# Sample path large deviations for unbounded additive functionals of the reflected random walk

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We prove a sample path large deviation principle (LDP) with sub-linear speed for unbounded functionals of certain Markov chains induced by the Lindley recursion. The LDP holds in the Skorokhod space $\mathbb{D}[0,1]$ equipped with the $M_1'$ topology. Our technique hinges on a suitable decomposition of the Markov chain in terms of regeneration cycles. Each regeneration cycle denotes the area accumulated during the busy period of the reflected random walk. We prove a large deviation principle for the area under the busy period of the MRW, and we show that it exhibits a heavy-tailed behavior.

*Key words*: Lindley recursion, busy period asymptotics, sample path large deviations, heavy tails.

## 1. Introduction

In this paper we develop sample path large deviation principles (LDP) for additive functionals of a Markov chain which is important in Operations Research (OR), namely, Lindley's recursion. This Markov chain describes the waiting time sequence in a single-server queue under a FIFO discipline and under independent and identically distributed (i.i.d.) inter-arrival times and service times. We focus on the case in which the input is light-tailed, i.e. the service times and inter-arrival times have a finite moment generating function in a neighborhood of the origin.

While the model that we consider is vital to many OR applications, and therefore important in its own right, our main contributions are also fundamental from a methodological standpoint. We contribute, as we shall explain, to the development of key tools in the study of sample- path large deviations for additive functionals of light-tailed geometrically and ergodic Markov chains.

A rich body of theory, pioneered by Donsker and Varadhan in classical work which goes back over forty years (see, for example, [6]) provides powerful tools designed to study large deviations for additive functionals of light-tailed and geometrically ergodic Markov chains. Roughly speaking, these are chains which converge exponentially fast to stationarity and whose stationary distribution is light-tailed.

Unfortunately, despite remarkable developments in the area, including the more recent contributions in [11], the prevailing assumptions in the literature are often not applicable to natural functionals of well-behaved geometrically ergodic models, such as Lindley's recursion with light-tailed input.

In particular, every existing general result describing sample path large deviations of functionals of a process such as Lindley's recursion, must assume the function of interest to be bounded. Hence, the current state-of-the-art rules out very important cases, such as the sample-path behavior of the empirical average of the waiting time sequence in single-server queue over large time scales. Our development allows one to study sample path large deviations for the cumulative waiting time sequence of a single-server queue. In particular, we provide methodological ideas which, we believe, will be useful in further development of the general theory of sample path large deviations for additive functionals of geometrically ergodic Markov processes. More precisely, our contributions are summarized as follows,

A) Let $\{X_k, k \geq 0\}$ follow Lindley's recursion. Assume that the associated increments have a finite moment generating function in a neighborhood of the origin and the traffic intensity is less than one, and let $f(x) = x^p$ for any $p > 0$. We establish a sample path large deviations principle for $\bar{Y}_n(\cdot) = \sum_{k=1}^{\lfloor n\cdot \rfloor} f(X_k)/n$ as $n \to \infty$ under the $M_1'$ topology on $\mathbb{D}[0,1]$ with a good rate function and a sublinear speed function which is fully characterized in Theorem 2.1. Though our result only pertains to a specific Markov chain, they can be extended to more general stochastic recursions, and diffusions; this will be pursued in a future study.

B) We believe that our overall strategy for establishing Theorem 2.1 can be applied generally to the sample-path large deviations analysis of additive functionals of geometrically ergodic Markov chains. Our strategy involves splitting the sample path in cycles, roughly corresponding to returns to a compact set (in the case of the Lindley recursion, the origin). Then, we show that the additive functional in a cycle has a Weibullian tail. Finally, we use ideas similar to those developed in [1], involving sample-path large deviations for random walks with Weibullian increments for the analysis. The result in [1], however, cannot be applied directly to our setting here because of two reasons. First, the cycle in progress at the end of the time interval is different from the rest. Second, the number of cycles (and thus the number of terms in the decomposition) is random.

The sublinear speed of convergence highlighted in A) underscores the main qualitative difference between our result and those traditionally obtained in the Donsker-Varadhan setting. In our setting, as hinted in B), the large deviations behavior of $\bar{Y}_n$ is characterized by heavy-tailed phenomena (in the form of Weibullian tails) which arise when studying the tails of the additive functional over a given busy period. Our choice of $f(\cdot)$ (growing slowly if $p > 0$) underscores the frailty of the assumptions required to apply the Donsker-Varadhan type theory (i.e. just a small amount of growth derails the application of the standard theory).

The choice of topology is an important aspect of our result. In [1] it is argued that $M_1'$ is a natural topology to consider for developing a full sample path large deviation principle for random walks with Weibullian increments. It is explained that such a result is impossible in the context of the $J_1$ topology in $\mathbb{D}[0,1]$. Actually, to be precise, the topology that we consider is a slightly stronger variation of the one considered by [16] and [17], who introduced the $M_1'$ topology in $\mathbb{D}[0,\infty)$, but in such a way that its direct projection onto $\mathbb{D}[0,1]$ loses important continuous functions (such as the maximum of the path in the interval). The key aspect in our variation is the evaluation of the metric at the right endpoint. The version that we consider merges the jumps, in the same way in which it is done at the left endpoint in the standard $M_1'$ description. This variation results in a stronger topology when restricted to functions on compact intervals and it includes the maximum as a continuous function. An important reason for why to use the $M_1'$ topology is that it allows to merge jumps. This seems to be particularly relevant given that in our setting the large deviations behavior will eventually merge the increments within the busy periods.

In addition to the two elements mentioned in B), which make the result in [1] not directly applicable, our choice of a strong topology also makes the approach in [1] difficult to use. In fact, in contrast to [1], in this paper, we use a projective limit strategy to directly obtain our large deviations principle. A direct approach, using the result in [1], which we explored, consisted in replacing the random number of busy periods by its fluid limits (for which there is a large deviations companion with a linear speed rate). Then, we tried to verify that this replacement results in an exponentially good approximation. This would have been a successful strategy if we had used the version of $M_1'$ considered by [16], but unfortunately such exponential approximation does not hold in the presence of our stronger topology.

The development of Theorem 2.1 highlights interesting and somewhat surprising qualitative insights. For example, consider the case $f(x) = x$, corresponding to the area drawn under the waiting time as a curve. As we show, deviations of order $O(1)$ upwards from the typical behavior of the process $\bar{Y}_n(\cdot)$ occur due to extreme behavior in a single busy period of duration $O(n^{1/2})$. A somewhat surprising insight involves the busy period in process at time $n$, which is split into two parts of size $O(n^{1/2})$ involving the age and forward life time of the cycle (the former contributes to the area calculations, while the later does not). This asymmetry, relative to the other busy periods during the time horizon $[0,n]$, which are completely accounted for inside the area calculation, raises the question of whether a correction in the LDP is needed, due to this effect, at the end of the time horizon. The answer is, no, the contribution to the current busy period and the ones inside the time horizon are symmetric. This result is highlighted in Theorems 2.2 and 2.3, which characterize the variational problem governing extreme busy periods.

There are several related works that deal with large deviations for the area under the waiting time sequence in a busy period. But they focus on queue length as in [2], or assume that the moment generating function of the increment is finite everywhere, as in [8]. None of these works obtain sample path results. Instead, we do not assume that the moment generating function of the service times or inter-arrival times is finite everywhere. To handle this level of generality, we employ recently developed sampled path LDP's [3, 4, 18]. This level of generality requires us to put in a substantial amount of work to rule out discontinuous solutions of the functional optimization problems that appear in the large deviations analysis.

Another hurdle in developing tail asymptotics for the additive functional in a busy period (reported in Theorems 2.2 and 2.3), is the fact that the functional describing the area under the busy period is not continuous. To deal with this, we exploit path properties of the most probable—in asymptotic sense–trajectories of the busy period along with the continuity of the area functional over a fixed time horizon. In particular, we rigorously show how to approximate the area over the busy period (which has a random endpoint) with the area over a large, *fixed* horizon. This is counter-intuitive at first, because the former approach allows one to remove the reflection operator. However, the latter approach does not have a first passage time (which is a discontinuous function) as horizon, and this turns out to carry more weight. In an earlier version of our proof, we attempted to exploit that the most likely path leading to a large area is concave, as the area functional is continuous at such paths. Though we obtain concavity as a by-product of our analysis (extending an idea from [7], who study a related problem), our method does not rely on this property, and can therefore be extended to situations where the optimal path is not concave; we will exhibit this in more detail in a forthcoming study where we consider additive functionals of diffusion processes.

This paper is organized as follows. We give a detailed model description as well as the main results in Section 2. Section 3 focuses on the technical details behind deriving the tail asymptotics for the area under a busy period. The proof of our sample path LDP is provided in Section 4. The paper is closed with two appendices covering auxilary duality results for Markov chains as well as large deviations results.

## 2. Model description and main results

### 2.1. Preliminaries

We consider the time-homogeneous Markov chain $\{X_n\}_{n\geq 0}$ that is induced by the Lindley recursion, i.e. $X_{n+1} \triangleq [X_n + U_{n+1}]^+$, $n \geq 0$, such that $X_0 = 0$. Note that the r.v.'s $\{U_i\}_{i\geq 1}$ are i.i.d. such that $\mathbf{E}(U_1) = \mu < 0$. The state space of the Markov chain $X_n$ is the half-line of non-negative real numbers. We make the following technical but necessary assumptions:

Assumption 2.1. *Let $\theta_+, \theta_-$ be respectively, the supremum and infimum of the set $\{\theta : \mathbf{E}(e^{\theta U}) < \infty\}$. We assume that $-\infty \leq \theta_- < 0 < \theta_+ \leq \infty$.*

Assumption 2.2. *For $\theta_+$ and $\theta_-$ in Assumption 2.1,*

$$\lim_{n\to\infty}\frac{\log \mathbf{P}(U\ge n)}{n}=-\theta_+,\ \lim_{n\to\infty}\frac{\log \mathbf{P}(-U\ge n)}{n}=\theta_-.$$

Assumption 2.3. $\mathbf{P}(U>0)>0$.

The purpose of this paper is to prove a sample path LDP for $\bar{Y}_n$, where

$$\bar{Y}_n(\cdot)\triangleq\frac{1}{n}\sum_{i=1}^{\lfloor n\cdot\rfloor}f(X_i),\quad f(x)=x^p$$

and $p>0$ is a fixed constant. We introduce basic notions that are used in the statement of one of our main results (Theorem 2.1). First, we set $\alpha\triangleq 1/(1+p)$. Let $\mathbb{D}[0,1]$ denote the Skorokhod space—the space of càdlàg functions from $[0,1]$ to $\mathbb{R}$. When the domain $[0,1]$ of a path space is clear from the context, we will omit $[0,1]$ and just write $\mathbb{D}$. We also consider the space $\mathbb{D}[0,\infty)$ of cádlág functions from $[0,\infty)$ to $\mathbb{R}$. Let $\mathcal{T}_{M_1'}$ denote the $M_1'$ Skorokhod topology, whose precise definition will be provided below. Unless specified otherwise, we assume that $\mathbb{D}[0,1]$ is equipped with $\mathcal{T}_{M_1'}$ throughout the rest of this paper.

Definition 2.1. For $\xi\in\mathbb{D}$, define the extended completed graph $\Gamma'(\xi)$ of $\xi$ as

$$\Gamma'(\xi)\triangleq\{(u,t)\in\mathbb{R}\times[0,1]: u\in[\xi(t-)\wedge\xi(t),\ \xi(t-)\vee\xi(t)]\}$$

where $\xi(0-)\triangleq 0$. Define an order on the graph $\Gamma'(\xi)$ by setting $(u_1,t_1)<(u_2,t_2)$ if either $t_1<t_2$; or $t_1=t_2$ and $|\xi(t_1-)-u_1|<|\xi(t_2-)-u_2|$. We call a continuous nondecreasing function $(u,t)=((u(s),t(s)),\ s\in[0,1])$ from $[0,1]$ to $\mathbb{R}\times[0,1]$ a parametrization of $\Gamma'(\xi)$ if $\Gamma'(\xi)=\{(u(s),t(s)):s\in[0,1]\}$. We also call such $(u,t)$ a parametrization of $\xi$.

Definition 2.2. Define the $M_1'$ metric on $\mathbb{D}$ as follows

$$d_{M_1'}(\xi,\zeta)\triangleq\inf_{\substack{(u,t)\in\Gamma'(\xi)\\(v,r)\in\Gamma'(\zeta)}}\{\|u-v\|_\infty+\|t-r\|_\infty\}.$$

We say that $\xi\in\mathbb{D}[0,1]$ is a pure jump function if $\xi=\sum_{i=1}^{\infty}x_i\mathbb{I}_{[u_i,1]}$ for some $x_i$'s and $u_i$'s such that $x_i\in\mathbb{R}$ and $u_i\in[0,1]$ for each $i$ and $u_i$'s are all distinct. Let $\mathbb{D}_p^{\uparrow}[0,1]$ be the subspace of $\mathbb{D}[0,1]$ consisting of non-decreasing pure jump functions that assume non-negative values at the origin. Let $\mathbb{BV}[0,1]$ be the subspace of $\mathbb{D}[0,1]$ consisting of cádlág paths with finite variation. Every $\xi\in\mathbb{BV}[0,1]$ has a Lebesgue decomposition with respect to the Lebesgue measure. That is, $\xi=\xi^{(a)}+\xi^{(s)}$ where $\xi^{(a)}$ denotes the absolutely continuous part of $\xi$, and $\xi^{(s)}$ denotes the singular part of $\xi$. Subsequently, using Hahn's decomposition theorem we can decompose $\xi^{(s)}$ into its non-decreasing singular part $\xi^{(u)}$ and non-increasing singular part $\xi^{(d)}$ so that $\xi^{(s)}=\xi^{(u)}+\xi^{(d)}$. Without loss of generality (w.l.o.g.), we assume that $\xi^{(s)}(0)=\xi^{(u)}=\xi^{(d)}=0$. We will also consider the space $\mathbb{BV}[0,\infty)$ of cádlág paths that are of bounded variation on any compact interval.

