geq 1\right),\end{aligned}\tag{4.1}$$

where the second equality follows from Lemma A.1 with $g(y_0,\dots,y_n)=\mathbb{1}(\sum_{\max\{i\leq n:y_i=0\}}^n y_i^p>nb)$, the second inequality follows from the upper bound in Lemma A.2, and the last equality follows from part (i) of Proposition 4.3.

From the tower property, we have that

$$\begin{aligned}&\mathbf{P}_\pi\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du\geq 1\right)\\&=\mathbf{E}_\pi\left[\left(1\{X_0\geq x\overline{y}\}+\sum_{i=1}^k\mathbb{1}\left\{X_0\in\left[\frac{i-1}{k}x\overline{y},\ \frac{i}{k}x\overline{y}\right]\right\}\right)\mathbf{P}\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du\geq 1\middle|X_0\right)\right]\\&\leq\ \mathbf{E}_\pi\mathbb{1}\{X_0\geq x\overline{y}\}+\sum_{i=1}^k\mathbf{E}_\pi\left[1\left\{X_0\in\left[\frac{i-1}{k}x\overline{y},\ \infty\right)\right\}\mathbf{P}_{\frac{i}{k}x\overline{y}}\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du\geq 1\right)\right]\\&\leq\ \pi[x\overline{y},\infty)+\sum_{i=1}^k\pi\left[\frac{i-1}{k}x\overline{y},\ \infty\right)\mathbf{P}_{\frac{i}{k}x\overline{y}}\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du\geq 1\right),\end{aligned}\tag{4.2}$$

where, in the first inequality, we used that the Markov chain $\{X_n, n \geq 1\}$ is monotone with respect to the initial state. Therefore, by the principle of the maximum term and part (i) of Proposition 4.2,

$$\begin{aligned}
&\limsup_{x\to\infty}\frac{1}{x}\log \mathbf{P}_\pi\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du \geq 1\right)\\
&\leq \limsup_{x\to\infty}\frac{1}{x}\log \pi[x\overline{y},\infty)\\
&\quad\vee \max_{i=1,\dots,k}\left\{\limsup_{x\to\infty}\frac{1}{x}\log\left(\pi\left[\frac{i-1}{k}x\overline{y},\infty\right)\mathbf{P}_{\frac{i}{k}x\overline{y}}\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du \geq 1\right)\right)\right\}\\
&= (-\beta\overline{y}) \vee \max_{i=1,\dots,k}\left\{-\frac{i-1}{k}\beta\overline{y} + \limsup_{x\to\infty}\frac{1}{x}\log \mathbf{P}_{\frac{i}{k}x\overline{y}}\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du \geq 1\right)\right\}.\\
&\leq (-\beta\overline{y}) \vee \max_{i=1,\dots,k}\left\{-\frac{i-1}{k}\beta\overline{y} - \mathcal{B}^*_{\frac{i}{k}\overline{y}}\right\}.
\end{aligned}$$

Note that, because $\mathcal{B}^*_y = 0$ for $y \geq \overline{y}$ because of part (i) of Lemma 4.1,

$$(-\beta\overline{y}) \vee \max_{i=1,\dots,k}\left\{-\frac{i-1}{k}\beta\overline{y} - \mathcal{B}^*_{\frac{i}{k}\overline{y}}\right\} = \max_{i\geq 1}\left\{-\frac{i-1}{k}\beta\overline{y} - \mathcal{B}^*_{\frac{i}{k}\overline{y}}\right\} = -\min_{i\geq 1}\left\{\frac{i-1}{k}\beta\overline{y} + \mathcal{B}^*_{\frac{i}{k}\overline{y}}\right\}.$$

Taking $k \to \infty$ and applying parts (ii) and (iii) of Proposition 4.3,

$$\limsup_{x\to\infty}\frac{1}{x}\log \mathbf{P}_\pi\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du \geq 1\right) \leq -\mathcal{B}^*_\pi = -\mathcal{B}^*_0.$$

From this, along with (4.1), we arrive at the desired upper bound:

$$\limsup_{x\to\infty}\frac{1}{n^\alpha}\log \mathbf{P}_0\left(\frac{1}{n}V_n \geq b\right) \leq \limsup_{x\to\infty}\frac{1}{x}\log \mathbf{P}_\pi\left(\int_0^{T_1/x}(X_{\lfloor ux\rfloor}/x)^p du \geq 1\right)\cdot b^\alpha \leq -\mathcal{B}^*_0\cdot b^\alpha.$$

Next, for n sufficiently large, using the lower bound of Lemma A.2 for $n \geq n_0$,

$$\begin{aligned}
\mathbf{P}_0\left(\frac{1}{n}V_n \geq b\right) &= \mathbf{P}_0\left(\sum_{i=T_{N(n)}+1}^{n} X_i^p \geq nb\right) = \frac{1}{\pi(0)}\mathbf{P}_\pi\left(\sum_{i=0}^{T_1^*-1}(X_i^*)^p \geq nb, X_n^* = 0\right)\\
&\geq \frac{\pi(0)}{2}\mathbf{P}_0\left(\sum_{i=1}^{T_1} X_i^p > nb\right) - \mathcal{O}(e^{-cn}) = \frac{\pi(0)}{2}\mathbf{P}_0(W_1 > nb) - \mathcal{O}(e^{-cn}). \qquad (4.3)
\end{aligned}$$

We can now apply part (ii) of Proposition 4.2 to (4.3) and obtain the matching lower bound:

$$\liminf_{n\to\infty}\frac{1}{n^\alpha}\log \mathbf{P}_0\left(\frac{1}{n}V_n \geq b\right) \geq -\mathcal{B}^*_0\cdot b^\alpha. \quad \square$$

4.3. Proof of Proposition 4.2

We first state a number of preliminary results. These results are analytic in nature, and their proofs can be found in Sections 5.4 and 5.5. For a fixed $M > 0$, let $B_y^{\mathbb{AC};M} \triangleq B_y^{\mathbb{AC}} \cap \{\xi \in \mathbb{D}[0,\infty) : \mathcal{T}(\xi) \leq M\}$, and let $B_y^M \triangleq B_y \cap \{\xi \in \mathbb{D}[0,\infty) : \mathcal{T}(\xi) \leq M\}$.

Lemma 4.3. *For any given $y \geq 0$, there exists a constant $M = M(y) > 0$ such that*

i. *For each $\xi \in B_y^{\mathbb{AC}}$, there exists a path $\zeta \in B_y^{\mathbb{AC};M}$ satisfying $I_y(\zeta) \leq I_y(\xi)$.*

ii. *It holds that*

$$\inf_{\xi \in B_y^{\mathbb{A}\mathbb{C}}} I_y(\xi) = \inf_{\xi \in B_y^{\mathbb{A}\mathbb{C};M}} I_y(\xi). \tag{4.4}$$

iii. *Moreover,* $M(y) \leq cy + d$ *for some* $c > 0$ *and* $d > 0$.

Lemma 4.4. *Let* $M > 0$ *be the constant in Lemma 4.3. Then,* $\mathcal{B}_y^* = \mathcal{V}_y^{T*}$ *for any* $T \geq M$.

Set

$$K_t \triangleq \left\{ \xi \in \mathbb{D}[0,t] : \xi(0) = 0, \int_0^t (\Psi(\xi)(s))^p ds \geq 1, \ \xi(s) \geq 0 \text{ for } s \in [0,t] \right\}.$$

The following corollary is immediate from the previous lemma and Lemma 4.3.

Corollary 4.1. *Let* $M > 0$ *be the constant in Lemma 4.3. For any* $y \geq 0$,

$$\inf_{t \in [0,M]} \inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi) = \mathcal{V}_0^{M*} = \mathcal{B}_0^*.$$

Next, we formulate a key preparatory lemma. This lemma is motivated by a result of Vysotsky [21], stated as Result B.3(ii). To apply this result, we need to verify a uniform continuity result. The next lemma provides the desired uniform continuity, whose proof is deferred to Section 5.6.

Recall the function $\Phi_T : \mathbb{D}[0,T] \to [0,\infty)$ defined as $\Phi_T(\xi) = \int_0^T (\Psi(\xi)(s))^p ds$ and $I_0^{\mathbb{BV}[0,T]}$ defined as

$$I_0^{\mathbb{BV}[0,T]}(\xi) \triangleq \begin{cases} \int_0^T \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+(\xi^{(u)}(T)) + \theta_- |\xi^{(d)}(T)| & \text{if } \xi \in \mathbb{BV}[0,T] \text{ and } \xi(0) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 4.5. *For each* $\gamma \geq 0$, Φ_T *is uniformly continuous on the set* $\{\xi : I_0^{\mathbb{BV}[0,T]}(\xi) \leq \gamma\}$ *w.r.t. the* M_1' *metric.*

We apply this lemma in our next and final preparatory lemma.

Lemma 4.6.

i. *For any* $t, y \geq 0$ *and* $T > 0$,

$$\limsup_{x \to \infty} \frac{1}{x} \log \mathbf{P}_{xy}(T_1/x > T) \leq ty + T \log \mathbf{E}e^{tU}. \tag{4.5}$$

ii. *For any* $y \geq 0$ *and* $T > 0$,

$$\limsup_{x \to \infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^T (X_{\lfloor sx \rfloor}/x)^p ds \geq 1 \right) \leq -\mathcal{V}_y^{T*}. \tag{4.6}$$

Proof of Lemma 4.6. For part (i), note that

$$\mathbf{P}_{xy}(T_1 > xT) \leq \mathbf{P}_{xy}(X_{\lfloor xT \rfloor} > 0) = \mathbf{P}\left(\sum_{i=1}^{\lfloor xT \rfloor} U_i > -xy \right) \leq e^{txy} \mathbf{E}(e^{tU})^{\lfloor xT \rfloor},$$

where the last inequality is from the Markov inequality. Taking logarithms, dividing both sides by x, and taking lim sup, we get (4.5).

For part (ii), note that, conditional on $X_0 = xy$, $\int_0^T (X_{\lfloor sx \rfloor}/x)^p ds = \Phi_T(\overline{K}_x + y)$. From Lemma 5.2, we know that Φ_T is uniformly continuous over the sublevel sets of the rate function $I_y^{\mathbb{BV}[0,T]}$ of $\overline{K}_x + y$. Hence, we can apply Result B.3(ii) to obtain

$$\lim_{x \to \infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^T (X_{\lfloor sx \rfloor}/x)^p ds \geq 1 \right) \leq -\inf_{a \in [1,\infty)} J_y(a),$$

where $J_y(a) \triangleq \inf\{I_0^{\mathbb{BV}[0,T]}(\xi) : \xi \in \mathbb{D}[0,T], \xi(0) = y, \Phi_T(\xi) = a\}$. Obviously, $\inf_{a \in [1,\infty)} J_y(a) = \mathcal{V}_y^{T*}$, and (4.6) follows. □

Now we are ready to prove Proposition 4.2.

Proof of Proposition 4.2. For part (i), consider a small enough $t_0 > 0$ so that $\mathbf{E}e^{t_0 U} < 1$, which is possible because $\mathbf{E}U < 0$ and U is light-tailed. Then, thanks to Lemma 4.4, we can pick a sufficiently large $T > 0$ so that $\mathcal{B}_y^* = \mathcal{V}_y^{T*}$ and $t_0 y + T \log \mathbf{E}e^{t_0 U} < -\mathcal{B}_y^*$. Considering the case $T_1/x \le T$ and $T_1/x > T$ separately and then applying the principle of the maximum term,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^{T_1/x} (X_{\lfloor ux \rfloor}/x)^p du \ge 1\right) \\
&\le \limsup_{x\to\infty} \frac{1}{x} \log \left\{ \mathbf{P}_{xy}\left(\int_0^{T_1/x} (X_{\lfloor ux \rfloor}/x)^p du \ge 1,\, T_1/x \le T\right) + \mathbf{P}_{xy}(T_1/x > T) \right\} \\
&\le \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^{T} (X_{\lfloor ux \rfloor}/x)^p du \ge 1\right) \vee \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}(T_1/x > T) \\
&\le (-\mathcal{V}_y^{T*}) \vee (t_0 y + T \log \mathbf{E}e^{t_0 U}) = (-\mathcal{B}_y^*) \vee (t_0 y + T \log \mathbf{E}e^{t_0 U}) = -\mathcal{B}_y^*,
\end{aligned} \tag{4.7}$$

where we used Lemma 4.6 for the third inequality.