### 2.2. Sample path large deviations

In this subsection, we present the sample path large deviation principle for $\bar{Y}_n$ and the main ideas of its proof. We start with a few definitions. Let $R$ be the reflection map i.e; $R(\xi)(t) = \xi(t) - \inf_{0\le s\le t}\{\xi(s)\vee 0\}$, $\forall t\ge 0$. Define $\mathcal{T}(\xi) = \inf\{t>0 : R(\xi)(t)\le 0\}$, $B_y \triangleq \left\{\xi\in\mathbb{BV}[0,\infty): \xi(0)=y,\ \int_0^{\mathcal{T}(\xi)} R(\xi)(s)^p ds \ge 1\right\}$, and $\Lambda^*(y)\triangleq \sup_{\theta\in\mathbb{R}}\{\theta y - \log \mathbf{E}(e^{\theta U})\}$. Set

$$I_y(\xi) \triangleq \begin{cases} \int_0^{\mathcal{T}(\xi)} \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+\xi^{(u)}(\mathcal{T}(\xi)) + \theta_-\xi^{(d)}(\mathcal{T}(\xi)) & \text{if } \xi(0)=y \text{ and } \xi\in\mathbb{BV}[0,\infty) \\ \infty & \text{otherwise} \end{cases},$$

and denote with $\mathcal{B}_y^*$ the optimal value of the following variational problem $\mathcal{B}_y$:

$$\mathcal{B}_y^* \triangleq \inf_{\xi\in B_y} I_y(\xi). \tag{$\mathcal{B}_y$}$$

Similarly, denote with $\mathcal{B}_\pi^*$ the optimal value of the following variational problem $\mathcal{B}_\pi$:

$$\mathcal{B}_\pi^* \triangleq \inf_{y\in[0,\infty),\,\xi\in B_y} \{\beta y + I_y(\xi)\}, \tag{$\mathcal{B}_\pi$}$$

where $\beta \triangleq \sup\{\theta\ge 0 : \mathbf{E}(e^{\theta U})\le 1\}$. Note that $\beta\le\theta_+$ and $\beta$ is strictly positive in view of Assumption 1 and the assumption that $\mu<0$. Note also that $\mathcal{B}_\pi^* = \inf_{y\in[0,\infty)}\left\{\beta y + \mathcal{B}_y^*\right\}$.

Let $T_0 \triangleq 0$ and $T_i \triangleq \inf\{k > T_{i-1} : X_k = 0\}$ for $i\ge 1$, and subsequently, define $\lambda = \mathbf{E}(\sum_{i=1}^{T_1} X_i^p)/\mathbf{E}(T_1)$. Define $\mathbb{D}^{(\lambda)}[0,1] = \{\xi\in\mathbb{D}[0,1] : \xi(t) = \lambda t + \zeta(t), \forall t\in[0,1], \zeta\in\mathbb{D}_p^{\uparrow}[0,1]\}$. i.e., the subspace of increasing functions with slope $\lambda$ and countable upward jumps. Recall that $\alpha = 1/(1+p)$.

Theorem 2.1. *The stochastic process $\bar{Y}_n$ satisfies a large deviation principle in $(\mathbb{D}[0,1],\mathcal{T}_{M_1'})$ with speed $n^\alpha$ and rate function $I_{\boldsymbol{Y}} : \mathbb{D}\to\mathbb{R}_+$*

$$I_{\boldsymbol{Y}}(\zeta) \triangleq \begin{cases} \mathcal{B}_0^* \sum_{t:\zeta(t)\ne\zeta(t-)} (\zeta(t)-\zeta(t-))^\alpha & \text{if } \zeta\in\mathbb{D}^{(\lambda)}[0,1], \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

*That is, for any measurable set $A$,*

$$-\inf_{A^\circ} I_{\boldsymbol{Y}}(\xi) \le \liminf_{n\to\infty} \frac{\log \mathbf{P}\left(\bar{Y}_n\in A\right)}{n^\alpha} \le \limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\bar{Y}_n\in A\right)}{n^\alpha} \le -\inf_{\bar{A}} I_{\boldsymbol{Y}}(\xi). \tag{2.2}$$

The full proof of Theorem 2.1 is deferred to Section 4. The strategy relies on a suitable representation for $\bar{Y}_n$ using renewal theory, which is presented next. The sequence $\{T_j, j\ge 1\}$ induces a renewal process $N(t) = \max\{k\ge 0 : T_k\le t\}$. We decompose the process $\bar{Y}_n$ as follows:

$$\bar{Y}_n(t) \triangleq \frac{1}{n}\sum_{j=1}^{N(nt)} \sum_{i=T_{j-1}+1}^{T_j} f(X_i) + \frac{1}{n}\sum_{i=T_{N(n)}+1}^{\lfloor nt\rfloor} f(X_i), \tag{2.3}$$

with the convention that $\sum_{i=T_{N(n)}}^{\lfloor nt\rfloor} f(X_i)$ is zero in case the superscript $\lfloor nt\rfloor$ is strictly smaller than the subscript $T_{N(n)}$. We introduce some notation for the analysis of $\bar{Y}_n$. Define

- $\tau_j = T_j - T_{j-1}$, the inter-arrival times of the renewal process $N$,
- $W_j = \sum_{i=T_{j-1}+1}^{T_j} f(X_i)$, the area under $f(X_i)$ during a busy period of $X_n$,
- $\bar{\mathcal{Z}}_n(\cdot) = \frac{1}{n}\sum_{j=1}^{N(n\cdot)} W_j$, the process up to the last regeneration point,
- $\bar{\mathcal{R}}_n(t) = \frac{1}{n}\sum_{i=T_{N(n)}+1}^{\lfloor nt\rfloor} f(X_i)$, the area under $f(X_i)$ starting from the previous regeneration,
- $\bar{V}_n = \frac{1}{n}\sum_{i=T_{N(n)}+1}^{n} f(X_i)$, the area starting from the last regeneration point,
- $\bar{S}_n(t) = \bar{V}_n \mathbb{1}_{\{1\}}(t)$, the stochastic process with one jump of size $\bar{V}_n$ at the end of the time horizon.

We derive our main result (Theorem 2.1) by proving that;

1) the tail behavior of $W_1$ and $\bar{V}_n$ is asymptotically Weibull-like;
2) $\bar{\mathcal{R}}_n$ and $\bar{S}_n$ are exponentially equivalent in the $M_1'$ topology;
3) $\bar{\mathcal{Z}}_n$ and $\bar{S}_n$ satisfy an LDP in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$; and lastly
4) $\bar{\mathcal{Z}}_n + \bar{S}_n$ satisfies an LDP in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$ with the rate function $I_{\mathbf{Y}}$.

Regarding the first step, we first derive the logarithmic asymptotics of $\bar{V}_n$ and $W_1$, which are presented in Theorems 2.2 and 2.3 in the next section. For the sample path LDP of $\bar{Y}_n$, we prove the exponential equivalence of $\bar{\mathcal{R}}_n$ and $\bar{\mathcal{S}}_n$ in Lemma 4.1 by pushing the last cycle $\bar{\mathcal{R}}_n$ to the end of the time horizon. Consequently, the LDP of $\bar{\mathcal{R}}_n$ is deduced, due to the LDP of $\bar{S}_n$ in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$. We derive an LDP for $\bar{\mathcal{Z}}_n$ in $\mathbb{D}$ with respect to the $M_1'$ topology by obtaining an LDP with the point-wise convergence topology which is strengthened to the $M_1'$ topology using the continuity of the identity map in the subspace of increasing cádlág paths. In the last step, we infer an LDP for $\bar{\mathcal{Z}}_n + \bar{S}_n$ through the use of a continuous mapping approach, and hence, we obtain the LDP for $\bar{Y}_n$.

Before embarking on the execution of this technical program, it is worth commenting on the role of the $\bar{\mathcal{R}}_n$, since this element will allow us to expose the importance of a careful analysis involving the area during a busy period. As mentioned in the introduction, one may wonder if the contribution of $\bar{\mathcal{R}}_n(t)$ may end up counting different in the form of the LDP. The typical path for $\bar{Y}_n(\cdot)$ is a straight line with drift equal to the steady-state waiting time. Our development indicates that most likely large deviations behavior away from the most likely path occur due to isolated busy periods which exhibit extreme behavior. For example, in the case $f(x) = x$, substantially extreme busy periods (leading to large deviations of order $O(n)$) have a duration of order $O\left(n^{1/2}\right)$ and exhibit excursions of order $O\left(n^{1/2}\right)$, therefore accumulating an area of order $O\left(n\right)$.

Our results in the next section characterize the variational problem which governs such extreme busy periods. But each busy period, including the one in progress at the end of the time horizon contributes the same way in the rate function may be somewhat remarkable. The reason is that when the cycle in progress at the end of the time horizon is extreme, as indicated in the introduction, its duration is of order $O\left(n^{1/2}\right)$. This suggests that the remaining of the cycle is also of order $O\left(n^{1/2}\right)$ and hence one may wonder if this long time duration may have a significant contribution to the total area. It turns out that this does not happen and the reason is the following. While the remaining of the cycle in progress may be large, the position of the chain is actually $o\left(n^{1/2}\right)$ from the end of the time horizon, so the total contribution to the area of the remaining portion of the cycle is negligible. This calculation is exposed in Proposition 2.1 below, and a time-reverse argument given in Appendix A.

### 2.3. Busy period asymptotics

It is clear that a large deviations analysis of the area under a busy period is an indispensable component for deriving the sample path LDP of $\bar{Y}_n$ in Theorem 2.1. Our next two theorems provide the asymptotic estimation for the tails of $W_1$ and $\bar{V}_n$, showing that they exhibit Weibull behavior.

THEOREM 2.2. *Let* $W_1 \triangleq \sum_{k=1}^{T_1} X_k^p$. *It holds that*

$$\lim_{t\to\infty} \frac{1}{t^{1/1+p}} \log \mathbf{P}\left(W_1 \geq t\right) = -\mathcal{B}_0^*. \tag{2.4}$$

For $\bar{V}_n$, we notice again a Weibull-like asymptotic behavior similar to $W_1$ except that the prefactor associated with $\bar{V}_n$ is $\mathcal{B}_\pi^*$ (instead of $\mathcal{B}_0^*$). It turns out that (see Proposition 2.1) the prefactor $\mathcal{B}_\pi^*$ is equal to $\mathcal{B}_0^*$. This leads to the conclusion that every busy period, including the one in progress at the end of the time horizon, has the same tail asymptotics.

THEOREM 2.3. *For the area of the busy period starting from the steady state* $(\pi)$, *we have that*

$$\lim_{n\to\infty} \frac{1}{n^{1/(p+1)}} \log \mathbf{P}_\pi(\bar{V}_n \geq b) = -\mathcal{B}_0^* \cdot b^{1/(1+p)}. \tag{2.5}$$

The tail asymptotics for $W_1$ and $\bar{V}_n$ are derived using a recently developed LDP for random walks with light-tailed increments due to [3, 4, 18], cf. Result 3 below. Specifically, $W_1$ is the image of the unrestricted random walk $\bar{K}_n = \frac{1}{n}\sum_{i=1}^n U_i$ to which the functional $\Phi(\xi) \triangleq \int_0^{\mathcal{T}(\xi)} (R(\xi)(s))^p ds$ is applied. Note that $\Phi : \mathbb{D}[0,\infty) \to \mathbb{R}_+$ is not continuous, and hence, the proof for the tail asymptotics of $W_1$ gets more involved than simply applying the contraction principle. We derive large deviations upper and lower bounds and show that they coincide.

For the upper bound, we replace the hitting time $T_1$ with a sufficiently large value $T$. This enables us to study the area of $X_n$ over the finite time horizon $[0,T]$. For $T$ large enough, we show that the area of the reflected random walk over the whole time horizon $[0,T]$ serves as an asymptotic upper bound for $W_1$, and it is expressed as a functional of $\bar{K}_n$. This functional is shown to be uniformly continuous in the (standard) $M_1$ topology on level sets of the rate function associated with the LDP for $\bar{K}_n$. Invoking Result 3, recently established in [18], we get a large deviation upper bound.

For the lower bound, we confine the functional of the area under the busy period, over a fixed time horizon by imposing an extra condition. Subsequently, we derive a variational problem associated with the lower bound. Lastly, we show that $\mathcal{B}_0$ has the same value as the variational problem associated with the large deviation upper and lower bound.

For $\bar{V}_n$ we follow the same approach with some slight modifications. In order to carry out our analysis for $\bar{V}_n$, we associate the tail of $W_1$ with the tail of $\bar{V}_n$ through Lemmas A.1, and A.2. We prove that $\bar{V}_n$ has similar tail asymptotics to that of $W_1$, initialized from the steady state of $X_n$ i.e;

$$\lim_{n\to\infty} \frac{\log \mathbf{P}_0(\bar{V}_n > x)}{n^{1/1+p}} = \lim_{n\to\infty} \frac{\log \mathbf{P}_\pi(W_1 > nx)}{n^{1/1+p}}.$$

For this reason, it is necessary to invoke tail asymptotics for the steady state distribution $\pi$ of $X_n$. To this end, we use a result in [15] (see Result 2) regarding the asymptotic behavior of the invariant measure of homogeneous Markov chains. Lastly, we repeat the same steps as in the analysis of $W_1$. Namely, we derive large deviation upper and lower bounds and we show that they coincide.