Next, we move on to part (ii). For any given $t > 0$, let

- $A_{t,\epsilon} \triangleq \{\xi \in \mathbb{D}[0,t] : \xi(0) = \epsilon,\ \int_0^t \Psi(\xi)(s)^p ds > 1,\ \xi(s) > 0,\ \ \forall s \in [0,t]\}$.
- $\tilde{A}_{t,\epsilon} \triangleq \{\xi \in \mathbb{D}[0,t] : \xi(0) = \epsilon,\ \int_0^t \Psi(\xi)(s)^p ds > 1,\ \xi(s) > \epsilon/2,\ \ \forall s \in [0,t]\}$.

Set $u = x^{1+p}$. Let ϵ be small enough such that $\mathbf{P}(U_1 > \sqrt{\epsilon}) > 0$. Define the event $B_{x,\epsilon} \triangleq \{U_i > \sqrt{\epsilon}, i = 1, \ldots, \lceil x\sqrt{\epsilon} \rceil\}$. Setting $k^* = \lceil x\sqrt{\epsilon} \rceil + 1$, we obtain

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{u^\alpha} \log \mathbf{P}_0(W_1 > u) \\
&= \liminf_{x\to\infty} \frac{1}{x} \log \mathbf{P}_0\left(\sum_{k=0}^{T_1} X_k^p > u\right) \\
&\ge \liminf_{x\to\infty} \frac{1}{x} \log \mathbf{P}_0\left(\sum_{k=k^*}^{T_1} X_k^p > x^{1+p},\, B_{x,\epsilon}\right) \\
&= \liminf_{x\to\infty} \frac{1}{x} \log \left[\mathbf{P}_0\left(\sum_{k=k^*}^{T_1} X_k^p > x^{1+p} \,|\, B_{x,\epsilon}\right) \mathbf{P}_0(B_{x,\epsilon})\right] \\
&\ge \liminf_{x\to\infty} \frac{1}{x} \log \left[\mathbf{P}_{\epsilon x}\left(\sum_{k=0}^{T_1} X_k^p > x^{1+p}\right) \mathbf{P}_0(B_{x,\epsilon})\right] \\
&= \liminf_{x\to\infty} \frac{1}{x} \log \left[\mathbf{P}_{\epsilon x}\left(\int_0^{T_1/x} (X_{\lfloor sx \rfloor}/x)^p ds > 1\right) \mathbf{P}_0(B_{x,\epsilon})\right] \\
&\ge \liminf_{x\to\infty} \frac{1}{x} \log \left[\mathbf{P}_{\epsilon x}\left(\int_0^{t} (X_{\lfloor sx \rfloor}/x)^p ds > 1,\, T_1 > xt\right) \mathbf{P}_0(B_{x,\epsilon})\right] \\
&= \liminf_{x\to\infty} \frac{1}{x} \log \left[\mathbf{P}_{\epsilon x}\left(\int_0^{t} (X_{\lfloor sx \rfloor}/x)^p ds > 1,\, X_{\lfloor sx \rfloor}/x > 0,\ \ \forall s \in [0,t]\right) \mathbf{P}_0(B_{x,\epsilon})\right] \\
&\ge \liminf_{x\to\infty} \frac{1}{x} \log [\mathbf{P}_\epsilon(\overline{K}_x \in A_{t,\epsilon}) \mathbf{P}_0(B_{x,\epsilon})] \\
&\ge -\inf_{\xi \in (A_{t,\epsilon})^\circ} I_\epsilon^{\mathbb{BV}[0,t]}(\xi) + \sqrt{\epsilon} \log \mathbf{P}(U_1 > \sqrt{\epsilon}) \\
&\ge -\inf_{\xi \in \tilde{A}_{t,\epsilon}} I_\epsilon^{\mathbb{BV}[0,t]}(\xi) + \sqrt{\epsilon} \log \mathbf{P}(U_1 > \sqrt{\epsilon}),
\end{aligned}$$

where the third equality is from part (i) of Proposition 4.3. The second-to-last inequality follows from part (i) of Result B.3 because the integral and the infimum are both continuous in the M_1 topology (see Whitt [22, respectively, theorems 11.5.1 and 13.4.1]). Recall that

$$K_t = \left\{ \xi \in \mathbb{D}[0,t] : \xi(0) = 0,\ \int_0^t (\Psi(\xi)(s))^p ds \ge 1,\ \xi(s) \ge 0 \text{ for } s \in [0,t] \right\}.$$

Note that, for all $\epsilon > 0$,

$$\inf_{\xi \in \tilde{A}_{t,\epsilon}} I_\epsilon^{\mathbb{BV}[0,t]}(\xi) \le \inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi). \tag{4.8}$$

To see this, suppose that $\xi \in K_t$. Then, $\tilde{\xi} = \epsilon + \xi$ belongs to $\tilde{A}_{t,\epsilon}$ and $I_\epsilon^{\mathbb{BV}[0,t]}(\tilde{\xi}) = I_0^{\mathbb{BV}[0,t]}(\xi)$. Because the construction holds for every $\xi \in K_t$, we have that $\inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi) \ge \inf_{\xi \in \tilde{A}_{t,\epsilon}} I_\epsilon^{\mathbb{BV}[0,t]}(\xi)$. Therefore,

$$\liminf_{u\to\infty} \frac{1}{u^\alpha} \log \mathbf{P}_0(W_1 > u) \ge -\inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi) + \sqrt{\epsilon} \log \mathbf{P}(U_1 > \sqrt{\epsilon}).$$

Because ϵ and t are arbitrary, taking $\epsilon \to 0$ and taking the infimum over $t \in [0, M]$, Corollary 4.1 gives

$$\liminf_{u\to\infty} \frac{1}{u^\alpha} \log \mathbf{P}_0(W_1 > u) \ge -\inf_{t\in[0,M]} \inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi) = -\mathcal{B}_0^*. \quad \square$$

4.4. Proof of Proposition 4.3

For part (i), note that

$$\frac{1}{x^{1+p}} \sum_{k=0}^{m-1} X_k^p = \frac{1}{x^{1+p}} \int_0^m X_{\lfloor u \rfloor}^p du = \frac{1}{x^{1+p}} \int_0^{m/x} x X_{\lfloor xs \rfloor}^p ds = \int_0^{m/x} \left(\frac{X_{\lfloor xs \rfloor}}{x} \right)^p ds,$$

where the second equality is from the change of variable $u = xs$. The claimed equivalence is immediate from this.

For part (iii), note that

$$\lim_{k\to\infty} \min_{i\ge1} \left\{ \frac{i-1}{k} \beta \overline{y} + \mathcal{B}^*_{\frac{i}{k}\overline{y}} \right\} = \lim_{k\to\infty} \left(\min_{i\ge1} \left\{ \beta \frac{i}{k} \overline{y} + \mathcal{B}^*_{\frac{i}{k}\overline{y}} \right\} - \frac{1}{k} \beta \overline{y} \right) = \lim_{k\to\infty} \min_{i\ge1} \left\{ \beta \frac{i}{k} \overline{y} + \mathcal{B}^*_{\frac{i}{k}\overline{y}} \right\}.$$

Moreover, from part (ii) of Proposition 4.1,

$$\lim_{k\to\infty} \min_{i\ge1} \left\{ \frac{i}{k} \beta \overline{y} + \mathcal{B}^*_{\frac{i}{k}\overline{y}} \right\} = \inf_{y\in[0,\infty)} \{ \beta y + \mathcal{B}_y^* \}.$$

For part (ii), note that, by definition, $\mathcal{B}_0^* \ge \mathcal{B}_\pi^*$. Therefore, we only have to prove that $\mathcal{B}_0^* \le \mathcal{B}_\pi^*$. Recall that $\beta = \sup\{\theta > 0 : \mathbf{E}(e^{\theta U}) \le 1\}$ and $\theta_+ = \sup\{\theta \in \mathbb{R} : \mathbf{E}(e^{\theta U}) < \infty\}$. For the rest of this proof, let Λ be the log-moment-generating function, and let D_Λ denote the effective domain of Λ, that is, $D_\Lambda = \{x : \Lambda(x) < \infty\}$. We start with a claim: for any $\epsilon > 0$, there exists a $u > 0$ such that

$$\Lambda^*(u)/u \le \beta + \epsilon. \tag{4.9}$$

To prove (4.9), we distinguish between the cases $\beta < \theta_+$ and $\beta = \theta_+$. For the first case, note that $\beta \in D_\Lambda^\circ$. In view of the convexity and continuity of $\mathbf{E}(e^{\theta U})$, $\mathbf{E}(e^{\beta U}) = 1$. Because of Dembo and Zeitouni [8, lemma 2.2.5(c)], Λ is a differentiable function in D_Λ° with $\Lambda'(\eta) = \frac{\mathbf{E}(Ue^{\eta U})}{\mathbf{E}(e^{\eta U})}$. Because $\beta \in D_\Lambda^\circ$, we have that $\Lambda'(\beta) = \mathbf{E}(Ue^{\beta U}) < \infty$. In addition, $\Lambda'(0) = \mathbf{E}(U) < 0$ implies that $\Lambda(\eta)$ is decreasing for small values of η. Now, the strict convexity and differentiability of Λ over its effective domain implies that Λ' is increasing at β, and thus, $\mathbf{E}(Ue^{\beta U}) > 0$. It can be checked that, for $u = \mathbf{E}(Ue^{\beta U})$,

$$\frac{\Lambda^*(u)}{u} = \frac{\beta \mathbf{E}(Ue^{\beta U}) - \log \mathbf{E}(e^{\beta U})}{\mathbf{E}(Ue^{\beta U})} = \beta,$$

and hence, our claim is proved. Consider now the case $\beta = \theta_+$. In view of Mogulskii [15, equation (5.5)], $\lim_{x\to\infty} \frac{\Lambda^*(x)}{x} = \theta_+$. That is, for any $\epsilon > 0$, we can choose a u so that $\Lambda^*(u)/u \le \theta_+ + \epsilon = \beta + \epsilon$. We proved the claim (4.9).

Back to the inequality $\mathcal{B}_0^* \le \mathcal{B}_\pi^*$, we show that, for any given $\epsilon > 0$ and any given path $\xi \in B_y$, we can construct a path $\zeta \in B_0$ so that $I_0(\zeta) \le I_y(\xi) + \beta y + \epsilon$. To this end, let $u > 0$ be such that $\Lambda^*(u)/u \le \beta + \epsilon/y$ and set

$$\zeta(s) \triangleq us \mathbb{1}_{\{s \le y/u\}} + \xi(s - y/u) \mathbb{1}_{\{s > y/u\}}.$$

Then, $\zeta(0)=0$, $\zeta(y/u)=y$, and $\zeta\in B_0$. Also, one can see that $I_y(\zeta)=(y/u)\Lambda^*(u)+\int_0^{\mathcal{T}(\xi)}\Lambda^*(\dot\xi(s))ds+\theta_+\xi^{(u)}(\mathcal{T}(\xi))=(y/u)\Lambda^*(u)+I_y(\xi)$. From the construction of u,

$$I_0(\zeta)\le\beta y+\epsilon+I_y(\xi)$$

as desired. This concludes the proof of part (iv). □

5. Additional Technical Lemmas and Proofs

5.1. Proof of Lemma 4.1

We first state a lemma on the monotonicity of the reflection map that is useful in the proof of Lemma 4.1.

Lemma 5.1. *Suppose that $\alpha,\beta,\gamma\in\mathbb{D}[0,T]$, $\alpha(s)=\beta(s)+\gamma(s)$, and $\gamma(s)$ is nonnegative and nondecreasing. Then, $\Psi(\alpha)(t)\ge\Psi(\beta)(t)$ for all $t\in[0,T]$.*

Proof. The proof of this lemma is an immediate consequence of Ramasubramanian [19, theorem 14.2.2]. □

Proof of Lemma 4.1. For (i), consider $\xi^*(t)\triangleq y+\mu\cdot t$ and note that $I_y(\xi^*)=0$. Also, $y\ge\overline{y}$ implies that $\xi^*\in B_y$, and hence, $\mathcal{B}_y^*=0$. For part (ii), we first show that $\mathcal{B}_y^*=\inf_{\xi\in B_y\cap J_+}I_y(\xi)$, where $J_+\triangleq\{\xi\in\mathbb{D}[0,\infty):\xi^{(d)}\equiv0\}$. To prove this, let $\xi\in B_y$ be given arbitrarily. We construct a path ξ_2 so that $\xi_2\in B_y\cap J_+$ and $I_y(\xi_2)\le I_y(\xi)$. Toward this, discard the downward jumps of ξ and let $\xi_1\triangleq\xi^{(a)}+\xi^{(u)}$. Then, set $\xi_2(t)\triangleq\xi_1(t\wedge\mathcal{T}(\xi))+\mu([t-\mathcal{T}(\xi)]^+)$, $t\ge0$. In view of Lemma 5.1, $\Psi(\xi_2)\ge\Psi(\xi)$ over $[0,\mathcal{T}(\xi)]$, and hence, $\xi_2\in B_y$ and $I_y(\xi)=I_y(\xi_1)+\theta_-\xi^{(d)}(\mathcal{T}(\xi))=I_y(\xi_2)+\theta_-\xi^{(d)}(\mathcal{T}(\xi))\ge I_y(\xi_2)$. This proves $\mathcal{B}_y^*=\inf_{\xi\in B_y\cap J_+}I_y(\xi)$.

Now, suppose that $\xi\in B_y\cap J_+$ is given. It is sufficient to prove that there exists a constant $c_\xi\in(0,\infty)$ such that, for any given $\epsilon>0$, one can construct a $\tilde\xi_\epsilon\in B_y^{\mathbb{AC}}$ such that

$$I_y(\tilde\xi_\epsilon)\le I_y(\xi)+\epsilon\cdot c_\xi. \tag{5.1}$$

In case $y\ge\overline{y}$, simply consider $\tilde\xi_\epsilon(t)\triangleq y+\mu\cdot t$ regardless of ξ and ϵ. Then, $\tilde\xi_\epsilon\in B_y^{\mathbb{AC}}$ and $I_y(\tilde\xi_\epsilon)=0\le I_y(\xi)+\epsilon\cdot c_\xi$ with $c_\xi=0$. In case $y<\overline{y}$, we first prove the following claim w.r.t. the local rate function Λ^*: for any $\epsilon>0$, there exists a $y_\epsilon\in\mathbb{R}_+$ such that

$$\Lambda^*(y)\le(\theta_++\epsilon)y,\quad\forall y\ge y_\epsilon.$$

Because this is trivially true if $\theta_+=\infty$, we prove the claim with the assumption that $\theta_+<\infty$. Note that, because of Jensen's inequality and the monotonicity of the map $x\longmapsto\log x$, we see that $\Lambda(\theta)=\log(\mathbf{E}(e^{\theta U_1}))\ge\log e^{\theta\mathbf{E}(U_1)}=\mu\cdot\theta$. This implies, for any $\theta<\theta_+$,

$$y\theta-\Lambda(\theta)\le y\theta-\mu\theta\le y\theta_+-\mu\theta_+\le(\theta_++\epsilon)y,\quad\forall y\ge-\mu\theta_+/\epsilon.$$

Therefore,

$$\Lambda^*(y)=\sup_{\theta\le\theta_+}\{y\theta-\Lambda(\theta)\}\le(\theta_++\epsilon)y,\qquad\forall y\ge-\mu\theta_+/\epsilon,$$

which is the claim with $y_\epsilon=-\mu\theta_+/\epsilon$.

To conclude the proof, pick $\xi\in B_y\cap J_+$, and assume w.l.o.g. that $I_y(\xi)<\infty$ (because (5.1) is trivial otherwise). For any given $\epsilon>0$, we construct a path $\tilde\xi_\epsilon$ such that $\tilde\xi_\epsilon\in B_y^{\mathbb{AC}}$ and $I_y(\tilde\xi_\epsilon)\le I_y(\xi)+\epsilon\cdot\xi^{(u)}(\mathcal{T}(\xi))$, which is (5.1) with $c_\xi=\xi^{(u)}(\mathcal{T}(\xi))$. Set $\tilde\xi_\epsilon(t)=y+y_\epsilon t$ for $t\in[0,\xi^{(u)}(\mathcal{T}(\xi))/y_\epsilon]$ and $\tilde\xi_\epsilon(t)=\xi^{(a)}(t-\xi^{(u)}(\mathcal{T}(\xi))/y_\epsilon)+\xi^{(u)}(\mathcal{T}(\xi))$ for $t>\xi^{(u)}(\mathcal{T}(\xi))/y_\epsilon$ with $y_\epsilon=-\mu\theta_+/\epsilon$. Then, it is straightforward to check that $\tilde\xi_\epsilon\in B_y^{\mathbb{AC}}$ from its construction and Lemma 5.1. In addition, $\tilde\xi_\epsilon(\mathcal{T}(\xi)+\xi^{(u)}(\mathcal{T}(\xi))/y_\epsilon)=\xi^{(a)}(\mathcal{T}(\xi))+\xi^{(u)}(\mathcal{T}(\xi))=0$ because $\xi\in B_y\cap J_+$, and hence, $\mathcal{T}(\tilde\xi_\epsilon)\le\mathcal{T}(\xi)+\xi^{(u)}(\mathcal{T}(\xi))/y_\epsilon$. Consequently,

$$\begin{aligned}I_y(\tilde\xi_\epsilon)&=\int_0^{\mathcal{T}(\tilde\xi_\epsilon)}\Lambda^*(\dot{\tilde\xi}_\epsilon(s))ds=\frac{\xi^{(u)}(\mathcal{T}(\xi))}{y_\epsilon}\Lambda^*(y_\epsilon)+\int_{\frac{\xi^{(u)}(\mathcal{T}(\xi))}{y_\epsilon}}^{\mathcal{T}(\xi)+\frac{\xi^{(u)}(\mathcal{T}(\xi))}{y_\epsilon}}\Lambda^*(\dot\xi^{(a)}(s-\xi^{(u)}(\mathcal{T}(\xi))/y_\epsilon))ds\\&\le\xi^{(u)}(\mathcal{T}(\xi))(\theta_++\epsilon)+\int_0^{\mathcal{T}(\xi)}\Lambda^*(\dot\xi^{(a)}(s))ds\le I_y(\xi)+\epsilon\cdot\xi^{(u)}(\mathcal{T}(\xi)).\end{aligned}$$

We have arrived at the desired inequality. □

5.2. Proof of Proposition 4.1

For part (i), let x, y be such that $0 \le x < y$. We show that, for any $\epsilon > 0$, there exists $\zeta \in B_y^{\mathbb{AC}}$ such that $I_y(\zeta) < \mathcal{B}_x^* + \epsilon$. To show this, note from Lemmas 4.1 and 4.3 that there exists $\xi \in B_x^{\mathbb{AC}}$ such that $I_x(\xi) < \mathcal{B}_x^* + \epsilon$ and $\mathcal{T}(\xi) < \infty$. Set

$$\zeta(t) \triangleq (y - x) + \xi(t \wedge \mathcal{T}(\xi)) + \mu \cdot [t - \mathcal{T}(\xi)]^+.$$

Because $\zeta(0) = y$ and $\zeta(t) - \xi(t)$ is nonnegative and nondecreasing on $t \in [0, \mathcal{T}(\xi)]$, Lemma 5.1 implies that $\Psi(\zeta)(t) > \Psi(\xi)(t)$ on $t \in [0, \mathcal{T}(\xi)]$, and hence, $\zeta \in B_y^{\mathbb{AC}}$. Moreover, note that $\mathcal{T}(\xi) \le \mathcal{T}(\zeta)$, and $\Lambda^*(\dot\zeta(s)) = \Lambda^*(\mu) = 0$ on $s \in [\mathcal{T}(\xi), \mathcal{T}(\zeta)]$. Therefore,

$$I_y(\zeta) = \int_0^{\mathcal{T}(\zeta)} \Lambda^*(\dot\zeta(s))ds = \int_0^{\mathcal{T}(\xi)} \Lambda^*(\dot\zeta(s))ds = \int_0^{\mathcal{T}(\xi)} \Lambda^*(\dot\xi(s))ds = I_x(\xi) < \mathcal{B}_x^* + \epsilon.$$

For part (ii), note that we only need to prove one side of the inequality thanks to part (i). That is, it is enough to show that, if $0 \le x < y$, then $\mathcal{B}_x^* \le \mathcal{B}_y^* + (y - x)\Lambda^*(1)$. Fix an $\epsilon > 0$ and pick $\zeta \in B_y^{\mathbb{AC}}$ such that $I_y(\zeta) \le \mathcal{B}_y^* + \epsilon$, which is always possible because of part (ii) of Lemma 4.1. Set

$$\xi(t) \triangleq (x + t)\mathbb{1}_{[0, y-x]}(t) + \zeta(t - (y - x))\mathbb{1}_{[y-x, \infty)}(t).$$

It follows that $\mathcal{T}(\xi) = \mathcal{T}(\zeta) + y - x$, and hence,

$$\begin{aligned} I_x(\xi) &= \int_0^{y-x} \Lambda^*(1)ds + \int_{y-x}^{\mathcal{T}(\xi)} \Lambda^*(\dot\xi(s))ds = (y - x)\Lambda^*(1) + \int_{y-x}^{\mathcal{T}(\zeta)+y-x} \Lambda^*(\dot\xi(s))ds \\ &= (y - x)\Lambda^*(1) + \int_0^{\mathcal{T}(\zeta)} \Lambda^*(\dot\xi(s + (y - x)))ds = (y - x)\Lambda^*(1) + \int_0^{\mathcal{T}(\zeta)} \Lambda^*(\dot\zeta(s))ds \\ &= (y - x)\Lambda^*(1) + I_y(\zeta) \le (y - x)\Lambda^*(1) + \mathcal{B}_y^* + \epsilon. \end{aligned}$$

Because $\xi \in B_x$, $\mathcal{B}_x^* \le (y - x)\Lambda^*(1) + \mathcal{B}_y^* + \epsilon$. Taking $\epsilon \to 0$ yields (ii).

For part (iii), we first note that $\mathcal{B}_y^* < \infty$: if we set $\xi(t) \triangleq (|\mu|(p+1))^\alpha(1 + \mu(t - 1))\mathbb{1}_{[1,\infty)}(t)$, then $\xi \in B_y^{\mathbb{AC}}$ and $I_y(\xi) < \infty$. To show that $\mathcal{B}_y^* > 0$, we can take $y \in (0, \overline{y})$ without loss of generality in view of (i). Define $\overline{y}(t) = \overline{y} + \mu t$ and set $T_\mu = \mathcal{T}(\overline{y}(\cdot)) = \overline{y}/|\mu|$. Write

$$\mathcal{B}_y^* = \min\left\{ \inf_{t \in [0, T_\mu]} \inf_{\xi \in B_y^{\mathbb{AC}}, \mathcal{T}(\xi) = t} I_y(\xi), \inf_{t \in [T_\mu, \infty]} \inf_{\xi \in B_y^{\mathbb{AC}}, \mathcal{T}(\xi) = t} I_y(\xi) \right\}. \tag{5.2}$$

We show that each of the two infima in this minimum is strictly positive. Observe that

$$\begin{aligned} \inf_{t \in [T_\mu, \infty]} \inf_{\xi \in B_y^{\mathbb{AC}}, \mathcal{T}(\xi) = t} I_y(\xi) &\ge \inf_{t \in [T_\mu, \infty]} \inf_{\xi : \xi(0) = y, \xi \in \mathbb{AC}, \mathcal{T}(\xi) = t} \int_0^t \Lambda^*(\dot\xi(s))ds \\ &\ge \inf_{t \in [T_\mu, \infty]} t\Lambda^*(-y/t) \ge \frac{\overline{y}}{|\mu|}\Lambda^*\left(\mu\frac{y}{\overline{y}}\right) > 0, \end{aligned}$$

using that Λ^* is a convex nonnegative function with $\Lambda^*(\mu) = 0$, and Jensen's inequality.