### 2.4. Computation of the decay rate $\mathcal{B}_0^*$

Before we prove our main theorems in the following sections, we conclude this section with a sketch of how to compute $\mathcal{B}_0^*$. We first note that it is not straightforward that the infimum in the representation ($\mathcal{B}_y$) of $\mathcal{B}_0^*$ is attained since the associated objective function does not have compact level sets unless the moment generating function of $U_1$ is finite everywhere, cf. [13]. The following proposition, however, facilitates the characterization of the optimal solution of $\mathcal{B}_0^*$.

Proposition 2.1. *Let* $B_y^{\mathbb{AC}} \triangleq B_y \cap \mathbb{AC}[0,\infty)$, $B_y^{\mathrm{CNCV}} \triangleq B_y^{\mathbb{AC}} \cap \{\xi \in \mathbb{AC}[0,\infty) : \xi$ *is concave*$\}$, *and recall that* $\mathcal{B}_y^* = \inf_{\xi\in B_y} I_{\mathcal{B}_y}(\xi)$. *Then,*

$$\mathcal{B}_y^* = \inf_{\xi\in B_y^{\mathbb{AC}}} I_y(\xi) = \inf_{\xi\in B_y^{\mathrm{CNCV}}} I_y(\xi).$$

We defer the proof of this proposition to Section 3.4. Now with this Proposition, we can significantly reduce the feasible region by writing

$$\mathcal{B}_0^* = \inf_{\xi\in B_0^{\mathrm{CNCV}}} I_0(\xi) = \inf_{z\geq\mu} \inf_{T\geq 0} \inf_{\xi\in F_{z,T}} I_0(\xi) \tag{2.6}$$

where $F_{z,T} = \{\xi : \xi \in B_0^{\mathrm{CNCV}}, \dot{\xi}(0) = z, \xi(T) = 0\}$. Every element in the set $F_{z,T}$ can be written as $\xi(t) = \int_0^t \dot{\xi}(s)ds$ with $\dot{\xi}(s) \in [\mu, z]$. Using this, it can be shown that $F_{z,T}$ is compact. Since $I_0(\xi)$ is lower semi-continuous, the inner infimum in (2.6) is attained by some function $\xi^*$. To characterize $\xi^*$, it is now (finally) convenient to remove the reflection operator. Given we require $\xi(T) = 0$, the concavity requirement implies that we can restrict our search to functions $\xi$ for which $\dot{\xi}(s) = \mu$ for $s > T$. Thus, the inner infimum of (2.6) is equivalent to minimizing $\int_0^T \Lambda^*(\dot{\xi}(s))ds$ subject to the constraints $\xi(t) = \int_0^t \dot{\xi}(s)ds$ with $\dot{\xi}$ decreasing, $\xi(0) = 0, \dot{\xi}(0) = z$, $\xi(T) = 0$ and $\int_0^T (R\xi)(s)^p ds \geq 1$. In turn, this is equivalent to requiring $\dot{\xi}$ decreasing, $\xi(0) = 0, \dot{\xi}(0) = z$, $\xi(T) = 0$ and $\int_0^T \xi(s)^p ds \geq 1$. Applying standard variational methods (see, for example, [12]), there exist constants $c$ and $\ell \geq 0$

such that $\xi^*$ satisfies the differential equation $\nabla\Lambda^*(\dot{\xi^*}(s)) = c - \ell p\left(\int_0^s \xi^*(t)^{p-1}dt\right)$. Since $\dot{\xi}(0) = z$, $c = \nabla\Lambda^*(z)$. Since $\nabla\Lambda^*(z) = (\nabla\Lambda)^{-1}(z)$, we can write

$$\dot{\xi^*}(s) = \nabla\Lambda\left(\nabla\Lambda^*(z) - \ell p\left(\int_0^s \xi^*(t)^{p-1}dt\right)\right). \tag{2.7}$$

To summarize the discussion in this section, we conclude that we can compute $\mathcal{B}_0^*$ by minimizing $\int_0^T \Lambda^*(\dot{\xi}(s))ds$ with $\dot{\xi}(s)$ satisfying (2.7), over $z \geq \mu$, $T \geq 0$, $\ell \geq 0$.

## 3. Proofs of the tail asymptotics of the areas below the busy periods

This section is organized as follows. Section 3.1 collects key technical results (Proposition 3.1, Proposition 3.2) required for the main proof and then provides the main proof of the tail asymptotics of $W_n$ and $\bar{V}_n$ (Theorem 2.2 and Theorem 2.3). Section 3.2 and Section 3.3 provide the proofs of the key technical results, i.e., Proposition 3.1 and Proposition 3.2, respectively. Section 3.4 proves Proposition 2.1.

### 3.1. Proof of Theorem 2.2 and Theorem 2.3

Before we prove the Theorem 2.2 and Theorem 2.3, we first state a couple of technical propositions that facilitates the proofs.

Proposition 3.1. *(i) Recall that $T_1 = \inf\{k > 0 : X_k = 0\}$.*

$$\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \geq 1\right) \leq -\mathcal{B}_y^*.$$

*(ii) Recall that $W_1 = \sum_{i=1}^{T_1} X_i^p$.*

$$\liminf_{u\to\infty} \frac{1}{u^{1/(1+p)}} \log \mathbf{P}_0 (W_1 > u) \geq -\mathcal{B}_0^*. \tag{3.1}$$

*Proof.* The proof of Proposition 3.1 is provided in Section 3.2. □

Proposition 3.2. *(i) $\sum_{k=0}^{m-1} X_k^p > x^{1+p} \iff \int_0^{m/x} \left(\frac{X_{\lfloor xs \rfloor}}{x}\right)^p ds > 1$*

*(ii) Let $\bar{y} = (|\mu|(p+1))^{1/1+p}$. For any $y \geq \bar{y}$,*

$$\mathcal{B}_y^* = 0.$$

*(iii) It holds that*

$$\mathcal{B}_0^* = \mathcal{B}_\pi^*.$$

*(iv) Finally,*

$$\lim_{k\to\infty} \min_{i\geq 1} \left\{\frac{i-1}{k}\beta\bar{y} + \mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\} = \inf_{y\in[0,\infty)} \left\{\beta y + \mathcal{B}_y^*\right\} = \mathcal{B}_\pi^*.$$

*Proof.* The proof of Proposition 3.2 is provided in Section 3.3. □

With Proposition 3.1 and Proposition 3.2 in our hands, we can readily prove Theorem 2.2.

*Proof of Theorem 2.2.* For the upper bound, setting $t = x^{p+1}$,

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t^{1/(1+p)}} \log \mathbf{P}(W_1 \ge t) &= \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_0 \left( \sum_{k=0}^{T_1-1} X_k^p \ge x^{1+p} \right) \\ &= \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_0 \left( \int_0^{T_1/x} \left( \frac{X(\lfloor ux \rfloor)}{x} \right)^p du \ge 1 \right) \\ &\le -\mathcal{B}_0^*.\end{aligned}$$

where we applied part *(i)* of Proposition 3.2 for the second equality and part *(i)* of Proposition 3.1 for the inequality. This together with the matching lower bound in part *(ii)* of Proposition 3.1, we arrive at the desired asymptotics (2.4). □

The proof of Theorem 2.3 is slightly more involved.

*Proof of Theorem 2.3.* We start with proving the large deviation upper bound for $\bar{V}_n$. Denote the time-reversed Markov process of $\{X_n\}_{n\ge0}$ with $\{X_n^*\}_{n\ge0}$, and let $T_1^* = \inf\{i > 0 : X_i^* \le 0\}$. Let $\bar{y} \triangleq (|\mu|(p+1))^{1/(1+p)}$ and fix $b > 0$. Setting $x^{p+1} = nb$,

$$\begin{aligned}\mathbf{P}_0(\bar{V}_n \ge b) = \mathbf{P}_0 \left( \frac{1}{n} \sum_{i=T_{N(n)}}^{n} X_i^p \ge b \right) &= \frac{1}{\pi(0)} \mathbf{P}_\pi \left( \frac{1}{n} \sum_{i=0}^{T_1^*} (X_i^*)^p \ge b, X_n^* = 0 \right) \\ &\le \frac{n+1}{\pi(0)} \mathbf{P}_\pi \left( \frac{1}{nb} \sum_{i=0}^{T_1-1} X_i^p \ge 1 \right) = \frac{n}{\pi(0)} \mathbf{P}_\pi \left( \int_0^{T_1/x} \left( \frac{X(\lfloor ux \rfloor)}{x} \right)^p du \ge 1 \right), \end{aligned} \tag{3.2}$$

where the second equality follows from Lemma A.1 with $g(y_0, ..., y_n) = I(\sum_{\max\{i\le n: y_i=0\}}^n y_i^p > nb)$, the inequality follows from the upper bound in Lemma A.2, and the last equality follows from part *(i)* of Proposition 3.2.

From the tower property, we have that

$$\begin{aligned}&\mathbf{P}_\pi \left( \int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \ge 1 \right) \\ &= \mathbf{E}_\pi \left[ \left( \mathbb{1}\{X(0) \ge x\bar{y}\} + \sum_{i=1}^{k} \mathbb{1}\left\{ X(0) \in \left[ \frac{i-1}{k} x\bar{y}, \frac{i}{k} x\bar{y} \right] \right\} \right) \mathbf{P}\Big( \int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \ge 1 \Big| X(0) \Big) \right] \\ &\le \mathbf{E}_\pi \mathbb{1}\{X(0) \ge x\bar{y}\} + \sum_{i=1}^{k} \mathbf{E}_\pi \left[ \mathbb{1}\left\{ X(0) \in \left[ \frac{i-1}{k} x\bar{y}, \infty \right) \right\} \mathbf{P}_{\frac{i}{k}x\bar{y}} \Big( \int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \ge 1 \Big) \right] \\ &\le \pi[x\bar{y}, \infty) + \sum_{i=1}^{k} \pi \left[ \frac{i-1}{k} x\bar{y}, \infty \right) \mathbf{P}_{\frac{i}{k}x\bar{y}} \Big( \int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \ge 1 \Big) \end{aligned} \tag{3.3}$$

where in the first inequality we used that the Markov chain $X_n$ is stochastically monotone. Therefore, by the principle of the maximum term and part *(i)* of Proposition 3.1,

$$\begin{aligned}&\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_\pi \left( \int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \ge 1 \right) \\ &\le \limsup_{x\to\infty} \frac{1}{x} \log \pi[x\bar{y}, \infty)\end{aligned}$$

$$\vee \max_{i=1,\dots,k}\left\{\limsup_{x\to\infty}\frac{1}{x}\log\left(\pi\left[\frac{i-1}{k}x\bar{y},\,\infty\right)\mathbf{P}_{\frac{i}{k}x\bar{y}}\Big(\int_0^{T_1/x}(X(\lfloor ux\rfloor)/x)^p du\geq 1\Big)\right)\right\}$$
$$=(-\beta\bar{y})\vee\max_{i=1,\dots,k}\left\{-\frac{i-1}{k}\beta\bar{y}+\limsup_{x\to\infty}\frac{1}{x}\log\mathbf{P}_{\frac{i}{k}x\bar{y}}\Big(\int_0^{T_1/x}(X(\lfloor ux\rfloor)/x)^p du\geq 1\Big)\right\}.$$
$$\leq(-\beta\bar{y})\vee\max_{i=1,\dots,k}\left\{-\frac{i-1}{k}\beta\bar{y}-\mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\}.$$

Note that since $\mathcal{B}^*_y=0$ for $y\geq\bar{y}$ due to part *(ii)* of Proposition 3.2,

$$(-\beta\bar{y})\vee\max_{i=1,\dots,k}\left\{-\frac{i-1}{k}\beta\bar{y}-\mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\}=\max_{i\geq 1}\left\{-\frac{i-1}{k}\beta\bar{y}-\mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\}=-\min_{i\geq 1}\left\{\frac{i-1}{k}\beta\bar{y}+\mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\}.$$

Taking $k\to\infty$ and applying part *(iii)* and *(iv)* of Proposition 3.2,

$$\limsup_{x\to\infty}\frac{1}{x}\log\mathbf{P}_\pi\left(\int_0^{T_1/x}(X(\lfloor ux\rfloor)/x)^p du\geq 1\right)\leq-\mathcal{B}^*_\pi=-\mathcal{B}^*_0.$$

From this along with (3.2), we arrive at the desired upper bound:

$$\begin{aligned}\limsup_{x\to\infty}\frac{1}{n^{1/(1+p)}}\log\mathbf{P}_0(\bar{V}_n\geq b)&\leq\limsup_{x\to\infty}\frac{1}{x}\log\mathbf{P}_\pi\left(\int_0^{T_1/x}(X(\lfloor ux\rfloor)/x)^p du\geq 1\right)\cdot b^{1/(1+p)}\\&\leq-\mathcal{B}^*_0\cdot b^{1/(1+p)}.\end{aligned}$$

Next, for $n$ sufficiently large, using the lower bound of Lemma A.2 for $n\geq n_0$:

$$\begin{aligned}\mathbf{P}_0(\bar{V}_n\geq b)=\mathbf{P}_0\left(\frac{1}{n}\sum_{i=T_{N(n)}}^{n}X_i^p\geq b\right)&=\frac{1}{\pi(0)}\mathbf{P}_\pi\left(\frac{1}{n}\sum_{i=0}^{T_1^*}(X_i^*)^p\geq b,X_n^*=0\right)\\&\geq\frac{\pi(0)}{2}\mathbf{P}_0\left(\frac{1}{nb}\sum_{i=0}^{T_1}{X_i}^p>1\right)=\frac{\pi(0)}{2}\mathbf{P}_0\left(W_1>nb\right).\end{aligned}\tag{3.4}$$

From here, we can directly apply part *(ii)* of Proposition 3.1 to (3.4) and obtain the matching lower bound:

$$\liminf_{n\to\infty}\frac{1}{n^{1/(1+p)}}\log\mathbf{P}_0(\bar{V}_n>b)\geq-\mathcal{B}^*_0\cdot b^{1/(1+p)}.$$

□

### 3.2. Proof of Proposition 3.1

Recall that $B_y^{\mathbb{AC}}=B_y\cap\mathbb{AC}[0,\infty)$. For a fixed $M>0$, let $B_y^{\mathbb{AC};M}\triangleq B_y^{\mathbb{AC}}\cap\{\xi\in\mathbb{D}[0,\infty):\mathcal{T}(\xi)\leq M\}$ and let $B_y^M\triangleq B_y\cap\{\xi\in\mathbb{D}[0,\infty):\mathcal{T}(\xi)\leq M\}$.