To lower bound the double infimum in (5.2), observe that, because $y < \overline{y}$, the area constraint $\int_0^t \xi(s)^p ds \ge 1$ can only be valid if there exists an $s \in [0, t]$ such that $\xi(s) \ge \overline{y} + \mu s$. Consequently,

$$\begin{aligned} \inf_{\xi \in B_y^{\mathbb{AC}}, \mathcal{T}(\xi) = t} I_y(\xi) &\ge \inf_{s \le t} \inf_{\xi \in \mathbb{AC}, \xi(0) = y, \xi(s) \ge \overline{y} + \mu s} I_y(\xi) \\ &\ge \inf_{s \le t} \inf_{\xi \in \mathbb{AC}, \xi(0) = y, \xi(s) \ge \overline{y} + \mu s} \int_0^s \Lambda^*(\dot\xi(u))du \\ &= \inf_{s \le t} \inf_{\xi \in \mathbb{AC}, \xi(s) - \xi(0) \ge \overline{y} - y + \mu s} \int_0^s \Lambda^*(\dot\xi(u))du \\ &\ge \inf_{s \le t} s\Lambda^*\left(\frac{\overline{y} - y}{s} + \mu\right), \end{aligned}$$

where we applied Jensen's inequality in the last step. Consequently,

$$\inf_{t\in[0,T_\mu]}\inf_{\xi\in B_y^{\mathbb{A}\mathbb{C}},\mathcal{T}(\xi)=t} I_y(\xi) \geq \inf_{s\leq T_\mu} s\Lambda^*\left(\frac{\overline{y}-y}{s}+\mu\right) = \frac{\overline{y}}{|\mu|}\Lambda^*\left(\mu+|\mu|\frac{\overline{y}-y}{\overline{y}}\right) > 0.$$

The equality holds because Λ^* is a strictly convex nonnegative function with $\Lambda^*(\mu)=0$. □

5.3. Proof of Lemma 4.2

For part (i), we first construct a new trajectory ξ_1 from ξ by discarding the downward jumps, that is, $\xi_1 = \xi^{(a)}+\xi^{(u)}$. Obviously, $I_y^{\mathbb{BV}[0,T]}(\xi_1) \leq I_y^{\mathbb{BV}[0,T]}(\xi)$. Note that $\xi_1 = \xi + (-\xi^{(d)})$, where $-\xi^{(d)}$ is nonnegative and nondecreasing. From Lemma 5.1, we have that $\Psi(\xi_1)(t) \geq \Psi(\xi)(t)$ for all $t\in[0,T]$, and hence, $\Phi_T(\xi_1)\geq\Phi_T(\xi)$. For each $t\in[0,T]$, let $l(t) \triangleq \inf\{s\in[0,T]: \Psi(\xi)(u)>0 \text{ for all } u\in[s,t]\}$, $r(t)\triangleq\sup\{s\in[0,T]: \Psi(\xi)(u)>0 \text{ for all } u\in[t,s]\}$, and $\sigma(t)\triangleq[l(t),r(t))$. Set $\mathcal{C}_1^+\triangleq\{\sigma(t)\subseteq[0,T]: t\in[0,T]\}$. Note that, by construction, the elements of $\mathcal{C}_1^+$ cannot overlap, and hence, there can be at most countable number of elements in $\mathcal{C}_1^+$. In view of this, we write $\mathcal{C}_1^+ = \{[l_i,r_i): i\in\mathbb{N}\}$ and let $\sigma_i\triangleq[l_i,r_i)$. The following observations are immediate from the construction of $\mathcal{C}_1^+$, the right continuity of ξ, and the fact that ξ_1 does not have any downward jumps.

O1. If $t\in[0,T)$ does not belong to any of the elements of $\mathcal{C}_1^+$, then $\Psi(\xi_1)(t)=0$.

O2. $\Psi(\xi_1)$ is continuous on the right end of the intervals σ_i except for the case $r_i=T$.

Note that O1 also implies that $\xi_1(t)=\xi_1(t-)$ for such t's. Let $s_n\triangleq\sum_{i=1}^{n-1}(r_i-l_i)$ for $n\in\mathbb{N}$. Note that $s_n\to s_\infty\in[0,T]$ as $n\to\infty$. Let $\dot{\xi}^{(a)}(t)$ denote the time derivative $\frac{d}{dt}\xi^{(a)}(t)$ of $\xi^{(a)}$ at t, and set

$$\zeta_1(t)\triangleq y+\int_0^t\dot{\zeta}_1(s)ds+\zeta_1^{(u)}(t),$$

where

$$\dot{\zeta}_1(t)\triangleq\sum_{i\in\mathbb{N}}\dot{\xi}^{(a)}(t-s_i+l_i)\mathbb{1}_{[s_i,s_{i+1})}(t)+\mu\mathbb{1}_{[s_\infty,T]}(t),$$

and

$$\zeta_1^{(u)}(t)\triangleq\sum_{i\in\mathbb{N}}(\xi^{(u)}(t\wedge s_{i+1}-s_i+l_i)-\xi^{(u)}(l_i-))\mathbb{1}_{[s_i,T]}(t).$$

That is, on the interval $[s_i,s_{i+1})$, ζ_1 behaves the same way as ξ_1 does on the interval $[l_i,r_i)$, whereas ζ_1 decreases linearly at the rate $|\mu|$ outside of those intervals. Given this, it can be checked that

O3. $\int_{s_i}^{s_{i+1}}(\Psi(\zeta_1)(s))^p\,ds\geq\int_{l_i}^{r_i}(\Psi(\xi_1)(s))^p ds$.

O4. $\int_{l_i}^{r_i}\Lambda^*(\dot{\xi}^{(a)}(s))ds=\int_{s_i}^{s_{i+1}}\Lambda^*(\dot{\zeta}_1^{(a)}(s))ds$.

O5. $\zeta_1^{(u)}(s_{i+1}-)-\zeta_1^{(u)}(s_i-)=\xi_1^{(u)}(r_i-)-\xi_1^{(u)}(l_i-)$.

Now, we verify the conditions i-1, i-2, i-3, and i-4. Note first that the conditions i-1 and i-4 are obvious from the construction of ζ_1. We can verify i-2 as follows:

$$\begin{aligned}\Phi_T(\xi_2) &= \int_0^T(\Psi(\zeta_1)(s))^p ds\geq\int_0^{s_\infty}(\Psi(\zeta_1)(s))^p ds=\sum_{i=1}^{\infty}\int_{s_i}^{s_{i+1}}(\Psi(\zeta_1)(s))^p ds\\ &\geq\sum_{i=1}^{\infty}\int_{l_i}^{r_i}(\Psi(\xi_1)(s))^p\,ds=\int_0^T(\Psi(\xi_1)(s))^p\,ds=\Phi_T(\xi_1),\end{aligned}$$

where the second inequality is from O3, and the second-to-last equality is from O1. Moving onto i-3, note that, because of the left continuity of ζ_1, $s_n\to s_\infty$ implies that $\xi(s_n-)\to\xi(s_\infty-)$. Also, $\zeta_1^{(u)}(s_\infty)-\zeta_1^{(u)}(s_\infty-)=0$ and $\zeta_1^{(u)}$ is constant on $[s_\infty,T]$. Therefore, $\sum_{i=1}^{\infty}(\zeta_1^{(u)}(s_{i+1}-)-\zeta_1^{(u)}(s_i-))=\lim_{n\to\infty}\zeta_1^{(u)}(s_{n+1}-)=\zeta_1^{(u)}(s_\infty-)=\zeta_1^{(u)}(T)$, where we adopted the convention that $\zeta_1^{(u)}(0-)=0$. From O4, O5, and this observation,

$$\begin{aligned}
I_y^{\mathbb{BV}[0,T]}(\zeta_1) &= \int_0^T \Lambda^*(\dot{\zeta}_1(t))ds + \theta_+\cdot\zeta_1^{(u)}(T)\\
&= \sum_{i=1}^{\infty}\int_{s_i}^{s_{i+1}} \Lambda^*(\dot{\zeta}_1(t))ds + \theta_+\cdot\sum_{i=1}^{\infty}(\zeta_1^{(u)}(s_{i+1}-)-\zeta_1^{(u)}(s_i-))\\
&= \sum_{i=1}^{\infty}\int_{l_i}^{r_i} \Lambda^*(\dot{\xi}^{(a)}(t))ds + \theta_+\cdot\sum_{i=1}^{\infty}(\xi^{(u)}(r_i-)-\xi^{(u)}(l_i-))\\
&\le \int_0^T \Lambda^*(\dot{\xi}^{(a)}(t))\,ds + \theta_+\cdot\xi^{(u)}(T) = I_y^{\mathbb{BV}[0,T]}(\xi_1).
\end{aligned}$$

For part (ii), we construct ζ_2 from ζ_1 by moving all the jumps of $\xi^{(u)}$ to time 0. This neither increases $I_y^{\mathbb{BV}[0,T]}$ nor decreases Φ_T. That is, if we set

$$\zeta_2(t) \triangleq y + \int_0^t \dot{\zeta}_1(s)ds + \zeta_1^{(u)}(T),$$

then $\Phi_T(\zeta_2)\ge\Phi_T(\zeta_1)$ obviously, and $\theta_+\cdot\zeta_1^{(u)}(T)+I^T_{y+\zeta_1^{(u)}(T)}(\zeta_2)\le I_y^T(\zeta_1)$. Noting that $\zeta_1^{(u)}(T)\le\xi^{(u)}(T)$, we see that ζ satisfies all the claims of the lemma.

For part (iii), let $\zeta\in\mathbb{AC}[0,T]$ be a concave majorant of ξ. Then, there exists a nonincreasing $\dot{\zeta}\in\mathbb{D}[0,T]$ such that $\zeta(t)=\xi(0)+\int_0^t\dot{\zeta}(s)ds$. (Because of the continuity of ξ, $\xi(0)$ and $\zeta(0)$ should coincide.) Let $\zeta_3(t)\triangleq\xi(0)+\int_0^t\mu\vee\dot{\zeta}(s)ds$. Note that iii-1, iii-2, and iii-4 are straightforward to check from the construction. To show that iii-3 is also satisfied, we construct $\mathcal{C}_2^+\triangleq\{(l_i',r_i')\subseteq[0,T]:i\in\mathbb{N}\}$ in a similar way to $\mathcal{C}_1^+$ so that the elements of $\mathcal{C}_2^+$ are nonoverlapping, and $\xi(s)<\zeta_3(s)$ if and only if $s\in(l_i',r_i')$ for some $i\in\mathbb{N}$. Note that, because of the continuity of ζ and ξ, $\zeta(l_i')=\xi(l_i')$ and $\zeta(r_i')=\xi(r_i')$, and ζ has to be a straight line on (l_i',r_i') for each $i\in\mathbb{N}$. Set $s_0\triangleq 0\vee\sup\{t\in[0,T]:\dot{\zeta}(t)\ge\mu\}$. Then, no interval in $\mathcal{C}_2^+$ contains s_0 because, otherwise, ζ has to be a straight line in a neighborhood of s_0, and hence, $\dot{\zeta}$ has to be constant there, but this is contradictory to the definition of s_0. Now, let $\dot{\xi}$ denote a derivative of ξ. Then, $\int_{l_i'}^{r_i'}\Lambda^*(\mu\wedge\dot{\zeta}(s))ds=\int_{l_i'}^{r_i'}\Lambda^*(\mu)ds=0$ for i's such that $r_i'>s_0$, and hence,

$$\begin{aligned}
I_y^{\mathbb{BV}[0,T]}(\xi)-I_y^{\mathbb{BV}[0,T]}(\zeta_3) &= \int_0^T\Lambda^*(\dot{\xi}(s))ds-\int_0^T\Lambda^*(\mu\vee\dot{\zeta}(s))ds\\
&\ge \sum_{i\in\mathbb{N}:\,r_i'\le s_0}\int_{l_i'}^{r_i'}(\Lambda^*(\dot{\xi}(s))-\Lambda^*(\dot{\zeta}(s)))ds.
\end{aligned}$$

Note that, from the construction of $\mathcal{C}_2^+$, if $s\in[l_i',r_i']$ for some i such that $r_i'\le s_0$, we have that $\dot{\zeta}(s)=(\zeta_3(r_i')-\zeta_3(l_i'))/(r_i'-l_i')=(\xi(r_i')-\xi(l_i'))/(r_i'-l_i')$, and hence, from Jensen's inequality,

$$\begin{aligned}
\int_{l_i'}^{r_i'}(\Lambda^*(\dot{\xi}(s))-\Lambda^*(\dot{\zeta}(s)))ds &= \int_{l_i'}^{r_i'}\Lambda^*(\dot{\xi}(s))ds-\int_{l_i'}^{r_i'}\Lambda^*((\xi(r_i')-\xi(l_i'))/(r_i'-l_i'))ds\\
&= \int_{l_i'}^{r_i'}\Lambda^*(\dot{\xi}(s))ds-(r_i'-l_i')\cdot\Lambda^*\left(\int_{l_i'}^{r_i'}\dot{\xi}(s)ds/(r_i'-l_i')\right)\\
&\ge 0.
\end{aligned}$$