LEMMA 3.1. *For any given $y\geq 0$, there exists a constant $M=M(y)>0$ such that*

- *for each $\xi\in B_y$, there exists a path $\zeta\in B_y^M$ such that $I_y(\zeta)\leq I_y(\xi)$;*
- *therefore,*

$$\inf_{\xi\in B_y}I_y(\xi)=\inf_{\xi\in B_y^M}I_y(\xi);\tag{3.5}$$

- *moreover,* $M(y) \le cy + d$ *for some* $c > 0$ *and* $d > 0$.

*Proof.* Let $\bar{y} \triangleq (|\mu|(p+1))^{1/1+p}$. In case $y \ge \bar{y}$, the equality in (3.5) holds with the optimal values of the LHS and RHS both being zero: to see this, set $M \triangleq -y/\mu$ and $\zeta(t) \triangleq y + \mu t$, and note that $\int_0^{\mathcal{T}(\zeta)} R(\zeta)(s)^p ds \ge 1$ and $\mathcal{T}(\zeta) = M$, and hence, $\zeta \in B_y^M$ while $I_y(\zeta) = 0$. Therefore, we assume for the rest of the proof that $y < \bar{y}$. It is enough to show that there exists $M > 0$ such that

$$\text{For any given } \xi \in B_y \setminus B_y^M, \text{ one can find } \zeta \in B_y^M \text{ such that } I_y(\zeta) \le I_y(\xi). \tag{3.6}$$

To construct such $M$, consider $w$ and $z$ such that $\mu < w < 0 < z$, $\Lambda^*(w) < \infty$ and $\Lambda^*(z) < \infty$. We consider a piece-wise linear path

$$\zeta(t) \triangleq (y + zt)\mathbb{1}_{[0,(\bar{y}-y)/z]}(t) + \big(\bar{y} + \mu\big(t - (\bar{y}-y)/z\big)\big)\mathbb{1}_{[(\bar{y}-y)/z,\infty)}(t)$$

and

$$M \triangleq \max\left\{\frac{(\bar{y}-y)\Lambda^*(z)}{z\Lambda^*(w)},\ (\bar{y}-y)/z - \bar{y}/\mu, \frac{y + \frac{\bar{y}-y}{z\theta_+}\Lambda^*(z)}{-w}\right\}.$$

Then, $\zeta \in B_y^M$ and $I_y(\zeta) = \Lambda^*(z)\frac{\bar{y}-y}{z}$. Suppose that $\xi \in B_y \setminus B_y^M$ so that $\mathcal{T}(\xi) > M$. If $\xi \notin \mathbb{BV}[0,\infty)$, $I(\xi) = \infty$, from which (3.6) is immediate. Suppose that $\xi \in \mathbb{BV}[0,\infty)$ so that $\xi = \xi^{(a)} + \xi^{(u)} + \xi^{(d)}$. Note that if we set $\xi' \triangleq \xi^{(a)} + \xi^{(u)}$, then $\mathcal{T}(\xi') \ge \mathcal{T}(\xi)$, $I_y(\xi') \le I_y(\xi)$, and $\xi' \in B_y$. Therefore, we assume w.l.o.g. $\xi^{(d)} = 0$. Note that if $\xi^{(u)}(\mathcal{T}(\xi)) \ge \Lambda^*(z)\frac{\bar{y}-y}{z\theta_+}$, then $I_y(\xi) \ge \theta_+\xi^{(u)}(\mathcal{T}(\xi)) > \Lambda^*(z)\frac{\bar{y}-y}{z} = I_y(\zeta)$. On the other hand, if $\xi^{(u)}(\mathcal{T}(\xi)) < \Lambda^*(z)\frac{\bar{y}-y}{z\theta_+}$, then $\xi^{(a)}(\mathcal{T}(\xi)) \ge -\Lambda^*(z)\frac{\bar{y}-y}{z\theta_+}$, and hence, by the construction of $M$, $\mu < w < -\Big(y + \frac{\bar{y}-y}{z\theta_+}\Lambda^*(z)\Big)/\mathcal{T}(\xi) < \Big(\xi^{(a)}(\mathcal{T}(\xi)) - y\Big)/\mathcal{T}(\xi)$. Therefore,

$$\begin{aligned} I_y(\xi) &\ge \int_0^{\mathcal{T}(\xi)} \Lambda^*(\dot{\xi}(s))ds \ge \mathcal{T}(\xi)\cdot\Lambda^*\Big(\Big(\xi^{(a)}(\mathcal{T}(\xi)) - y\Big)/\mathcal{T}(\xi)\Big) \\ &\ge \mathcal{T}(\xi)\cdot\Lambda^*\Big(-\Big(y + \frac{\bar{y}-y}{z\theta_+}\Lambda^*(z)\Big)/\mathcal{T}(\xi)\Big) \ge \mathcal{T}(\xi)\cdot\Lambda^*(w) \ge M\cdot\Lambda^*(w) \ge \frac{(\bar{y}-y)}{z}\Lambda^*(z) = I_y(\zeta), \end{aligned}$$

where the second inequality is from Jensen's inequality, the third and fourth inequalities are from the monotonicity of $\Lambda^*$ on $[\mu,\infty)$, and the fifth and the sixth inequalities are from the construction of $M$ and $\zeta$, respectively. This concludes the proof of (3.6) and (3.1).

To see the existence of $c > 0$ and $d > 0$, note that for the case $y \ge \bar{y}$, our construction of $M(y)$ is linear in $y$, whereas $M(y)$ is bounded for the case $y < \bar{y}$. ☐

Fix $T > 0$ and consider a functional $\Phi_T : \mathbb{D}[0,T] \to \mathbb{R}_+$, where $\Phi_T(\xi) = \int_0^T (R(\xi)(s))^p ds$. Now, let $\mathcal{V}_y^{T*}$ denote the optimal value of the following optimization problem $\mathcal{V}_y^T$:

$$\mathcal{V}_y^{T*} \triangleq \inf_{\xi \in V_y^T} I_y^{\mathbb{BV}[0,T]}(\xi), \tag{$\mathcal{V}_y^T$}$$

where

$$V_y^T \triangleq \{\xi \in \mathbb{D}[0,T] : \xi(0) = y,\ \Phi_T(\xi) \ge 1\},$$

and

$$I_y^{\mathbb{BV}[0,T]}(\xi) \triangleq \begin{cases} \int_0^T \Lambda^*(\dot{\xi}(s))ds + \theta_+ \xi^{(u)}(T) + \theta_- \xi^{(d)}(T) & \text{if } \xi(0) = y \text{ and } \xi \in \mathbb{BV}[0,T], \\ \infty & \text{otherwise.} \end{cases}$$

LEMMA 3.2. *Let $M > 0$ be the constant in Lemma 3.1. Then,*

$$\mathcal{B}_y^* = \mathcal{V}_y^{T*}$$

*for any $T \geq M$.*

*Proof.* The conclusion of the Lemma follows immediately from the following claims.

**Claim 1:** $\mathcal{V}_y^{T*}$ is nonincreasing in $T$.

*Proof of Claim 1.* Let $t_1 < t_2$. For each $\xi_1 \in V_y^{t_1}$, consider $\xi_2(s) \triangleq \xi_1(s \wedge t_1) + \mu(s - t_1)\mathbb{1}_{(t_1,t_2]}(t)$. Then, $\xi_2 \in V_y^{t_2}$ and $I_y^{\mathbb{BV}[0,t_1]}(\xi_1) = I_y^{\mathbb{BV}[0,t_2]}(\xi_2)$. Therefore, $\mathcal{V}_y^{t_2*}$ is at least as small as $\mathcal{V}_y^{t_1*}$.

**Claim 2:** If $M > 0$ is such that $\inf_{\xi \in B_y^M} I_y(\xi) = \inf_{\xi \in B_y} I_y(\xi)$ as in Lemma 3.1, then

$$\inf_{\xi \in B_y^M} I_y(\xi) \geq \mathcal{V}_y^{M*}.$$

*Proof of Claim 2.* Given an $\epsilon > 0$, consider $\xi_\epsilon \in B_y^M$ such that $I_y(\xi_\epsilon) \leq \inf_{\xi \in B_y^M} I_y(\xi) + \epsilon$. Set $\zeta_\epsilon(t) \triangleq \xi_\epsilon(t \wedge \mathcal{T}(\xi_\epsilon)) + \mu(t - \mathcal{T}(\xi_\epsilon))\mathbb{1}_{(\mathcal{T}(\xi_\epsilon),M]}(t)$. Then, $\zeta_\epsilon \in V_y^M$ and hence,

$$\mathcal{V}_y^{M*} = \inf_{\xi \in V_y^M} I_y^{\mathbb{BV}[0,M]}(\xi) \leq I_y^{\mathbb{BV}[0,M]}(\zeta_\epsilon) \leq I_y(\xi_\epsilon) \leq \inf_{\xi \in B_y^M} I_y(\xi) + \epsilon.$$

Taking $\epsilon \to 0$, we arrive at Claim 2.

**Claim 3:** For any $T > 0$,

$$\mathcal{V}_y^{T*} \geq \inf_{\xi \in B_y} I_y(\xi).$$

*Proof of Claim 3.* Given an $\epsilon > 0$, consider $\xi_\epsilon \in V_y^T$ such that $I_y^{\mathbb{BV}[0,T]}(\xi_\epsilon) \leq \inf_{\xi \in V_y^T} I_y^{\mathbb{BV}[0,T]}(\xi) + \epsilon$. Set $\zeta_\epsilon(t) \triangleq \xi_\epsilon(t \wedge T) + \mu(t - T)\mathbb{1}_{(T,\infty)}(t)$. Then, $\zeta_\epsilon \in B_y$ and hence,

$$\mathcal{V}_y^{T*} = \inf_{\xi \in V_y^T} I_y^{\mathbb{BV}[0,T]}(\xi) + \epsilon \geq I_y^{\mathbb{BV}[0,T]}(\xi_\epsilon) \geq I_y(\zeta_\epsilon) \geq \inf_{\xi \in B_y^T} I_y(\xi).$$

Taking $\epsilon \to 0$, we arrive at Claim 3. □

Set

$$K_t \triangleq \left\{ \xi \in \mathbb{D}[0,t] : \xi(0) = 0, \ \int_0^t (R(\xi)(s))^p ds \geq 1, \ \xi(s) \geq 0 \text{ for } s \in [0,t] \right\}.$$

The following corollary is immediate from the two previous lemmas.

COROLLARY 3.1. *Let $M > 0$ be the constant in Lemma 3.1. For any $y \geq 0$,*

$$\inf_{t \in [0,M]} \inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi) = \mathcal{V}_0^{M*} = \mathcal{B}_0^*.$$

PROPOSITION 3.3. *For the optimal value $\mathcal{B}_y^*$ associated with $\mathcal{B}_y$, we have that* *(i)* $y \mapsto \mathcal{B}_y^*$ *is non-increasing in* $y$*;*

*(ii)* $y \mapsto \mathcal{B}_y^*$ *is Lipschitz continuous.*

*Proof.* For part *(i)*, let $x, y$ be such that $0 \le x < y$. We will show that for any $\epsilon > 0$, there exists $\zeta \in B_y$ such that $I_y(\zeta) < \mathcal{B}_x^* + \epsilon$. Due to Lemma 3.1, we can pick $\xi \in B_x$ such that $I_x(\xi) < \mathcal{B}_x^* + \epsilon$ and $\mathcal{T}(\xi) < \infty$. Set

$$\zeta(t) \triangleq (y-x) + \xi(t \wedge \mathcal{T}(\xi)) + \mu \cdot [t - \mathcal{T}(\xi)]^+.$$

Then, since $\zeta(0) = y$, $R(\zeta)(t) \ge R(\xi)(t)$ on $t \in [0, \mathcal{T}(\xi)]$, we see that $\zeta \in B_y$. On the other hand, since $\zeta$ has no jump on $[\mathcal{T}(\xi), \infty)$, and $\Lambda^*(\dot{\zeta}^{(a)}(s)) = \Lambda^*(\mu) = 0$ on $s \in [\mathcal{T}(\xi), \infty)$, as well as $\mathcal{T}(\xi) \le \mathcal{T}(\zeta)$,

$$\begin{aligned}
I_y(\zeta) &= \int_0^{\mathcal{T}(\zeta)} \Lambda^*(\dot{\zeta}^{(a)}(s))ds + \theta_+ \zeta^{(u)}(\mathcal{T}(\zeta)) + \theta_- \zeta^{(d)}(\mathcal{T}(\zeta)) \\
&= \int_0^{\mathcal{T}(\xi)} \Lambda^*(\dot{\zeta}^{(a)}(s))ds + \theta_+ \zeta^{(u)}(\mathcal{T}(\xi)) + \theta_- \zeta^{(d)}(\mathcal{T}(\xi)) \\
&= \int_0^{\mathcal{T}(\xi)} \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+ \xi^{(u)}(\mathcal{T}(\xi)) + \theta_- \xi^{(d)}(\mathcal{T}(\xi)) \\
&= I_x(\xi) < \mathcal{B}_x^* + \epsilon.
\end{aligned}$$

For part *ii)*, note that we only need to prove one side of the inequality thanks to part *i)*. That is, it is enough to show that if $0 \le x < y$, then $\mathcal{B}_x^* \le \mathcal{B}_y^* + (y-x)\Lambda^*(1)$. Fix an $\epsilon > 0$ and pick $\zeta \in B_y$ such that $I_y(\zeta) \le \mathcal{B}_y^* + \epsilon$. Set

$$\xi(t) \triangleq (x+t)\mathbb{1}_{[0,y-x]}(t) + \zeta(t-(y-x))\mathbb{1}_{[y-x,\infty)}(t).$$

Then $\xi^{(u)}(s) = \zeta^{(u)}(s-(y-x))$ and $\xi^{(d)}(s) = \zeta^{(d)}(s-(y-x))$ on $s \in [y-x, \infty]$, and $\xi^{(u)}(s) = \xi^{(d)}(s) = 0$ on $s \in [0, y-x]$, and $\mathcal{T}(\xi) = \mathcal{T}(\zeta) + y - x$. Hence,

$$\begin{aligned}
I_x(\xi) &= \int_0^{y-x} \Lambda^*(1)ds + \int_{y-x}^{\mathcal{T}(\xi)} \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+ \cdot \xi^{(u)}(\mathcal{T}(\xi)) + \theta_- \cdot \xi^{(d)}(\mathcal{T}(\xi)) \\
&= (y-x)\Lambda^*(1) + \int_{y-x}^{\mathcal{T}(\zeta)+y-x} \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+ \cdot \xi^{(u)}(\mathcal{T}(\zeta)+y-x) + \theta_- \cdot \xi^{(d)}(\mathcal{T}(\zeta)+y-x) \\
&= (y-x)\Lambda^*(1) + \int_0^{\mathcal{T}(\zeta)} \Lambda^*(\dot{\xi}^{(a)}(s+(y-x)))ds + \theta_+ \cdot \zeta^{(u)}(\mathcal{T}(\zeta)) + \theta_- \cdot \zeta^{(d)}(\mathcal{T}(\zeta)) \\
&= (y-x)\Lambda^*(1) + \int_0^{\mathcal{T}(\zeta)} \Lambda^*(\dot{\zeta}^{(a)}(s))ds + \theta_+ \cdot \zeta^{(u)}(\mathcal{T}(\zeta)) + \theta_- \cdot \zeta^{(d)}(\mathcal{T}(\zeta)) \\
&= (y-x)\Lambda^*(1) + I_y(\zeta) \le (y-x)\Lambda^*(1) + \mathcal{B}_y^* + \epsilon.
\end{aligned}$$

Since $\xi \in B_x$, this implies that $\mathcal{B}_x^* \le (y-x)\Lambda^*(1) + \mathcal{B}_y^* + \epsilon$. Taking $\epsilon \to 0$, we arrive at the desired inequality. ∎

Next, we formulate our main preparatory result for the asymptotic upper bound. This relies on a result of [18], for which we need to verify a uniform continuity result. This is the goal of the next two lemmas. Let $\mathrm{TV}(\xi)$ be the total variation of $\xi$.

Lemma 3.3. *The function $H : \mathbb{D}[0,T] \to [0,\infty)$ given by $H(\xi) = \int_0^T \xi(s)ds$ is Lipschitz continuous on the set of $\{\xi : \mathrm{TV}(\xi) \le M\}$ for every $M < \infty$.*

*Proof.* Let $\xi$ be such that $\mathrm{TV}(\xi) \le M$ and let $\zeta$ be such that $d_{M_1}(\xi,\zeta) \le \epsilon$. Set $\eta(t) \triangleq \inf\{x : d((t,x),\Gamma(\xi)) \le \epsilon\}$ where $\Gamma(\xi)$ is the completed graph of $\xi$ and $d$ is the $L_1$ distance in $\mathbb{R}^2$, i.e., $d((t,x),(s,y)) = |t-s| + |x-y|$. Then $d_{M_1}(\xi,\zeta) \le \epsilon$ implies that $\zeta(t) \ge \eta(t)$ for all $t \in [0,T]$. Due to the construction of $\eta$ and the fact that $L_1$ balls are contained in $L_2$ balls of the same radius, the difference between the area below $\xi$ and the area below $\eta$ is bounded by $\mathrm{len}(\Gamma(\xi)) \times \epsilon$, where the lenght $\mathrm{len}(\Gamma(\xi))$ of $\Gamma(\xi)$ is bounded by $T + \mathrm{TV}(\xi)$. Putting everything together, we conclude that

$$\int_0^T \xi(s)ds - \int_0^T \zeta(s) \ge \int_0^T \xi(s)ds - \int_0^T \eta(s) \ge (T + M_\alpha)\epsilon. \tag{3.7}$$

The upper bound can be established in the same way. □

Recall the function $\Phi_T : \mathbb{D}[0,T] \to [0,\infty)$ defined as $\Phi_T(\xi) = \int_0^T R(\xi)(s)^p ds$.

Lemma 3.4. *$\Phi_T$ is Hölder continuous with index* $\min\{p,1\}$ *on the set* $\{\xi : I_K(\xi) \le \alpha\}$.

*Proof.* Let $\xi$ be such that $I_K(\xi) \le \alpha$. Let $\delta \in (0, \min\{\theta_+, |\theta_-|\})$. Observe that $\Lambda^*(\dot{\xi}(s)) \ge \delta|\dot{\xi}(s)| - \Lambda(\delta)$. Consequently,

$$\int_0^1 |\dot{\xi}(s)|ds + \xi^u(1) + |\xi^d(1)| \le (\alpha + \Lambda(\delta))/\delta \triangleq M_\alpha.$$

Consequently, if $I_K(\xi) \le \alpha$, then $\mathrm{TV}(\xi) \le M_\alpha$. The reflection map $R$ is a Lipschitz continuous map from $\mathbb{D}[0,T]$ to $\mathbb{D}[0,T]$ w.r.t. the $\mathcal{M}_1$ topology with Lipschitz constant 2 (cf. [19], Theorem 13.5.1), and if the total variation of $\xi$ is bounded by $M_\alpha$, the total variation of $R(\xi)$ is bounded by $2M_\alpha$. Consequently, the total variation of $R(\xi)^p$ is bounded by $2^p(2M_\alpha)^p \triangleq \tilde{M}_\alpha$. Moreover, the map $\xi \to R(\xi)^p$ is Hölder continuous on $\{\xi : I_K(\xi) \le \alpha\}$ with index $\min\{p,1\}$. Since the composition of a Lipschitz and Hölder continuous map is again Hölder continuous (in this case, with exponent $\min\{p,1\}$), the proof follows from Lemma 3.3. □

Lemma 3.5. *(i) For any $t, y \ge 0$ and $T > 0$,*

$$\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(T_1/x > T\right) \le ty + T \log \mathbf{E} e^{tU}. \tag{3.8}$$

*(ii) For any $y \ge 0$ and $T > 0$,*

$$\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^T (X(\lfloor sx \rfloor)/x)^p ds \ge 1\right) \le -\mathcal{V}_y^{T^*}. \tag{3.9}$$

*Proof.* For part *(i)*, note that

$$\mathbf{P}_{xy}\left(T_1 > xT\right) \le \mathbf{P}_{xy}\left(X_{\lfloor xT \rfloor} > 0\right) = \mathbf{P}\left(\sum_{i=1}^{\lfloor xT \rfloor} U_i > -xy\right) \le e^{txy} \mathbf{E}\left(e^{tU}\right)^{\lfloor xT \rfloor},$$

where the last inequality is from the Markov inequality. Taking logarithms, dividing both sides by $x$, and taking lim sup, we get (3.8).

For part *(ii)*, as a Hölder continuous map is uniformly continuous, Lemma 3.3 allows us to apply Result 3 (ii) to obtain

$$\lim_{x\to\infty}\frac{1}{x}\log\mathbf{P}_{xy}\left(\int_0^T (X(\lfloor sx\rfloor)/x)^p ds\geq 1\right)\leq -\inf_{a\in[1,\infty)} J_y(a),$$

where $J_y(a)=\inf\{I_K(\xi):\xi\in\mathbb{D}[0,T],\xi(0)=y,\Phi_T(\xi)=a\}$. Obviously, $\inf_{a\in[1,\infty)} J_y(a)=\mathcal{V}_y^{T*}$ and (3.9) follows. ☐

Now we are ready to prove Proposition 3.1.

*Proof of Proposition 3.1.* For part *(i)*, consider a small enough $t_0>0$ so that $\mathbf{E}e^{t_0U}<1$. Then, thanks to Lemma 3.2, we can pick a sufficiently large $T>0$ so that $\mathcal{B}_y^*=\mathcal{V}_y^{T*}$ and $t_0y+T\log\mathbf{E}e^{t_0U}<-\mathcal{B}_y^*$. Considering the case $T_1/x\leq T$ and $T_1/x>T$ separately and then applying the principle of the maximum term,

$$\begin{aligned}
&\limsup_{x\to\infty}\frac{1}{x}\log\mathbf{P}_{xy}\bigg(\int_0^{T_1/x}(X(\lfloor ux\rfloor)/x)^p du\geq 1\bigg)\\
&\leq \limsup_{x\to\infty}\frac{1}{x}\log\left\{\mathbf{P}_{xy}\bigg(\int_0^{T_1/x}(X(\lfloor ux\rfloor)/x)^p du\geq 1,\,T_1/x\leq T\bigg)+\mathbf{P}_{xy}\Big(T_1/x>T\Big)\right\}\\
&\leq \limsup_{x\to\infty}\frac{1}{x}\log\mathbf{P}_{xy}\left(\int_0^{T}(X(\lfloor ux\rfloor)/x)^p du\geq 1\right)\vee\limsup_{x\to\infty}\frac{1}{x}\log\mathbf{P}_{xy}\left(T_1/x>T\right)\\
&\leq\Big(-\mathcal{V}_y^{T*}\Big)\vee\Big(t_0y+T\log\mathbf{E}e^{t_0U}\Big)=\Big(-\mathcal{B}_y^*\Big)\vee\Big(t_0y+T\log\mathbf{E}e^{t_0U}\Big)=-\mathcal{B}_y^*,
\end{aligned}\tag{3.10}$$

where we used Lemma 3.5 for the third inequality.

Next, we move on to part *(ii)*. For any given $t>0$, let

- $A_{t,\epsilon}=\{\xi\in\mathbb{D}[0,t]:\xi(0)=\epsilon,\ \int_0^t R(\xi)(s)^p ds>1,\,\xi(s)>0,\ \forall\, s\in[0,t]\}$ and
- $\tilde{A}_{t,\epsilon}=\{\xi\in\mathbb{D}[0,t]:\xi(0)=\epsilon,\ \int_0^t R(\xi)(s)^p ds>1,\,\xi(s)>\epsilon/2,\,\forall s\in[0,t]\}$.

Set $u=x^{1+p}$. Let $\epsilon$ be small enough such that $\mathbf{P}(U_1>\sqrt{\epsilon})>0$. Define the event $B_{x,\epsilon}=\{U_i>\sqrt{\epsilon},i=1,\dots,\lceil x\sqrt{\epsilon}\rceil\}$. Setting $k^*=\lceil x\sqrt{\epsilon}\rceil+1$, we obtain

$$\begin{aligned}
&\liminf_{u\to\infty}\frac{1}{u^{1/(1+p)}}\log\mathbf{P}_0\left(W_1>u\right)\\
&=\liminf_{x\to\infty}\frac{1}{x}\log\mathbf{P}_0\left(\sum_{k=0}^{T_1}X_k^p>u\right)\\
&\geq\liminf_{x\to\infty}\frac{1}{x}\log\mathbf{P}_0\left(\sum_{k=k^*}^{T_1}X_k^p>x^{1+p},\,B_{x,\epsilon}\right)\\
&=\liminf_{x\to\infty}\frac{1}{x}\log\left[\mathbf{P}_0\left(\sum_{k=k^*}^{T_1}X_k^p>x^{1+p}|B_{x,\epsilon}\right)\mathbf{P}_0\left(B_{x,\epsilon}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \liminf_{x\to\infty} \frac{1}{x} \log \left[ \mathbf{P}_{\epsilon x}\left( \sum_{k=0}^{T_1} X_k^p > x^{1+p} \right) \mathbf{P}_0\left(B_{x,\epsilon}\right) \right] \\
&= \liminf_{x\to\infty} \frac{1}{x} \log \left[ \mathbf{P}_{\epsilon x}\left( \int_0^{T_1/x} \left(X_{\lfloor xs \rfloor}/x\right)^p ds > 1 \right) \mathbf{P}_0\left(B_{x,\epsilon}\right) \right] \\
&\geq \liminf_{x\to\infty} \frac{1}{x} \log \left[ \mathbf{P}_{\epsilon x}\left( \int_0^{t} \left(X_{\lfloor xs \rfloor}/x\right)^p ds > 1,\, T_1 > xt \right) \mathbf{P}_0\left(B_{x,\epsilon}\right) \right] \\
&= \liminf_{x\to\infty} \frac{1}{x} \log \left[ \mathbf{P}_{\epsilon x}\left( \int_0^{t} \left(X_{\lfloor xs \rfloor}/x\right)^p ds > 1,\, X_{\lfloor xs \rfloor}/x > 0,\, \forall s \in [0,t] \right) \mathbf{P}_0\left(B_{x,\epsilon}\right) \right] \\
&\geq \liminf_{x\to\infty} \frac{1}{x} \log \left[ \mathbf{P}_{\epsilon}\left( \bar{K}_x \in A_{t,\epsilon} \right) \mathbf{P}_0\left(B_{x,\epsilon}\right) \right] \\
&\geq - \inf_{\xi \in (A_{t,\epsilon})^\circ} I_\epsilon^{\mathbb{BV}[0,t]}(\xi) + \sqrt{\epsilon} \log \mathbf{P}(U_1 > \sqrt{\epsilon}) \\
&\geq - \inf_{\xi \in \tilde{A}_{t,\epsilon}} I_\epsilon^{\mathbb{BV}[0,t]}(\xi) + \sqrt{\epsilon} \log \mathbf{P}(U_1 > \sqrt{\epsilon}).
\end{aligned}
$$

where the third equality is from part *(i)* of Proposition 3.2. The second to last inequality follows from part *(i)* of Result 3 since the integral and the infimum are both continuous in the $\mathcal{M}_1$ topology (see, respectively Theorem 11.5.1 and Theorem 13.4.1 of [19]). Recall that

$$
K_t = \left\{ \xi \in \mathbb{D}[0,t] : \xi(0) = 0,\ \int_0^t (R(\xi)(s))^p ds \geq 1,\ \xi(s) \geq 0 \text{ for } s \in [0,t] \right\}.
$$