Therefore, ζ_3 satisfies iii-3 as well. □

5.4. Proof of Lemma 4.3

Recall that $\overline{y}\triangleq(|\mu|(p+1))^{1/1+p}$. If $y\ge\overline{y}$, the equality in (4.4) holds with the optimal values of the left- (LHS) and right-hand sides (RHS) both being zero: to see this, we invoke Lemma 4.1, part (i). Moving on to the case $y<\overline{y}$, it is enough to show that there exists $M>0$ such that

$$\text{for any given } \xi\in B_y^{\mathbb{AC}}\setminus B_y^{\mathbb{AC};M},\text{ one can find } \zeta\in B_y^{\mathbb{AC};M}\text{ such that } I_y(\zeta)\le I_y(\xi). \tag{5.3}$$

To construct such M, consider w and z such that $\mu < w < 0 < z$, $\Lambda^*(w) < \infty$, and $\Lambda^*(z) < \infty$. We consider a piecewise linear path

$$\zeta(t) \triangleq (y + zt)\mathbb{1}_{[0,(\overline{y}-y)/z]}(t) + (\overline{y} + \mu(t - (\overline{y} - y)/z))\mathbb{1}_{[(\overline{y}-y)/z,\,\infty)}(t)$$

so that $\zeta \in B_y^{\mathbb{AC};M}$ and $I_y(\zeta) = \Lambda^*(z)\frac{\overline{y}-y}{z}$. Let

$$M \triangleq \max\left\{\frac{(\overline{y}-y)\Lambda^*(z)}{z\Lambda^*(w)},\, (\overline{y}-y)/z - \overline{y}/\mu,\, -y/w\right\}.$$

Suppose that $\xi \in B_y^{\mathbb{AC}} \setminus B_y^{\mathbb{AC};M}$, and hence, $\mathcal{T}(\xi) > M$. Note that $\mu < w < -y/\mathcal{T}(\xi)$ by the construction of M. We can now estimate

$$I_y(\xi) = \int_0^{\mathcal{T}(\xi)} \Lambda^*(\dot{\xi}(s))ds \geq \mathcal{T}(\xi)\cdot\Lambda^*(-y/\mathcal{T}(\xi)) \geq \mathcal{T}(\xi)\cdot\Lambda^*(w) \geq M\cdot\Lambda^*(w) \geq \frac{(\overline{y}-y)}{z}\Lambda^*(z) = I_y(\zeta).$$

In this derivation, the first inequality is from Jensen's inequality. The second inequality follows because $\Lambda^*(x)$ is nondecreasing in $x \geq \mu$. The third and fourth inequalities are from the choice of ξ and the construction of M, respectively. This concludes the proof of (5.3).

To see the existence of $c > 0$ and $d > 0$, note that, for the case $y \geq \overline{y}$, our construction of $M(y)$ is linear in y, whereas $M(y)$ is bounded for the case $y < \overline{y}$. □

5.5. Proof of Lemma 4.4

The proof that $\mathcal{B}_y^* = \mathcal{V}_y^{*T}$ for sufficiently large t's follows immediately from the following claims along with Lemma 4.1.

Claim 5.1. $\mathcal{V}_y^{T*}$ is nonincreasing in T.

Proof of Claim 5.1. Let $t_1 < t_2$. For each $\xi_1 \in V_y^{t_1}$, consider $\xi_2(s) \triangleq \xi_1(s \wedge t_1) + \mu(s - t_1)\mathbb{1}_{(t_1,t_2]}(t)$. Then, $\xi_2 \in V_y^{t_2}$ and $I_y^{\mathbb{BV}[0,t_1]}(\xi_1) = I_y^{\mathbb{BV}[0,t_2]}(\xi_2)$. Therefore, $\mathcal{V}_y^{t_2*}$ is at least as small as $\mathcal{V}_y^{t_1*}$.

Claim 5.2. If $M > 0$ is such that $\inf_{\xi\in B_y^{\mathbb{AC};M}} I_y(\xi) = \inf_{\xi\in B_y^{\mathbb{AC}}} I_y(\xi)$ as in Lemma 4.3, then

$$\inf_{\xi\in B_y^{\mathbb{AC};M}} I_y(\xi) \geq \mathcal{V}_y^{M*}.$$

Proof of Claim 5.2. Given an $\epsilon > 0$, consider $\xi_\epsilon \in B_y^{\mathbb{AC};M}$ such that $I_y(\xi_\epsilon) \leq \inf_{\xi\in B_y^{\mathbb{AC};M}} I_y(\xi) + \epsilon$. Set $\zeta_\epsilon(t) \triangleq \xi_\epsilon(t \wedge \mathcal{T}(\xi_\epsilon)) + \mu(t - \mathcal{T}(\xi_\epsilon))\mathbb{1}_{(\mathcal{T}(\xi_\epsilon),M]}(t)$. Then, $\zeta_\epsilon \in V_y^M$, and hence,

$$\mathcal{V}_y^{M*} = \inf_{\xi\in V_y^M} I_y^{\mathbb{BV}[0,M]}(\xi) \leq I_y^{\mathbb{BV}[0,M]}(\zeta_\epsilon) = I_y(\xi_\epsilon) \leq \inf_{\xi\in B_y^M} I_y(\xi) + \epsilon.$$

Taking $\epsilon \to 0$, we arrive at Claim 5.2.

Claim 5.3. For any $T > 0$,

$$\inf_{\xi\in B_y} I_y(\xi) \leq \mathcal{V}_y^{T*}.$$

Proof of Claim 5.3. (Throughout this proof, we interpret $\theta_+\cdot z$ as zero if $\theta_+ = \infty$ and $z = 0$. Likewise, we interpret $\epsilon/z = \infty$ if $z = 0$.) First note that the claim is trivial if $\mathcal{V}_y^{T*} = \infty$, and hence, we only consider the case that $\mathcal{V}_y^{T*} < \infty$. Fix an $\epsilon > 0$ and consider $\xi_1 \in V_y^T$ such that $I_y^{\mathbb{BV}[0,T]}(\xi_1) < \mathcal{V}_y^{T*} + \epsilon$. (Note that this implies that $\xi_1^{(u)}(T) = 0$ if $\theta_+ = \infty$.) Because of Lemma 4.2, part (ii), there exists a path $\xi_2 \in \mathbb{AC}[0,T]$ such that $\xi_2(0) = y + z$, $0 \leq z \leq \xi_1^{(u)}(T)$, ξ_2 is nonnegative over $[0,t_0]$ for some $t_0 \in [0,T]$, ξ_2 is affine with slope μ over $[t_0,T]$, $\Phi_T(\xi_2) \geq \Phi_T(\xi_1) \geq 1$, and

$$\theta_+\cdot z + I_{y+z}^{\mathbb{BV}[0,T]}(\xi_2) \leq I_y^{\mathbb{BV}[0,T]}(\xi_1) \leq \mathcal{V}_y^{T*} + \epsilon. \tag{5.4}$$

Recall the well-known property of Λ^* that $\lim_{x\to\infty}\frac{\Lambda^*(x)}{x}=\theta_+$ (see, for example, Mogulskii [15, equation (5.5)]). Consequently, we can choose a $u>0$ large enough so that

$$\Lambda^*(u)/u \le \theta_+ + \epsilon/z. \tag{5.5}$$

Set $\hat{T} \triangleq z/u + T$ and consider $\xi_3 \in \mathbb{AC}[0,\hat{T}]$ such that

$$\xi_3(s) = (y+us)\mathbb{1}_{[0,z/u]}(s) + \xi_2(s-z/u)\mathbb{1}_{(z/u,\hat{T}]}(s),\ s\in[0,\hat{T}].$$

Then, $\xi_3(0)=y$ and $\Phi_{\hat{T}}(\xi_3)\ge\Phi_T(\xi_2)\ge 1$. Moreover, in view of (5.4) and (5.5),

$$\begin{aligned}
I_y^{\mathbb{BV}[0,\hat{T}]}(\xi_3) &= (z/u)\Lambda^*(u) + \int_{z/u}^{\hat{T}}\Lambda^*(\dot{\xi}_3(s))ds\\
&= (z/u)\Lambda^*(u) + \int_0^T \Lambda^*(\dot{\xi}_2(s))ds\\
&\le \theta_+\cdot z + \epsilon + I_{y+z}^{\mathbb{BV}[0,T]}(\xi_2)\\
&\le \mathcal{V}_y^{T*} + 2\epsilon.
\end{aligned}$$

Next, from part (iii) of Lemma 4.2, we know that there exists a path $\xi_4\in\mathbb{AC}[0,\hat{T}]$ such that $\xi_4(0)=y$, $\Phi_{\hat{T}}(\xi_4)\ge\Phi_{\hat{T}}(\xi_3)\ge 1$, $I_y^{\mathbb{BV}[0,\hat{T}]}(\xi_4)\le I_y^{\mathbb{BV}[0,\hat{T}]}(\xi_3)\le\mathcal{V}_y^{T*}+2\epsilon$, ξ_4 is concave on $[0,\hat{T}]$, and $\dot{\xi}_4$ is bounded by μ from below. Finally, define $\xi\in\mathbb{D}[0,\infty)$ as

$$\xi(t) \triangleq \xi_4(t\wedge\hat{T}) + \mu([t-\hat{T}]^+),\quad t\ge 0.$$

Note that, if $\mathcal{T}(\xi)\le\hat{T}$, because of the concavity of ξ, $\Psi(\xi)$ and $\Psi(\xi_4)$ are zero after $\mathcal{T}(\xi)$. Therefore, $\Phi(\xi)=\Phi_{\hat{T}}(\xi_4)\ge 1$. If $\mathcal{T}(\xi)>\hat{T}$, then $\Phi(\xi)>\Phi_{\hat{T}}(\xi_4)\ge 1$. That is, $\xi\in B_y$ in all cases. Moreover, because $\dot{\xi}(t)=\mu$ for $t\ge\hat{T}$, $I_y(\xi)=I_y^{\mathbb{BV}[0,\hat{T}]}(\xi_4)\le\mathcal{V}_y^{T*}+2\epsilon$. Therefore,

$$\inf_{\xi\in B_y} I_y(\xi) \le \mathcal{V}_y^{T*} + 2\epsilon.$$

Because ϵ is arbitrary, this proves Claim 5.3. □

5.6. Uniform Continuity: Proof of Lemma 4.5

We first state two preparatory lemmas. Let $\mathrm{TV}(\xi)$ be the total variation of ξ.