Note that for all $\epsilon > 0$,

$$
\inf_{\xi \in \tilde{A}_{t,\epsilon}} I_\epsilon^{\mathbb{BV}[0,t]}(\xi) \leq \inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi). \tag{3.11}
$$

To see this, suppose that $\xi \in K_t$. Then, $\tilde{\xi} = \epsilon + \xi$ belongs to $\tilde{A}_{t,\epsilon}$ and $I_\epsilon^{\mathbb{BV}[0,t]}(\tilde{\xi}) = I_0^{\mathbb{BV}[0,t]}(\xi)$. Since the construction holds for every $\xi \in K_t$, we have that $\inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi) \geq \inf_{\xi \in \tilde{A}_{t,\epsilon}} I_\epsilon^{\mathbb{BV}[0,t]}(\xi)$. Therefore,

$$
\liminf_{u\to\infty} \frac{1}{u^{1/(1+p)}} \log \mathbf{P}_0\left(W_1 > u\right) \geq - \inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi) + \sqrt{\epsilon} \log \mathbf{P}(U_1 > \sqrt{\epsilon}).
$$

Since $\epsilon$ and $t$ are arbitrary, taking $\epsilon \to 0$ and taking the infimum over $t \in [0,M]$, Corollary 3.1 gives

$$
\liminf_{u\to\infty} \frac{1}{u^{1/(1+p)}} \log \mathbf{P}_0\left(W_1 > u\right) \geq - \inf_{t\in[0,M]} \inf_{\xi \in K_t} I_0^{\mathbb{BV}[0,t]}(\xi) = -\mathcal{B}_0^*.
$$

□

### 3.3. Proof of Proposition 3.2

*Proof of Proposition 3.2.* For part *(i)*, note that

$$
\frac{1}{x^{1+p}} \sum_{k=0}^{m-1} X_k^p = \frac{1}{x^{1+p}} \int_0^m X_{\lfloor u \rfloor}^p du = \frac{1}{x^{1+p}} \int_0^{m/x} x X_{\lfloor xs \rfloor}^p ds = \int_0^{m/x} \left( \frac{X_{\lfloor xs \rfloor}}{x} \right)^p ds
$$

where the second equality is from the change of variable with $u = xs$. The claimed equivalence is immediate from this.

For part *(ii)*, note that if we set $\xi^*(t) \triangleq \bar{y} - \mu t$, then $I_y(\xi^*) = 0$ while $\xi^* \in B_{\bar{y}}$, and hence, $\mathcal{B}_{\bar{y}}^* = 0$.

For part *(iii)*, note that

$$\lim_{k\to\infty}\min_{i\geq 1}\left\{\frac{i-1}{k}\beta\bar{y}+\mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\}=\lim_{k\to\infty}\left(\min_{i\geq 1}\left\{\beta\frac{i}{k}\bar{y}+\mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\}-\frac{1}{k}\beta\bar{y}\right)=\lim_{k\to\infty}\min_{i\geq 1}\left\{\beta\frac{i}{k}\bar{y}+\mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\}.$$

Moreover, from part *(ii)* of Proposition 3.3,

$$\lim_{k\to\infty}\min_{i\geq 1}\left\{\frac{i}{k}\beta\bar{y}+\mathcal{B}^*_{\frac{i}{k}\bar{y}}\right\}=\inf_{y\in[0,\infty)}\left\{\beta y+\mathcal{B}^*_y\right\}.$$

For part *(iv)*, note that by definition, $\mathcal{B}^*_0\geq\mathcal{B}^*_\pi$. Therefore, we only have to prove that $\mathcal{B}^*_0\leq\mathcal{B}^*_\pi$. Recall that $\beta=\sup\{\theta>0:\mathbf{E}(e^{\theta U})\leq 1\}$ and $\theta_+=\sup\{\theta\in\mathbb{R}:\mathbf{E}(e^{\theta U})<\infty\}$. For the rest of this proof, let $\Lambda$ be the log-moment generating function and let $D_\Lambda$ denote the effective domain of $\Lambda$ i.e; $D_\Lambda=\{x:\Lambda(x)<\infty\}$. We start with a claim: for any $\epsilon>0$ there exists a $u>0$ such that

$$\Lambda^*(u)/u\leq\beta+\epsilon. \tag{3.12}$$

To prove (3.12) we distinguish between the cases $\beta<\theta_+$ and $\beta=\theta_+$. For the first case note that $\beta\in D^\circ_\Lambda$. In view of the convexity and continuity of $\mathbf{E}(e^{\theta U})$, $\mathbf{E}(e^{\beta U})=1$. Due to Lemma 2.2.5 $(c)$ of [5], $\Lambda$ is a differentiable function in $D^\circ_\Lambda$ with $\Lambda'(\eta)=\frac{\mathbf{E}(Ue^{\eta U})}{\mathbf{E}(e^{\eta U})}$. Since $\beta\in D^\circ_\Lambda$ we have that $\Lambda'(\beta)=\mathbf{E}(Ue^{\beta U})<\infty$. In addition, $\Lambda^{'}(0)=\mathbf{E}(U)<0$ implies that $\Lambda(\eta)$ is decreasing for small values of $\eta$. Now, the convexity and differentiability of $\Lambda$ over its effective domain implies that $\Lambda'$ should be increasing at $\beta$ and thus $\mathbf{E}(Ue^{\beta U})>0$. It can be checked that for $u=\mathbf{E}(Ue^{\beta U})$,

$$\frac{\Lambda^*(u)}{u}=\frac{\beta\mathbf{E}(Ue^{\beta U})-\log\mathbf{E}(e^{\beta U})}{\mathbf{E}(Ue^{\beta U})}=\beta,$$

and hence our claim is proved. Consider now the case $\beta=\theta_+$. In view of Equation (5.5) in [14], $\lim_{x\to\infty}\frac{\Lambda^*(x)}{x}=\theta_+$. That is, for any $\epsilon>0$ we can choose a $u$ so that $\Lambda^*(u)/u\leq\theta_++\epsilon=\beta+\epsilon$. We proved the claim (3.12).

Back to the inequality $\mathcal{B}^*_0\leq\mathcal{B}^*_\pi$, we will show that for any given $\epsilon>0$ and any given path $\xi\in B_y$, we can construct a path $\zeta\in B_0$ so that $I_0(\zeta)\leq I_y(\xi)+\beta y+\epsilon$. To this end, let $u>0$ be such that $\Lambda^*(u)/u\leq\beta+\epsilon/y$ and set

$$\zeta(s)\triangleq us\mathbb{1}_{\{s\leq y/u\}}+\xi(s-y/u)\mathbb{1}_{\{s>y/u\}}.$$

Then $\zeta(0)=0$, $\zeta(y/u)=y$, and $\zeta\in B_0$. Also, one can see that $I_y(\zeta)=(y/u)\Lambda^*(u)+\int_0^{\mathcal{T}(\xi)}\Lambda^*(\dot{\xi}(s))ds+\theta_+\xi^{(u)}(\mathcal{T}(\xi))=(y/u)\Lambda^*(u)+I_y(\xi)$. From the construction of $u$,

$$I_0(\zeta)\leq\beta y+\epsilon+I_y(\xi)$$

as desired. This concludes the proof of part *(iv)*. $\square$

### 3.4. Proof of Proposition 2.1

We start with a couple of lemmas that facilitate the proof of Proposition 2.1.

LEMMA 3.6. *Suppose that $\alpha, \beta, \gamma \in \mathbb{D}[0,T]$, $\alpha(s) = \beta(s) + \gamma(s)$, and $\gamma(s)$ is non-negative and non-decreasing. Then, $R(\alpha)(t) \geq R(\beta)(t)$ for all $t \in [0,T]$.*

*Proof.* Recall first that if $z \geq 0$ then $x \wedge (y+z) \leq (x \wedge y) + z$ for any $x, y \in \mathbb{R}$. From the non-negativity and monotonicity assumptions on $\gamma$, we have that

$$0 \wedge \alpha(s) \leq 0 \wedge \beta(s) + \gamma(s) \leq 0 \wedge \beta(s) + \gamma(t), \quad 0 \leq s \leq t$$

and hence,

$$0 \wedge \inf_{0 \leq s \leq t} \alpha(s) \leq 0 \wedge \alpha(s) \leq 0 \wedge \beta(s) + \gamma(t), \quad 0 \leq s \leq t.$$

Taking infimum over $s \in [0,t]$, we get $0 \wedge \inf_{s \in [0,t]} \alpha(s) \leq 0 \wedge \inf_{s \in [0,t]} \beta(s) + \gamma(t)$. Therefore,

$$R(\alpha)(t) = \alpha(t) - 0 \wedge \inf_{s \in [0,t]} \alpha(s) \geq \alpha(t) - 0 \wedge \inf_{s \in [0,t]} \beta(s) - \gamma(t) = \beta(t) - 0 \wedge \inf_{s \in [0,t]} \beta(s) = R(\beta)(t).$$

□

Recall that $\Phi_T(\xi) = \int_0^T (R(\xi)(s))^p \mathrm{d}s$.

LEMMA 3.7. *Suppose that $\xi \in \mathbb{BV}[0,T]$ and set $y \triangleq \xi(0)$. Then*

*(i) there exists a path $\zeta_1 \in \mathbb{BV}[0,T]$ such that*

*i-1) $\zeta_1(0) = y$;*

*i-2) $\Phi_T(\zeta_1) \geq \Phi_T(\xi)$;*

*i-3) $I_y^{\mathbb{BV}[0,T]}(\zeta_1) \leq I_y^{\mathbb{BV}[0,T]}(\xi)$;*

*i-4) For some $t \in [0,T]$, $\zeta_1$ is nonnegative over $[0,t]$ and $\zeta_1$ is linear with slope $\mu$ over $[t,T]$.*

*(ii) there exists a path $\zeta_2 \in \mathbb{AC}[0,T]$ such that*

*ii-1) $\zeta_2(0) = y + z$ for some $z \in [0, \xi^{(u)}(T)]$;*

*ii-2) $\Phi_T(\zeta_2) \geq \Phi_T(\xi)$;*

*ii-3) $\theta_+ \cdot z + I_{y+z}^{\mathbb{BV}[0,T]}(\zeta_2) \leq I_y^{\mathbb{BV}[0,T]}(\xi)$;*

*ii-4) For some $t \in [0,T]$, $\zeta_2$ is nonnegative over $[0,t]$ and $\zeta_2$ is linear with slope $\mu$ over $[t,T]$.*

*Suppose further that $\xi \in \mathbb{AC}[0,T]$. Then*

*(iii) there exists a path $\zeta_3 \in \mathbb{AC}[0,T]$ such that*

*iii-1) $\zeta_3(0) = y$;*

*iii-2) $\Phi_T(\zeta_3) \geq \Phi_T(\xi)$;*

*iii-3) $I_y^{\mathbb{BV}[0,T]}(\zeta_3) \leq I_y^{\mathbb{BV}[0,T]}(\xi)$;*

*iii-4) $\zeta_3$ is concave over $[0,T]$ and its derivative is bounded by $\mu$ from below.*

*Proof.* For part *(i)*, we first construct a new trajectory $\xi_1$ from $\xi$ by discarding the downward jumps, i.e., $\xi_1 = \xi^{(a)} + \xi^{(u)}$. Obviously, $I_y^{\mathbb{BV}[0,T]}(\xi_1) \le I_y^{\mathbb{BV}[0,T]}(\xi)$. Note that $\xi_1 = \xi + (-\xi^{(d)})$ where $-\xi^{(d)}$ is non-negative and non-decreasing. From Lemma 3.6 we have that $R(\xi_1)(t) \ge R(\xi)(t)$ for all $t \in [0,T]$, and hence, $\Phi_T(\xi_1) \ge \Phi_T(\xi)$. For each $t \in [0,T]$, let $l(t) \triangleq \inf\{s \in [0,T] : R(\xi)(u) > 0 \text{ for all } u \in [s,t]\}$, $r(t) \triangleq \sup\{s \in [0,T] : R(\xi)(u) > 0 \text{ for all } u \in [t,s]\}$, and $\sigma(t) \triangleq [l(t), r(t))$. Set $\mathcal{C}_1^+ \triangleq \{\sigma(t) \subseteq [0,T] : t \in [0,T]\}$. Note that, by construction, the elements of $\mathcal{C}_1^+$ cannot overlap, and hence, there can be at most countable number of elements in $\mathcal{C}_1^+$. In view of this, we write $\mathcal{C}_1^+ = \{[l_i, r_i) : i \in \mathbb{N}\}$ and let $\sigma_i \triangleq [l_i, r_i)$. The following observations are immediate from the construction of $\mathcal{C}_1^+$, the right continuity of $\xi$, and the fact that $\xi_1$ does not have any downward jumps.

O1. If $t \in [0,T)$ doesn't belong to any of the elements of $\mathcal{C}_1^+$, then $R(\xi_1)(t) = 0$.