Lemma 5.2. *For any $M<\infty$, the function $H:\mathbb{D}[0,T]\to[0,\infty)$ given by $H(\xi)\triangleq\int_0^T\xi(s)ds$ is Lipschitz continuous on the set $\{\xi:\mathrm{TV}(\xi)+\xi(0)\le M\}$ w.r.t. the M_1' metric.*

Proof. Let ξ be such that $\mathrm{TV}(\xi)+\xi(0)\le M$, and let ζ be such that $d_{M_1'}(\xi,\zeta)\le\epsilon$. Set $\eta_-(t)\triangleq\inf\{x:d((x,t),\Gamma(\xi))\le\epsilon\}$, where $\Gamma(\xi)$ is the completed graph of ξ and d is the L_1 distance in $\mathbb{R}^2$, that is, $d((x,t),(u,s))=|x-y|+|t-s|$. Then, $d_{M_1'}(\xi,\zeta)\le\epsilon$ implies that $\zeta(t)\ge\eta_-(t)$ for all $t\in[0,T]$. Note that, if we denote the arc length of $\Gamma(\xi)$ with $\mathrm{len}(\Gamma(\xi))$, then $\mathrm{len}(\Gamma(\xi))$ is bounded by $T+\mathrm{TV}(\xi)+\xi(0)$. Because of the construction of η and the fact that L_1 balls are contained in L_2 balls of the same radius, the difference between the area below ξ and the area below η is bounded by $\mathrm{len}(\Gamma(\xi))\cdot\epsilon$. Putting everything together, we conclude that

$$\int_0^T\xi(s)ds-\int_0^T\zeta(s)\le\int_0^T\xi(s)ds-\int_0^T\eta_-(s)\le\mathrm{len}(\Gamma(\xi))\cdot\epsilon\le(T+\mathrm{TV}(\xi)+\xi(0))\cdot\epsilon\le(T+M)\cdot\epsilon.$$

Similarly, by majorizing ζ with $\eta_+(t)\triangleq\inf\{x:d((x,t),\Gamma(\xi))\le\epsilon\}$, we also get

$$\int_0^T\xi(s)ds-\int_0^T\zeta(s)\ge(T+M)\cdot\epsilon,$$

proving the Lipschitz continuity of H with Lipschitz constant $(T+M)$. □

Lemma 5.3. *The reflection map Ψ is a Lipschitz continuous map from $\mathbb{D}[0,T]$ to $\mathbb{D}[0,T]$ w.r.t. the M_1' topology with Lipschitz constant two.*

Proof. The proof is a straightforward adaptation of the proof of Lipschitz continuity given in Whitt [22, theorem 13.5.1] for the M_1 topology. That theorem is based on elementary estimates, and the key in Whitt [22, lemma 13.5.3], which establishes that parametric representation of a path $\xi \in \mathbb{D}[0,T]$ is preserved under taking projections. The proof of this property for M_1, given in Whitt [22], extends to M_1' by using the definition of an extended completed graph Γ' for M_1' rather than the completed graph Γ for M_1. Along these lines, it follows that, if (u, t) is a parametric representation of $\Gamma'(\xi)$, then $(\Psi(u), t)$ is a parametric representation of $\Gamma'(\Psi(\xi))$. Using this result, the steps in the proof of Whitt [22, theorem 13.5.1] follow verbatim for M_1'. We omit the details. □

Now we are ready to prove Lemma 4.5.

Proof of Lemma 4.5. Suppose that $I_0^{\mathbb{BV}[0,T]}(\xi), I_0^{\mathbb{BV}[0,T]}(\zeta) \in [0,\gamma]$. Then, from Vysotsky [21, inequality (13)], we know that there exists γ' such that $TV(\xi), TV(\zeta) \in [0,\gamma']$. To prove the uniform continuity, suppose that $d_{M_1'}(\xi,\zeta) < \epsilon$. Then, $d_{M_1'}(\Psi(\xi),\Psi(\zeta)) < 2\epsilon$ by Lemma 5.3, and $TV(\Psi(\xi)), TV(\Psi(\zeta)) \in [0,2\gamma']$. In turn, we have that $d_{M_1'}(\Psi(\xi)\vee\epsilon, \Psi(\zeta)\vee\epsilon) < 2\epsilon$ and $TV(\Psi(\xi)\vee\epsilon), TV(\Psi(\zeta)\vee\epsilon) \in [0,2\gamma']$. Using the mean value theorem, we obtain the following inequality for $x,y,a,b \in [0,\infty)$ such that $x,y \in [a,b]$:

$$|x^p - y^p| \le p(a^{p-1} \vee b^{p-1})|x-y|. \tag{5.6}$$

Now, suppose that (u, t) and (v, s) are the parametric representations of $\Psi(\xi)\vee\epsilon$ and $\Psi(\zeta)\vee\epsilon$, respectively. Then, there exists $r_\xi \in [0,1]$ such that $u(r) \le \epsilon$ for $r \le r_\xi$ and $u(r) \ge \epsilon$ for $r \ge r_\xi$. Likewise, there exists $r_\zeta \in [0,1]$ such that $v(r) \le \epsilon$ for $r \le r_\zeta$ and $v(r) \ge \epsilon$ for $r \ge r_\zeta$. We assume w.l.o.g. that $r_\xi \le r_\zeta$. Note that, because $u(r), v(r) \in [0,\epsilon]$ on $r \in [0,r_\xi]$, we get

$$\sup_{[0,r_\xi]} |u^p(r) - v^p(r)| \le \epsilon^p.$$

Also, because $u(r) \in [\epsilon, 2\gamma']$ and $v(r) \in [0,\epsilon]$ on $r \in [r_\xi, r_\zeta]$, we get from (5.6) that

$$\begin{aligned}
\sup_{[r_\xi,r_\zeta]} |u^p(r) - v^p(r)| &\le \sup_{[r_\xi,r_\zeta]} \{|u^p(r) - \epsilon^p| + |\epsilon^p - v^p(r)|\} \le \sup_{[r_\xi,r_\zeta]} \{|u^p(r) - \epsilon^p| + \epsilon^p\} \\
&\le \sup_{[r_\xi,r_\zeta]} \{p(\epsilon^{p-1} \vee (2\gamma')^{p-1})|u(r) - \epsilon| + \epsilon^p\} \\
&\le \sup_{[r_\xi,r_\zeta]} \{p(\epsilon^{p-1} \vee (2\gamma')^{p-1})(|u(r) - v(r)| + |v(r) - \epsilon|) + \epsilon^p\} \\
&\le \sup_{[r_\xi,r_\zeta]} \{p(\epsilon^{p-1} \vee (2\gamma')^{p-1})(|u(r) - v(r)| + \epsilon) + \epsilon^p\} \\
&\le p(\epsilon^{p-1} \vee (2\gamma')^{p-1})(\|u - v\|_\infty + \epsilon) + \epsilon^p.
\end{aligned}$$

Finally, because $u(r), v(r) \in [\epsilon, 2\gamma']$ on $r \in [r_\zeta, 1]$, we get again from (5.6) that

$$\sup_{[r_\zeta,1]} |u^p(r) - v^p(r)| \le \sup_{[r_\zeta,1]} p(\epsilon^{p-1} \vee (2\gamma')^{p-1})|u(r) - v(r)| \le p(\epsilon^{p-1} \vee (2\gamma')^{p-1})\|u - v\|_\infty.$$

From these inequalities, we see that, if $(u,t) \in \Gamma'(\Psi(\xi)\vee\epsilon)$ and $(v,s) \in \Gamma'(\Psi(\zeta)\vee\epsilon)$,

$$\begin{aligned}
\|u^p - v^p\|_\infty &= \sup_{[0,r_\xi]} |u^p(r) - v^p(r)| \vee \sup_{[r_\xi,r_\zeta]} |u^p(r) - v^p(r)| \vee \sup_{[r_\zeta,1]} |u^p(r) - v^p(r)| \\
&\le p(\epsilon^{p-1} \vee (2\gamma')^{p-1})(\|u - v\|_\infty + \epsilon) + \epsilon^p.
\end{aligned}$$

Now, we can bound the M_1' distance between $(\Psi(\xi)\vee\epsilon)^p$ and $(\Psi(\zeta)\vee\epsilon)^p$ as follows:

$$
\begin{aligned}
& d_{M_1'}((\Psi(\xi)\vee\epsilon)^p, (\Psi(\zeta)\vee\epsilon)^p) \\
&= \inf_{\substack{(u,t)\in\Gamma'((\Psi(\xi)\vee\epsilon)^p)\\(v,s)\in\Gamma'((\Psi(\zeta)\vee\epsilon)^p)}} \{\|u-v\|_\infty + \|t-s\|_\infty\} \\
&= \inf_{\substack{(u,t)\in\Gamma'(\Psi(\xi)\vee\epsilon)\\(v,s)\in\Gamma'(\Psi(\zeta)\vee\epsilon)}} \{\|u^p-v^p\|_\infty + \|t-s\|_\infty\} \\
&\le \inf_{\substack{(u,t)\in\Gamma'(\Psi(\xi)\vee\epsilon)\\(v,s)\in\Gamma'(\Psi(\zeta)\vee\epsilon)}} \{p(\epsilon^{p-1}\vee(2\gamma')^{p-1})(\|u-v\|_\infty+\epsilon)+\epsilon^p+\|t-s\|_\infty\} \\
&\le (1\vee(p(\epsilon^{p-1}\vee(2\gamma')^{p-1}))) \inf_{\substack{(u,t)\in\Gamma'(\Psi(\xi)\vee\epsilon)\\(v,s)\in\Gamma'(\Psi(\zeta)\vee\epsilon)}} \{\|u-v\|_\infty+\|t-s\|_\infty\} + p(\epsilon^{p-1}\vee(2\gamma')^{p-1})\epsilon+\epsilon^p \\
&= (1\vee(p(\epsilon^{p-1}\vee(2\gamma')^{p-1})))d_{M_1'}(\Psi(\xi)\vee\epsilon, \Psi(\zeta)\vee\epsilon) + p(\epsilon^{p-1}\vee(2\gamma')^{p-1})\epsilon+\epsilon^p \\
&\le (1\vee(p(\epsilon^{p-1}\vee(2\gamma')^{p-1})))2\epsilon + p(\epsilon^{p-1}\vee(2\gamma')^{p-1})\epsilon+\epsilon^p \\
&\triangleq \delta(\epsilon).
\end{aligned}
$$

Note that $\delta(\epsilon)\to 0$ as $\epsilon\to 0$.

To apply Lemma 5.2, we examine the total variations of $(\Psi(\xi)\vee\epsilon)^p$ and $(\Psi(\zeta)\vee\epsilon)^p$. Recall the notation $\mathbf{T} = \cup_{d=1}^\infty \{(t_1,\dots,t_d): 0\le t_1 < t_2 < \cdots < t_d \le 1\}$. From (5.6),

$$
\begin{aligned}
TV((\Psi(\xi)\vee\epsilon)^p) &= \sup_{\mathbf{t}\in\mathcal{P}} \sum_{i=1}^{n_{\mathbf{t}}} |(\Psi(\xi)\vee\epsilon)^p(t_i) - (\Psi(\xi)\vee\epsilon)^p(t_{i-1})| \\
&\le p(\epsilon^{p-1}\vee(2\gamma')^{p-1}) \sup_{(t_0,\dots,t_k)\in\mathbf{T}} \sum_{i=1}^{k} |(\Psi(\xi)\vee\epsilon)(t_i) - (\Psi(\xi)\vee\epsilon)(t_{i-1})| \\
&= p(\epsilon^{p-1}\vee(2\gamma')^{p-1}) TV(\Psi(\xi)\vee\epsilon) \\
&\le p(\epsilon^{p-1}\vee(2\gamma')^{p-1})2\gamma'.
\end{aligned}
$$

Similarly,

$$
TV((\Psi(\zeta)\vee\epsilon)^p) \le p(\epsilon^{p-1}\vee(2\gamma')^{p-1})2\gamma'.
$$

These two bounds allow us to apply Lemma 5.2 to H to obtain

$$
\begin{aligned}
d_{M_1'}(H((\Psi(\xi)\vee\epsilon)^p), H((\Psi(\zeta)\vee\epsilon)^p)) &\le (T+2\gamma' p(\epsilon^{p-1}\vee(2\gamma')^{p-1}))\cdot d_{M_1'}(G(\xi)\vee\epsilon^p, G(\zeta)\vee\epsilon^p) \\
&\le (T+2\gamma' p(\epsilon^{p-1}\vee(2\gamma')^{p-1}))\delta(\epsilon).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& d_{M_1'}(\Phi_T(\xi), \Phi_T(\zeta)) \\
&= d_{M_1'}(H(G(\xi)), H(G(\zeta))) \\
&\le d_{M_1'}(H(G(\xi)), H(G(\xi)\vee\epsilon^p)) + d_{M_1'}(H(G(\xi)\vee\epsilon^p), H(G(\zeta)\vee\epsilon^p)) + d_{M_1'}(H(G(\zeta)\vee\epsilon^p), H(G(\zeta))) \\
&\le \epsilon^p T + \delta(\epsilon) + \epsilon^p T.
\end{aligned}
$$

This concludes the proof of the desired uniform continuity. □

Acknowledgments

The authors are grateful to Oliver Tse and Vladislav Vysotsky for useful discussions and to the referees for comments that helped to improve the exposition of the paper.

Appendix A. Results on the Theory of Markov Chains

Let $\{X_m, -\infty < m < \infty\}$ be a geometrically ergodic stationary Markov chain on the state space S (which includes an element zero) and invariant distribution π, such that $\pi(\{0\}) = \pi(0) > 0$. Let $\{X^*_m, -\infty < m < \infty\}$ be the time-reversed stationary version of $\{X_m, -\infty < m < \infty\}$. It is well-known that $(X^*_0, \dots, X^*_k) =^d (X_k, \dots, X_0)$ for any $k \geq 0$; see Sforzo [20] and references therein for a discussion on reversibility for general state-space Markov chains. The following lemma follows directly by applying this identity.

Lemma A.1. *It holds that*

$$\mathbf{P}_0(X_i \in A_i : 1 \leq i \leq k) = \frac{1}{\pi(0)} \mathbf{P}_\pi(X^*_i \in A_{k-i} : 0 \leq i \leq k-1, X^*_k = 0), \tag{A.1}$$

$$\mathbf{E}_0[g(X_0, X_1, \dots, X_k)] = \frac{1}{\pi(0)} \mathbf{E}_\pi[g(X^*_k, X^*_{k-1}, \dots, X^*_0) I(X^*_k = 0)], \tag{A.2}$$

for any nonnegative integer k and measurable $g : \mathbb{R}^{k+1}_+ \to \mathbb{R}$.

Using the previous result, we can now establish the following lemma.

Lemma A.2. *Define* $T = \inf\{n \geq 1 : X_n = 0\}$, $T^* = \inf\{n \geq 1 : X^*_n = 0\}$, *and suppose that* $\mathbf{P}_0(T > n) = \mathcal{O}(e^{-cn})$ *for some* $c > 0$. *In addition, let* n_0 *be such that* $\inf_{k \geq n_0} \mathbf{P}_0(X_k = 0) \geq \pi(0)/2$. *Then,*

$$\mathbf{P}_\pi\left(\sum_{k=0}^{T^*-1} (X^*_k)^p \geq x, X^*_n = 0\right) \leq (n+1)\mathbf{P}_\pi\left(\sum_{k=0}^{T-1} X^p_k \geq x\right), \tag{A.3}$$

and

$$\mathbf{P}_\pi\left(\sum_{k=0}^{T^*-1} (X^*_k)^p \geq x, X^*_n = 0\right) \geq (\pi(0)^2/2)\mathbf{P}_0\left(\sum_{k=1}^{T} X^p_k \geq x\right) - \mathcal{O}(e^{-cn}). \tag{A.4}$$

Proof. We first derive the upper bound by noting that

$$\begin{aligned}
\mathbf{P}_\pi\left(\sum_{k=0}^{T^*-1}(X^*_k)^p \geq x,\ X^*_n = 0\right) &= \sum_{m=0}^{n} \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X^*_k)^p \geq x,\ T^* = m,\ X^*_n = 0\right) \\
&= \sum_{m=0}^{n} \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X^*_k)^p \geq x,\ X^*_1 > 0, \dots, X^*_{m-1} > 0,\ X^*_m = 0,\ X^*_n = 0\right) \\
&\leq \sum_{m=0}^{n} \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X^*_k)^p \geq x,\ X^*_1 > 0, \dots, X^*_{m-1} > 0\right) \\
&= \sum_{m=0}^{n} \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}X^p_{m-1-k} \geq x, X_{m-1} > 0, \dots, X_1 > 0\right) \\
&= \sum_{m=0}^{n} \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}X^p_k \geq x,\ T \geq m\right) \leq \sum_{m=0}^{n} \mathbf{P}_\pi\left(\sum_{k=0}^{T-1}X^p_k \geq x,\ T \geq m\right) \\
&\leq (n+1)\mathbf{P}_\pi\left(\sum_{k=0}^{T-1}X^p_k \geq x\right).
\end{aligned}$$

For the lower bound, first write

$$\begin{aligned}
\mathbf{P}_\pi\left(\sum_{k=0}^{T^*-1}(X^*_k)^p \geq x,\ X^*_n = 0\right) &= \sum_{m=1}^{n} \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X^*_k)^p \geq x,\ T^* = m,\ X^*_n = 0\right) \\
&= \sum_{m=1}^{n} \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X^*_k)^p \geq x,\ T^* = m\right)\mathbf{P}_0(X_{n-m} = 0).
\end{aligned}$$

Applying Lemma A.1 with $k = m$ and $g(y_0, \ldots, y_m) = I(\sum_{i=0}^{m} y_i^p > x,\ y_0 > 0,\ y_1 > 0, \ldots, y_{m-1} > 0)$, we obtain

$$\begin{aligned}
\mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X_k^*)^p \geq x, T^* = m\right) &= \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X_k^*)^p \geq x,\ X_0^* > 0, \ldots, X_{m-1}^* > 0,\ X_m^* = 0\right)\\
&= \mathbf{P}_\pi\left(\sum_{k=0}^{m}(X_k^*)^p \geq x,\ X_0^* > 0, \ldots, X_{m-1}^* > 0,\ X_m^* = 0\right)\\
&= \pi(0)\mathbf{P}_0\left(\sum_{k=1}^{m}X_k^p \geq x, X_i^* > 0, i = 1, \ldots, m-1\right)\\
&= \pi(0)\mathbf{P}_0\left(\sum_{k=1}^{m}X_k^p \geq x, T \geq m\right)\\
&\geq \pi(0)\mathbf{P}_0\left(\sum_{k=1}^{m}X_k^p \geq x, T = m\right).
\end{aligned}$$

Consequently, for every fixed n_0 such that $\inf_{k \geq n_0} \mathbf{P}_0(X_k = 0) \geq \pi(0)/2$,

$$\begin{aligned}
\mathbf{P}_\pi\left(\sum_{k=0}^{T^*-1}(X_k^*)^p \geq x, X_n^* = 0\right) &\geq \pi(0)\sum_{m=1}^{n-n_0}\mathbf{P}_0\left(\sum_{k=1}^{T}X_k^p \geq x, T = m\right)\mathbf{P}_0(X_{n-m} = 0)\\
&\geq \pi(0)\mathbf{P}_0\left(\sum_{k=1}^{T}X_k^p \geq x, T \leq n - n_0\right)\inf_{k \geq n_0}\mathbf{P}_0(X_k = 0).\\
&\geq (\pi(0)^2/2)\mathbf{P}_0\left(\sum_{k=1}^{T}X_k^p \geq x\right) - \mathcal{O}(e^{-cn}). \quad \square
\end{aligned}$$

Appendix B. LDP Results

We collect some LDP results that have appeared in the literature. A straightforward adaptation of Bazhba et al. [3, Corollary 3.2] to our context is the following.

Result B.1. *Let $\overline{\mathcal{K}}_n \triangleq \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor}\mathcal{U}_i$, $t \in [0,1]$ be the scaled random walk driven by an i.i.d. sequence $\{\mathcal{U}_i, i \geq 1\}$. Assume that $\mathbf{E}[e^{s\mathcal{U}_1}] < \infty$ for some $s < 0$, and $\mathbf{P}(\mathcal{U}_1 \geq x) = e^{-L(x)x^\alpha}$ for $\alpha \in (0,1)$. Suppose that L is a slowly varying function, and $L(x)x^{\alpha-1}$ is eventually decreasing. Then, $\overline{\mathcal{K}}_n$ satisfies the LDP in $(\mathbb{D}[0,T], \mathcal{T}_{M_1'})$ with the speed $L(n)n^\alpha$ and the rate function $I_{M_1'} : \mathbb{D}[0,T] \to [0,\infty]$,*

$$I_{M_1'}(\xi) \triangleq \begin{cases} \sum_{t \in [0,1]}(\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}^{(\mathbf{E}\mathcal{U}_1)}[0,T] \text{ with } \xi(0) \geq 0,\\ \infty & \text{otherwise}.\end{cases}$$

The following result, which is folklore but explicitly stated in Nuyens and Zwart [16], provides the logarithmic asymptotics for the invariant distribution π of $\{X_n\}_{n \geq 0}$ with $X_{n+1} = [X_n + U_{n+1}]^+$, $n \geq 0$, with $\{U_i, i \geq 1\}$ i.i.d. such that $\mathbf{E}(U_1) = \mu < 0$.

Result B.2 (Nuyens and Zwart [16]). *Recall that $\beta = \sup\{s : \mathbf{E}[e^{sU}] \leq 1\}$. It holds that*

$$\lim_{n\to\infty}\frac{\log \pi([n,\infty))}{n} = -\beta.$$

Finally, we mention a recent sample-path LDP for random walks with light-tailed increments, developed in Vysotsky [21], that we use in this paper.

Result B.3. *Recall that $\overline{K}_x^y = y + \frac{1}{x}\sum_{i=1}^{\lfloor xt \rfloor}U_i$, $t \in [0,1]$ is the scaled random walk driven by the i.i.d. sequence $\{U_i, i \geq 1\}$, which satisfies* Assumptions 2.1 *and* 2.2. *Recall also that*

$$I_y^{\mathbb{BV}[0,T]}(\xi) = \begin{cases} \int_0^T \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+(\xi^{(u)}(T)) + \theta_-|\xi^{(l)}(T)| & \text{if } \xi \in \mathbb{BV}[0,1] \text{ and } \xi(0) = y,\\ \infty & \text{otherwise}.\end{cases} \tag{B.1}$$

i. *(Borovkov and Mogulskii [5, 6]) $\overline{K}_x^y$ (as $x \to \infty$) satisfies a large deviations lower bound in the M_1 topology with the rate function $I_y^{\mathbb{BV}[0,T]}$.*

ii. *(Vysotsky [21]) Let ϕ be a real-valued function on $\mathbb{D}[0,T]$, which is uniformly continuous in the M_1' topology on the level sets $\{\xi : I_y^{\mathbb{BV}[0,T]}(\xi) \leq \alpha\}, \alpha < \infty$. Then, $\phi(\overline{K}_x^0)$ satisfies an LDP with the rate function $cl(J_\phi)$, where $cl(J_\phi)$ is the lower semicontinuous regularization of $J_\phi(u) = \inf_{\xi:\phi(\xi)=u} I_0^{\mathbb{BV}[0,T]}(\xi)$.*

Appendix C. Computing $\mathcal{B}_0^*$: Finding a Smooth Minimizer

In this appendix, we provide some details that could facilitate the computation of $\mathcal{B}_0^*$. Note that it is not straightforward that the infimum in the representation of $\mathcal{B}_0^*$ in ($\mathcal{B}_y$) is attained because the associated objective function does not have compact level sets unless the moment-generating function of U_1 is finite everywhere; compare this with Lynch and Sethuraman [14]. The following proposition, however, facilitates the characterization of $\mathcal{B}_0^*$.

Proposition C.1. *Let* $B_y^{\mathbb{AC}} \triangleq B_y \cap \mathbb{AC}[0,\infty)$, $B_y^{\mathbb{CNCV}} \triangleq B_y^{\mathbb{AC}} \cap \{\xi \in \mathbb{AC}[0,\infty) : \xi$ *is concave*$\}$, *and recall that* $\mathcal{B}_y^* = \inf_{\xi\in B_y} I_y(\xi)$. *Then,*

$$\mathcal{B}_y^* = \inf_{\xi\in B_y^{\mathbb{AC}}} I_y(\xi) = \inf_{\xi\in B_y^{\mathbb{CNCV}}} I_y(\xi).$$

We defer the proof of this proposition to the end of this appendix. We apply Proposition C.1 to write

$$\mathcal{B}_y^* = \inf_{\xi\in B_y^{\mathbb{CNCV}}} I_y(\xi) = \inf_{z_0,\, z_T : z_0 \ge z_T} \inf_{T\ge 0} \inf_{\xi\in F^y_{z_0,z_T,T}} I_y(\xi), \tag{C.1}$$

where $F^y_{z_0,z_T,T} \triangleq \{\xi : \xi \in B_y^{\mathbb{CNCV}}, \dot\xi(0) = z_0, \dot\xi(T) = z_T, \xi(T) = 0\}$. We next ensure compactness of $F^y_{z_0,z_T,T}$ in the following lemma, which is also proven at the end of this appendix.

Lemma C.1. $F^y_{z_0,z_T,T}$ *is a compact set with respect to the* J_1 *topology.*

On the set $F^y_{z_0,z_T,T}$, the conditions $\xi(0) = y, \xi(T) = 0$, and concavity imply that $\xi(s) > 0$ for $s \in (0,T)$. Therefore, the identity $I_y(\xi) = \int_0^T \Lambda^*(\dot\xi(s))ds$ holds. The RHS of this identity is lower semicontinuous in ξ on the compact set $F^y_{z_0,z_T,T}$ and, therefore, attains a minimum ξ^* as long as the set $F^y_{z_0,z_T,T}$ is nonempty. The latter property holds if the solution ξ with slope $z_0 > 0$ on $[0, T|z_T|/(z_0 + |z_T|)]$ and slope $z_T < 0$ on $[T|z_T|/(z_0 + |z_T|, T]$ yields an area of at least one.