O2. $R(\xi_1)$ is continuous on the right end of the intervals $\sigma_i$ except for the case $r_i = T$.

Note that O1 also implies that for such $t$'s, $\xi_1(t) = \xi_1(t-)$. Let $s_n \triangleq \sum_{i=1}^{n-1}(r_i - l_i)$ for $n \in \mathbb{N}$. Note that $s_n \to s_\infty \in [0,T]$ as $n \to \infty$. Let $\dot{\xi}^{(a)}(t)$ denote the time derivative $\frac{d}{dt}\xi^{(a)}(t)$ of $\xi^{(a)}$ at $t$, and set

$$\zeta_1(t) \triangleq y + \int_0^t \dot{\zeta}_1(s)ds + \zeta_1^{(u)}(t),$$

where

$$\dot{\zeta}_1(t) \triangleq \sum_{i\in\mathbb{N}} \dot{\xi}^{(a)}(t - s_i + l_i)\mathbb{1}_{[s_i, s_{i+1})}(t) + \mu\mathbb{1}_{[s_\infty, T]}(t),$$

and

$$\zeta_1^{(u)}(t) \triangleq \sum_{i\in\mathbb{N}} \big(\xi^{(u)}(t \wedge s_{i+1} - s_i + l_i) - \xi^{(u)}(l_i-)\big)\mathbb{1}_{[s_i, T]}(t).$$

That is, on the interval $[s_i, s_{i+1})$, $\zeta_1$ behaves the same way as $\xi_1$ does on the interval $[l_i, r_i)$; whereas $\zeta_1$ decreases linearly at the rate $|\mu|$ outside of those intervals. Given this, it can be checked that

O3. $\int_{s_i}^{s_{i+1}} \big(R(\zeta_1)(s)\big)^p \mathrm{d}s \ge \int_{l_i}^{r_i} \big(R(\xi_1)(s)\big)^p \mathrm{d}s$

O4. $\int_{l_i}^{r_i} \Lambda^*(\dot{\xi}^{(a)}(s))\mathrm{d}s = \int_{s_i}^{s_{i+1}} \Lambda^*(\dot{\zeta}_1^{(a)}(s))\mathrm{d}s$

O5. $\zeta_1^{(u)}(s_{i+1}-) - \zeta_1^{(u)}(s_i-) = \xi_1^{(u)}(r_i-) - \xi_1^{(u)}(l_i-)$

Now, we verify the conditions *i-1), i-2), i-3), i-4)*. Note first that the conditions *i-1)* and *i-4)* are obvious from the construction of $\zeta_1$. We can verify *i-2)* as follows:

$$\begin{aligned}\Phi_T(\xi_2) &= \int_0^T \big(R(\zeta_1)(s)\big)^p \mathrm{d}s \ge \int_0^{s_\infty} \big(R(\zeta_1)(s)\big)^p \mathrm{d}s = \sum_{i=1}^{\infty}\int_{s_i}^{s_{i+1}} \big(R(\zeta_1)(s)\big)^p \mathrm{d}s \\ &\ge \sum_{i=1}^{\infty}\int_{l_i}^{r_i} \big(R(\xi_1)(s)\big)^p \mathrm{d}s = \int_0^T \big(R(\xi_1)(s)\big)^p \mathrm{d}s = \Phi_T(\xi_1),\end{aligned}$$

where the second inequality is from O3, and the second last equality is from O1. Moving onto *i-3)*, note that due to the left continuity of $\zeta_1$, $s_n \to s_\infty$ implies that $\xi(s_n-) \to \xi(s_\infty-)$. Also, $\zeta_1^{(u)}(s_\infty) - \zeta_1^{(u)}(s_\infty-) = 0$ and $\zeta_1^{(u)}$ is constant on $[s_\infty, T]$. Therefore, $\sum_{i=1}^{\infty}\Big(\zeta_1^{(u)}(s_{i+1}-) - \zeta_1^{(u)}(s_i-)\Big) =$

$\lim_{n\to\infty}\zeta_1^{(u)}(s_{n+1}-)=\zeta_1^{(u)}(s_\infty-)=\zeta_1^{(u)}(T)$ where we adopted the convention that $\zeta_1^{(u)}(0-)=0$. From O4, O5, and this observation,

$$\begin{aligned}
I_y^{\mathbb{BV}[0,T]}(\zeta_1) &= \int_0^T \Lambda^*(\dot\zeta_1(t))\mathrm{d}s + \theta_+\cdot\zeta_1^{(u)}(T)\\
&= \sum_{i=1}^\infty \int_{s_i}^{s_{i+1}} \Lambda^*(\dot\zeta_1(t))\mathrm{d}s + \theta_+\cdot\sum_{i=1}^\infty\Big(\zeta_1^{(u)}(s_{i+1}-)-\zeta_1^{(u)}(s_i-)\Big)\\
&= \sum_{i=1}^\infty \int_{l_i}^{r_i} \Lambda^*(\dot\xi^{(a)}(t))\mathrm{d}s + \theta_+\cdot\sum_{i=1}^\infty\Big(\xi^{(u)}(r_i-)-\xi^{(u)}(l_i-)\Big)\\
&\le \int_0^T \Lambda^*(\dot\xi^{(a)}(t))\mathrm{d}s + \theta_+\cdot\xi^{(u)}(T) = I_y^{\mathbb{BV}[0,T]}(\xi_1).
\end{aligned}$$

For part *(ii)*, we construct $\zeta_2$ from $\zeta_1$ by moving all the jumps of $\xi^{(u)}$ to time 0. This neither increases $I_y^{\mathbb{BV}[0,T]}$ nor decreases $\Phi_T$. That is, if we set

$$\zeta_2(t)\triangleq y+\int_0^t\dot\zeta_1(s)ds+\zeta_1^{(u)}(T),$$

then $\Phi_T(\zeta_2)\ge\Phi_T(\zeta_1)$ obviously, and $\theta_+\cdot\zeta_1^{(u)}(T)+I^T_{y+\zeta_1^{(u)}(T)}(\zeta_2)\le I_y^T(\zeta_1)$. Noting that $\zeta_1^{(u)}(T)\le\xi^{(u)}(T)$, we see that $\zeta$ satisfies all the claims of the lemma.

For part *(iii)*, let $\zeta\in\mathbb{AC}[0,T]$ be a concave majorant of $\xi$. Then there exists a non-increasing $\dot\zeta\in\mathbb{D}[0,T]$ such that $\zeta(t)=\xi(0)+\int_0^t\dot\zeta(s)ds$. (Due to the continuity of $\xi$, $\xi(0)$ and $\zeta(0)$ should coincide.) Let $\zeta_3(t)\triangleq\xi(0)+\int_0^t\mu\vee\dot\zeta(s)ds$. Note that *iii-1)*, *iii-2)*, and *iii-4)* are straightforward to check from the construction. To show that *iii-3)* is also satisfied, we construct $\mathcal{C}_2^+=\{(l_i',r_i')\subseteq[0,T]:i\in\mathbb{N}\}$ in a similar way to $\mathcal{C}_1^+$ so that the elements of $\mathcal{C}_2^+$ are non-overlapping, and $\xi(s)<\zeta_3(s)$ if and only if $s\in(l_i',r_i')$ for some $i\in\mathbb{N}$. Note that due to the continuity of $\zeta$ and $\xi$, $\zeta(l_i')=\xi(l_i')$ and $\zeta(r_i')=\xi(r_i')$, and $\zeta$ has to be a straight line on $(l_i',r_i')$ for each $i\in\mathbb{N}$. Set $s_0\triangleq0\vee\sup\{t\in[0,T]:\dot\zeta(t)\ge\mu\}$. Then, no interval in $\mathcal{C}_2^+$ contains $s_0$, because otherwise, $\zeta$ has to be a straight line in a neighborhood of $s_0$, and hence, $\dot\zeta$ has to be constant there, but this is contradictory to the definition of $s_0$. Now, let $\dot\xi$ denote a derivative of $\xi$. Then $\int_{l_i'}^{r_i'}\Lambda^*(\mu\wedge\dot\zeta(s))ds=\int_{l_i'}^{r_i'}\Lambda^*(\mu)ds=0$ for $i$'s such that $r_i'>s_0$, and hence,

$$\begin{aligned}
I_y^{\mathbb{BV}[0,T]}(\xi)-I_y^{\mathbb{BV}[0,T]}(\zeta_3) &= \int_0^T\Lambda^*(\dot\xi(s))ds-\int_0^T\Lambda^*(\mu\vee\dot\zeta(s))ds\\
&\ge \sum_{i\in\mathbb{N}:\ r_i'\le s_0}\int_{l_i'}^{r_i'}\Big(\Lambda^*(\dot\xi(s))-\Lambda^*(\dot\zeta(s))\Big)ds.
\end{aligned}$$

Note that from the construction of $\mathcal{C}_2^+$, if $s\in[l_i',r_i']$ for some $i$ such that $r_i'\le s_0$, we have that $\dot\zeta(s)=(\zeta_3(r_i')-\zeta_3(l_i'))/(r_i'-l_i')=(\xi(r_i')-\xi(l_i'))/(r_i'-l_i')$, and hence, from Jensen's inequality,

$$\begin{aligned}
\int_{l_i'}^{r_i'}\Big(\Lambda^*(\dot\xi(s))-\Lambda^*(\dot\zeta(s))\Big)ds &= \int_{l_i'}^{r_i'}\Lambda^*(\dot\xi(s))ds-\int_{l_i'}^{r_i'}\Lambda^*\big((\xi(r_i')-\xi(l_i'))/(r_i'-l_i')\big)ds\\
&= \int_{l_i'}^{r_i'}\Lambda^*(\dot\xi(s))ds-(r_i'-l_i')\cdot\Lambda^*\Big(\int_{l_i'}^{r_i'}\dot\xi(s)ds/(r_i'-l_i')\Big)\\
&\ge 0.
\end{aligned}$$

Therefore, $\zeta_3$ satisfies *iii-3)* as well. $\square$

Now we are ready to prove Proposition 2.1.

*Proof of Proposition 2.1.* Since $B_y^{\mathrm{CNCV}} \subseteq B_y^{\mathbb{AC}} \subseteq B_y$, we only have to prove that $\mathcal{B}_y^* \geq \inf_{\xi \in B_y^{\mathrm{CNCV}}} I_y(\xi)$. For this, we show that for any given $\xi \in B_y$ and any given $\epsilon > 0$, there is $\zeta \in B_y^{\mathrm{CNCV}}$ such that $I_y(\zeta) \leq I_y(\xi) + \epsilon$. To construct such $\zeta$, we first note that we can find $\xi_1 \in B_y$ such that $\mathcal{T}(\xi_1) < \infty$ and $I_y(\xi_1) \leq I_y(\xi)$ thanks to Lemma 3.1. Now set $T = \mathcal{T}(\xi_1)$ and denote the restriction of $\xi_1$ on $[0,T]$ with $\check{\xi}_1$—i.e., $\check{\xi}_1 \in \mathbb{D}[0,T]$ and $\check{\xi}_1(t) = \xi_1(t)$ for $t \in [0,T]$. We appeal to Lemma 3.7 to pick a path $\xi_2 \in \mathbb{AC}[0,T]$ such that $\xi_2(0) = y + z$, $0 \leq z \leq \check{\xi}_1^{(u)}(T) = \xi_1^{(u)}(T)$, $\Phi_T(\xi_2) \geq \Phi_T(\check{\xi}_1) \geq 1$, and $\theta_+ \cdot z + I_{y+z}^{\mathbb{BV}[0,T]}(\xi_2) \leq I_y^{\mathbb{BV}[0,T]}(\check{\xi}_1) = I_y(\xi_1) \leq I_y(\xi)$. Due to Equation (5.5) in [14], $\lim_{x\to\infty} \frac{\Lambda^*(x)}{x} = \theta_+$. As a consequence, we can choose a $u > 0$ large enough so that

$$\Lambda^*(u)/u \leq \theta_+ + \epsilon/z. \tag{3.13}$$

Set

$$\xi_3(s) \triangleq (y + us)\mathbb{1}_{[0,z/u]}(s) + \xi_2\left(s - z/u\right)\mathbb{1}_{(z/u, z/u+T]}(s).$$

Then, $\xi_3 \in \mathbb{AC}[0, z/u + T]$, $\xi_3(0) = y$, $\xi_3(z/u) = y + z$, and that $\Phi_{z/u+T}(\xi_3) \geq \Phi_T(\xi_2) \geq 1$. Moreover,

$$I_y^{\mathbb{BV}[0,z/u+T]}(\xi_3) = (z/u)\Lambda^*(u) + \int_0^T \Lambda^*(\dot{\xi}_2(s))ds \leq \theta_+ z + \epsilon + I_{y+z}^{\mathbb{BV}[0,T]}(\xi_2) \leq I_y(\xi) + \epsilon.$$

Next, we appeal to the part *(iii)* of Lemma 3.7 to find a $\check{\zeta} \in \mathbb{AC}[0, z/u + T]$ such that $\check{\zeta}(0) = y$, $\Phi_{z/u+T}(\check{\zeta}) \geq 1$, $I_y^{\mathbb{BV}[0,z/u+T]}(\check{\zeta}) \leq I_y^{\mathbb{BV}[0,z/u+T]}(\xi_3) \leq I_y(\xi) + \epsilon$, and $\check{\zeta}$ is concave on $[0, z/u + T]$ with the derivative bounded by $\mu$ from below. Now, if we set

$$\zeta(t) = \check{\zeta}(t \wedge (z/u + T)) + \mu([t - (z/u + T)]^+), \quad t \geq 0,$$

then $\zeta \in B_y^{\mathrm{CNCV}}$ and $I_y(\zeta) = I_y^{\mathbb{BV}[0,z/u+T]}(\check{\zeta}) \leq I_y(\xi) + \epsilon$. $\square$

## 4. Proof of the sample path LDP

In this section, we prove Theorem 2.1. We begin our analysis in this section with a lemma that establishes the large-deviations behavior of the busy period active at time $n$. To this end, define $\mathbb{D}^{\leqslant 1} \triangleq \{\xi \in \mathbb{D} : \xi = x\mathbb{1}_{\{1\}}$ for some $x \geq 0\}$ and recall the definition of $\bar{S}_n = \bar{V}_n \mathbb{I}_{\{1\}}(t)$ and $\bar{V}_n = \frac{1}{n}\sum_{i=T_{N(n)}+1}^{n} f(X_i)$.