We now characterize the minimizer of the inner infimum of the RHS in Lemma C.1 through Euler–Lagrange equations. Such a characterization is usually only possible if the minimum is sufficiently smooth (e.g., not just AC, but C^1). This requires additional assumptions. We use Lagrange duality and note that the feasible region of admissible paths is only convex (and, hence, the absence of a duality gap is only guaranteed) if $p \le 1$. In addition, we utilize sufficient conditions for smoothness of optimal solutions of variational problems developed in Cesari [7, chapter 2.6], which seems to exclude the case $p<1$, so in what follows, we assume $p=1$. We make the additional assumption that $\Lambda(\theta) = \log E[\exp\{\theta U\}]$ is steep at θ_+ and θ_-, that is, $\lim_{\theta\uparrow\theta_+} \nabla\Lambda(\theta) = \infty$ and $\lim_{\theta\downarrow\theta_-} \nabla\Lambda(\theta) = -\infty$. Under these assumptions, $\Lambda^*(z) = z\theta(z) - \Lambda(\theta(z))$ with $\theta(z)$ the unique solution of $z = \nabla\Lambda(\theta)$. The steepness assumptions make Λ^* a smooth (C^∞) function on the entire real line. Its derivative satisfies $\nabla\Lambda^*(z) = (\nabla\Lambda)^{-1}(z)$. To reduce our setting to the framework in Cesari [7, chapter 2.6], we first incorporate our area constraint into a Lagrangian. Fix $\ell \ge 0$ and define

$$f_\ell(\xi(t), \dot\xi(t)) = \Lambda^*(\dot\xi(t)) - \ell[\xi(s) - 1/T]. \tag{C.2}$$

The Lagrangian $L_\ell(\xi)$ of our problem w.r.t. the constraint $\int_0^T \xi(s)ds \ge 1$ is $\int_0^T f_\ell(\xi(s), \dot\xi(s))ds$. We show that the problem of minimizing $L_\ell(\xi)$ over the set of concave absolutely continuous paths ξ such that $\xi(0) = \xi(T) = 0$, $\xi'(0) = z_0, \xi'(T) = z_T, z_0 > 0 > z_T$ produces a solution that is C^1 for every $\ell \ge 0$. Because $\nabla\Lambda^*$ is strictly increasing, f_ℓ satisfies Cesari [7, (2.6.1)], which demands that $f_\ell^{(y)}(x,y) = \frac{d}{dy} f_\ell^{(y)}(x,y) = \nabla\Lambda^*(y)$ is strictly increasing. Cesari [7, property (2.6.4)], which demands that $|f_\ell^{(y)}(x,y)| = |\nabla\Lambda^*(y)| \to \infty$ as $|y| \to \infty$, uniform in $x \ge 0$, follows from the fact that Λ is steep. We next propose an AC candidate solution ξ^*, which satisfies the Euler–Lagrange equation almost everywhere. In our setting, this equation is given by, for some constant c,

$$\nabla\Lambda^*(\dot{\xi^*}(s)) = c - \ell s, s \in [0,T]. \tag{C.3}$$

Because $\dot\xi(0) = z_0$, $c = \nabla\Lambda^*(z_0)$. Apply $\nabla\Lambda^*(z) = (\nabla\Lambda)^{-1}(z)$ to write, for almost every s,

$$\dot{\xi^*}(s) = \nabla\Lambda(\nabla\Lambda^*(z) - \ell s), s \in [0,T]. \tag{C.4}$$

Because f_ℓ is C^1 and $\dot\xi^*(t) \in [z_T, z_0]$ on $[0,T]$, we can now conclude from Cesari [7, theorem 2.6.ii] that ξ^* is C^1 on $[0,T]$, so that (C.4) is valid for all $t \in [0,T]$. This expression can now be substituted into the Lagrangian $L_\ell(\xi)$. Maximizing this over ℓ gives an expression for the inner infimum in (C.1), which can then be optimized further over z_0, z_T, T.

Proof of Proposition C.1. Because of Lemma 4.1(ii), $\mathcal{B}_y^* = \inf_{\xi\in B_y^{\mathbb{AC}}} I_y(\xi)$. Because $B_y^{\mathbb{CNCV}} \subseteq B_y^{\mathbb{AC}}$, we only have to prove that $\inf_{\xi\in B_y^{\mathbb{AC}}} I_y(\xi) \ge \inf_{\xi\in B_y^{\mathbb{CNCV}}} I_y(\xi)$. For this, we show that, for any given $\xi \in B_y^{\mathbb{AC}}$, there is $\zeta \in B_y^{\mathbb{CNCV}}$ such that $I_y(\zeta) \le I_y(\xi)$. To construct such ζ, we first note that we can find $\xi_1 \in B_y^{\mathbb{AC}}$ such that $\mathcal{T}(\xi_1) < \infty$ and $I_y(\xi_1) \le I_y(\xi)$ thanks to Lemma 4.3. Now, set $T = \mathcal{T}(\xi_1)$ and denote the restriction of ξ_1 on $[0,T]$ with $\check\xi_1$—that is, $\check\xi_1 \in \mathbb{AC}[0,T]$ and $\check\xi_1(t) = \xi_1(t)$ for $t \in [0,T]$. We appeal to (iii) of Lemma 4.2 to pick a path $\xi_2 \in \mathbb{AC}[0,T]$ such that $\xi_2(0) = y$, $\Phi_T(\xi_2) \ge \Phi_T(\check\xi_1) \ge 1$, $I_y^{\mathbb{BV}[0,T]}(\xi_2) \le I_y^{\mathbb{BV}[0,T]}(\check\xi_1) = I_y(\xi_1) \le I_y(\xi)$, and ξ_2 concave on $[0,T]$ with the derivative bounded by μ from below. Now, set $\zeta = \xi_2(t \wedge T) + \mu([t-T]^+)$, $t \ge 0$. Then, $\zeta \in B_y^{\mathbb{CNCV}}$ and $I_y(\zeta) = I_y^{\mathbb{BV}[0,T]}(\xi_2) \le I_y(\xi)$. □

We end this section with the proof of compactness of $F^y_{z_0,z_T,T}$.

Proof of Lemma C.1. Recall that $F^{y}_{z_0,z_T,T} = \{\xi : \xi \in B^{\mathbb{CNCV}}_{y}, \dot{\xi}(0) = z_0, \dot{\xi}(T) = z_T, \xi(T) = 0\}$. Let $\mathcal{P}([0,T])$ denote the space of all Borel probability measures. Consider $\varphi : \mathcal{P}([0,T]) \to \mathbb{D}[0,T]$, a mapping defined for each $\nu \in \mathcal{P}([0,T])$ as follows:

$$\varphi(\nu)(t) \triangleq y + \int_0^t (z_0 + (z_T - z_0)\nu([0,s]))ds.$$

We consider the weak topology on $\mathcal{P}([0,T])$ and the J_1 topology on $\mathbb{D}[0,T]$. Note that $\mathcal{P}([0,T])$ is a compact space because of Prokhorov's theorem. Now, observe that

$$F^{y}_{z_0,z_T,T} = \varphi(\mathcal{P}([0,T])) \cap \{\xi \in \mathbb{D}[0,T] : \xi(T) = 0\}.$$

Because $\{\xi \in \mathbb{D}[0,T] : \xi(T) = 0\}$ is closed, the proof of this lemma is complete if φ is continuous. To confirm the continuity of φ, consider a sequence ν_n that converges to ν in $\mathcal{P}([0,T])$. Note that

$$\begin{aligned} d_{J_1}(\varphi(\nu_n), \varphi(\nu)) \le \|\varphi(\nu_n) - \varphi(\nu)\|_\infty &= \sup_{t\in[0,T]} \left| (z_T - z_0) \int_0^t (\nu_n([0,s]) - \nu([0,s]))ds \right| \\ &\le |z_0 - z_T| \int_0^T |\nu_n([0,s]) - \nu([0,s])|ds, \end{aligned} \tag{C.5}$$

and $|\nu_n([0,s]) - \nu([0,s])|$ is bounded by two for each n. Because the (weak) convergence of ν_n to ν implies that $|\nu_n([0,s]) - \nu([0,s])| \to 0$ for almost every $s \in [0,T]$, the bounded convergence theorem guarantees that (C.5) converges to zero as $n \to \infty$, proving the continuity of φ. This concludes the proof of the desired compactness of $F^{y}_{z_0,z_T,T}$. □

References

[1] Bazhba M, Blanchet J, Laeven RJA, Zwart B (2022) Large deviations asymptotics for unbounded additive functionals of diffusion processes. Preprint, submitted February 22, https://arxiv.org/abs/2202.10799.
[2] Bazhba M, Blanchet J, Rhee CH, Zwart B (2019) Queue length asymptotics for the multiple-server queue with heavy-tailed Weibull service times. *Queueing Systems* 93(3–4):195–226.
[3] Bazhba M, Blanchet J, Rhee CH, Zwart B (2020) Sample path large deviations for Lévy processes and random walks with Weibull increments. *Ann. Appl. Probab.* 30(6):2695–2739.
[4] Blanchet J, Glynn P, Meyn S (2013) Large deviations for the empirical mean of an $M/M/1$ queue. *Queueing Systems* 73(4):425–446.
[5] Borovkov AA, Mogulskii AA (2013) Large deviation principles for random walk trajectories. II. *Theory Probab. Appl.* 57(1):1–27.
[6] Borovkov AA, Mogulskii AA (2014) Large deviation principles for random walk trajectories. III. *Theory Probab. Appl.* 58(1):25–37.
[7] Cesari L (1983) Problems with ordinary differential equations. *Optimization—Theory and Applications*, Applications of Mathematics, vol. 17 (Springer-Verlag, New York).
[8] Dembo A, Zeitouni O (2010) *Large Deviations Techniques and Applications.* Stochastic Modelling and Applied Probability, vol. 38 (Springer-Verlag, Berlin).
[9] Donsker M, Varadhan S (1975) Asymptotic evaluation of certain Markov process expectations for large time, II. *Comm. Pure Appl. Math.* 28(2):279–301.
[10] Duffy KR, Meyn SP (2014) Large deviation asymptotics for busy periods. *Stochastic Systems* 4(1):300–319.
[11] Ganesh A, O'Connell N, Wischik D (2004) *Big Queues*. Lecture Notes in Mathematics, vol. 1838 (Springer-Verlag, Berlin).
[12] Gantert N (1998) Functional Erdos-Renyi laws for semiexponential random variables. *Ann. Probab.* 26(3):1356–1369.
[13] Kontoyiannis I, Meyn SP (2003) Spectral theory and limit theorems for geometrically ergodic Markov processes. *Ann. Appl. Probab.* 13(1):304–362.
[14] Lynch J, Sethuraman J (1987) Large deviations for processes with independent increments. *Ann. Probab.* 15(2):610–627.
[15] Mogulskii AA (1993) Large deviations for processes with independent increments. *Ann. Probab.* 21(1):202–215.
[16] Nuyens M, Zwart B (2006) A large-deviations analysis of the $GI/GI/1$ SRPT queue. *Queueing Systems* 54(2):85–97.
[17] Puhalskii AA, Whitt W (1997) Functional large deviation principles for first-passage-time processes. *Ann. Appl. Probab.* 7(2):362–381.
[18] Puhalskii AA, Whitt W (1998) Functional large deviation principles for waiting and departure processes. *Probab. Engrg. Inform. Sci.* 12(4):479–507.
[19] Ramasubramanian S (2000) A subsidy-surplus model and the Skorokhod problem in an orthant. *Math. Oper. Res.* 25(3):509–538.
[20] Sforzo R (2005) Reversible Markov processes on general spaces and spatial migration processes. *Adv. Appl. Probab.* 37(3):801–818.
[21] Vysotsky V (2021) Contraction principle for trajectories of random walks and Cramér's theorem for kernel-weighted sums. *ALEA Latin Amer. J. Probab. Math. Statist.* 18(2):1103–1125.
[22] Whitt W (2002) *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Application to Queues* (Springer Science & Business Media, New York).