Lemma 4.1. *$\bar{S}_n$ satisfies the LDP in $(\mathbb{D}, \mathcal{T}_{M_1'})$ with speed $n^\alpha$ and the rate function $I_S : \mathbb{D} \to \mathbb{R}_+$ where*

$$I_S(\zeta) \triangleq \begin{cases} \mathcal{B}_0^*(\zeta(1) - \zeta(1-))^\alpha, & \text{if } \zeta \in \mathbb{D}^{\leqslant 1}, \\ \infty, & \textit{otherwise.} \end{cases} \tag{4.1}$$

*Proof.* Define a function $T:\mathbb{R}_+\to\mathbb{D}^{\leqslant 1}$ as $T(x)\triangleq x\cdot\mathbb{1}_{\{1\}}$. Then, $\bar{S}_n=T(\bar{V}_n)$ and $T$ is a continuous function w.r.t. the $M_1'$ topology. Therefore, the desired LDP follows from the contraction principle if we prove that $\bar{V}_n$ satisfies an LDP in $\mathbb{R}_+$ with sub-linear speed $n^\alpha$ and the rate function $\mathcal{I}_v:\mathbb{R}_+\to\mathbb{R}_+$ where $\mathcal{I}_v(x)=\mathcal{B}_0^*\cdot x^\alpha$. To prove the LDP for $\bar{V}_n$, note first that since $\mathbf{P}(\bar{V}_n\in\cdot)$ is exponentially tight (w.r.t. the speed $n^\alpha$) from Theorem 2.3, it is enough to establish the weak LDP. For the weak LDP, we start with showing that for any $a,b\in\mathbb{R}$, $B\triangleq(a,b)\cap\mathbb{R}_+$ satisfies $\limsup_{n\to\infty}\frac{\log\mathbf{P}(\bar{V}_n\in B)}{n^\alpha}=\liminf_{n\to\infty}\frac{\log\mathbf{P}(\bar{V}_n\in B)}{n^\alpha}$. Since this holds trivially if $b\le 0$ or $a\ge b$, we assume that $0\vee a<b$. Note that from Theorem 2.3, $\mathbf{P}(\bar{V}_n\ge b)/\mathbf{P}(\bar{V}_n\ge 0\vee a)\to 0$. Therefore,

$$\limsup_{n\to\infty}\frac{\log\mathbf{P}\left(\bar{V}_n\in B\right)}{n^\alpha}\le\limsup_{n\to\infty}\frac{\log\mathbf{P}\left(\bar{V}_n\ge 0\vee a\right)}{n^\alpha}\le-\mathcal{B}_0^*\cdot(0\vee a)^\alpha.$$

Similarly, for small enough $\epsilon>0$,

$$\begin{aligned}\liminf_{n\to\infty}\frac{\log\mathbf{P}\left(\bar{V}_n\in B\right)}{n^\alpha}&\ge\liminf_{n\to\infty}\frac{\log\left\{\mathbf{P}\left(\bar{V}_n\ge 0\vee a+\epsilon\right)\left(1-\frac{\mathbf{P}\left(\bar{V}_n\ge b\right)}{\mathbf{P}\left(\bar{V}_n\ge 0\vee a+\epsilon\right)}\right)\right\}}{n^\alpha}\\&=\liminf_{n\to\infty}\frac{\log\mathbf{P}\left(\bar{V}_n\ge 0\vee a+\epsilon\right)}{n^\alpha}=-\mathcal{B}_0^*\cdot(0\vee a+\epsilon)^\alpha.\end{aligned}$$

Taking $\epsilon\to 0$, we see that the limit supremum and the limit infimum coincide. Since $\mathcal{C}=\{(a,b)\cap\mathbb{R}_+ : a,b\in\mathbb{R},\ a\le b\}$ forms a base of the Euclidean topology on $\mathbb{R}_+$, Theorem 4.1.11 of [5] applies, and hence, proves the desired weak LDP. This concludes the proof. $\square$

We next work towards a sample path LDP for $\bar{\mathcal{Z}}_n$. We employ a well-known technique, based on the projective limit theorem by Dawnson and Gärtner; see Theorem 4.6.1 in [5]. The following three lemmas lead to the first key step in this approach, which consists of obtaining the finite dimensional LDP for $\bar{\mathcal{Z}}_n$.

LEMMA 4.2. *For any given $0=t_0<t_1<t_2<\ldots<t_k$, let $\Delta t_i=t_i-t_{i-1}$ for $i=1,\ldots,k$. Then,*

$$\limsup_{n\to\infty}\frac{1}{n^\alpha}\log\mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j\ge na_1,\ldots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\ge na_k\right)\le-\mathcal{B}_0^*\left(\sum_{i=1}^{k}(a_i-\lambda\Delta t_i)_+^\alpha\right),\tag{4.2}$$

$$\liminf_{n\to\infty}\frac{1}{n^\alpha}\log\mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j\ge na_1,\ldots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\ge na_k\right)\ge-\mathcal{B}_0^*\left(\sum_{i=1}^{k}(a_i-\lambda\Delta t_i)_+^\alpha\right),\tag{4.3}$$

*where $(x)_+\triangleq x\vee 0$.*

*Proof.* Firstly, for notational convenience, let $E_i^{(n)}(\epsilon)\triangleq n[t_i/\mathbf{E}\tau-\epsilon,t_i/\mathbf{E}\tau+\epsilon]$. We will use this notation throughout the proof of this lemma. For the upper bound in equation (4.2), notice that

$$\mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j\ge na_1,\ldots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\ge na_k\right)\le\underbrace{\sum_{i=1}^{k}\mathbf{P}\left(N(nt_i)\notin E_i^{(n)}(\epsilon)\right)}_{=\text{(I)}}$$

$$+\underbrace{\mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \geq na_1, \ldots, \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \geq na_k, N(nt_i) \in E_i^{(n)}(\epsilon) \text{ for } i=1,\ldots,k\right)}_{=(\mathrm{II})}.$$

For (I), by ([16]),

$$\limsup_{n\to\infty} \frac{\log(\mathrm{I})}{n^\alpha} = -\infty. \tag{4.4}$$

Shifting our attention to (II),

$$\begin{aligned}
&\mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \geq na_1, \ldots, \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \geq na_k, N(nt_i) \in E_i^{(n)}(\epsilon) \text{ for } i=1,\ldots,k\right)\\
&\leq \sum_{i_1=\lceil n(t_1/\mathbf{E}\tau-\epsilon)\rceil}^{\lfloor n(t_1/\mathbf{E}\tau+\epsilon)\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}\tau-\epsilon)\rceil}^{\lfloor n(t_k/\mathbf{E}\tau+\epsilon)\rfloor} \mathbf{P}\left(\sum_{j=1}^{i_1} W_j \geq na_1, \ldots, \sum_{j=i_{k-1}+1}^{i_k} W_j \geq na_k, N(nt_l)=i_l \text{ for } l=1,\ldots,k\right)\\
&\leq \sum_{i_1=\lceil n(t_1/\mathbf{E}\tau-\epsilon)\rceil}^{\lfloor n(t_1/\mathbf{E}\tau+\epsilon)\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}\tau-\epsilon)\rceil}^{\lfloor n(t_k/\mathbf{E}\tau+\epsilon)\rfloor} \mathbf{P}\left(\sum_{j=1}^{i_1} W_j \geq na_1, \ldots, \sum_{j=i_{k-1}+1}^{i_k} W_j \geq na_k\right) I(i_1 \leq \ldots \leq i_k)\\
&= \sum_{i_1=\lceil n(t_1/\mathbf{E}\tau-\epsilon)\rceil}^{\lfloor n(t_1/\mathbf{E}\tau+\epsilon)\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}\tau-\epsilon)\rceil}^{\lfloor n(t_k/\mathbf{E}\tau+\epsilon)\rfloor} \mathbf{P}\left(\sum_{j=1}^{i_1} W_j \geq na_1\right) \cdots \mathbf{P}\left(\sum_{j=i_{k-1}+1}^{i_k} W_j \geq na_k\right)\\
&\leq (2\epsilon n)^k\, \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/\mathbf{E}\tau+\epsilon)\rfloor} W_j \geq na_1\right) \cdots \mathbf{P}\left(\sum_{j=\lceil n(t_{k-1}/\mathbf{E}\tau-\epsilon)\rceil}^{\lfloor n(t_k/\mathbf{E}\tau+\epsilon)\rfloor} W_j \geq na_k\right).
\end{aligned}$$

Now, we have that from Result 1,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n^\alpha}\log(\mathrm{II}) &\leq \sum_{i=1}^{k} \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\sum_{j=\lceil n(t_{i-1}/\mathbf{E}(\tau)-\epsilon)\rceil}^{\lfloor n(t_i/\mathbf{E}(\tau)+\epsilon)\rfloor} W_j \geq na_i\right) + \limsup_{n\to\infty} \frac{\log(2\epsilon n)^k}{n^\alpha}\\
&\leq -\mathcal{B}_0^* \sum_{i=1}^{k} (a_i - \lambda(\Delta t_i + 2\epsilon \mathbf{E}\tau))_+^\alpha.
\end{aligned}$$

Taking $\epsilon \to 0$, we arrive at

$$\limsup_{n\to\infty} \frac{1}{n^\alpha}\log(\mathrm{II}) \leq -\mathcal{B}_0^* \sum_{i=1}^{k} (a_i - \lambda\Delta t_i)_+^\alpha. \tag{4.5}$$

In view of (4.4) and (4.5),

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j \geq na_1, \ldots, \sum_{j=N(nt_{i-1})}^{N(nt_i)-1} W_j \geq na_i, \ldots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j \geq na_k\right)\\
&\leq \max\left\{\limsup_{n\to\infty} \frac{\log(\mathrm{I})}{n^\alpha}, \limsup_{n\to\infty} \frac{\log(\mathrm{II})}{n^\alpha}\right\} \leq -\mathcal{B}_0^* \sum_{i=1}^{k} (a_i - \lambda\Delta t_i)_+^\alpha.
\end{aligned}$$

For the lower bound in Equation (4.3), notice that

$$\mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \ldots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k\right)$$

$$
\begin{aligned}
&\geq \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \ldots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k, N(nt_i) \in E_i^{(n)}(\epsilon) \text{ for } i=1,\ldots,k\right)\\
&\geq \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E\tau-\epsilon)\rfloor-1} W_j > na_1, \ldots, \sum_{j=\lceil n(t_{k-1}/E\tau+\epsilon)\rceil}^{\lfloor n(t_k/E\tau-\epsilon)\rfloor-1} W_j > na_k, N(nt_i) \in E_i^{(n)}(\epsilon) \text{ for } i=1,\ldots,k\right)\\
&\geq \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E\tau-\epsilon)\rfloor-1} W_j > na_1, \ldots, \sum_{j=\lceil n(t_{k-1}/E\tau+\epsilon)\rceil}^{\lfloor n(t_k/E\tau-\epsilon)\rfloor-1} W_j > na_i\right) - (\mathrm{I})\\
&= \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E\tau-\epsilon)\rfloor-1} W_j > na_1\right)\prod_{i=2}^{k}\mathbf{P}\left(\sum_{j=\lceil n(t_{i-1}/E\tau+\epsilon)\rceil}^{\lfloor n(t_i/E\tau-\epsilon)\rfloor-1} W_j > na_i\right) - (\mathrm{I})\\
&= \underbrace{\mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E\tau-\epsilon)\rfloor-1} W_j > na_1\right)\prod_{i=2}^{k}\mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_i/E\tau-\epsilon)\rfloor-\lceil n(t_{i-1}/E\tau+\epsilon)\rceil} W_j > na_i\right)}_{=(\mathrm{III})} - (\mathrm{I}). \qquad (4.6)
\end{aligned}
$$

From Theorem 2.2, Result 1 and (4.4), we get $\frac{(\mathrm{I})}{(\mathrm{III})} \to 0$ as $n \to \infty$. Therefore, (4.6) leads to

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \ldots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k\right)\\
&\geq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log\left\{(\mathrm{III})\left(1-\frac{(\mathrm{I})}{(\mathrm{III})}\right)\right\} = \liminf_{n\to\infty}\frac{1}{n^\alpha}\log(\mathrm{III})\\
&= -\mathcal{B}_0^* \sum_{i=1}^{k} (a_i - \lambda(\Delta t_i - 2\epsilon \mathbf{E}\tau))_+^\alpha.
\end{aligned}
$$

Taking $\epsilon \to 0$, we arrive at (4.3) concluding the proof. □

Our next lemma establishes the weak LDP for $\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$.

LEMMA 4.3. *For any given $\mathbf{t} = (t_1, \ldots, t_k)$ such that $0 = t_0 \leq t_1 < \ldots < t_k \leq 1$, the probability measures $\mu_n$ of $\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ satisfy the LDP in $\mathbb{R}_+^k$ w.r.t. Euclidean topology with speed $n^\alpha$ and the good rate function $I_{\mathbf{t}} : \mathbb{R}_+^k \to \mathbb{R}_+$:*

$$
I_{\mathbf{t}}(x_1, \ldots, x_k) = \begin{cases} \mathcal{B}_0^* \sum_{i=1}^k (x_i - \lambda \Delta t_i)^\alpha & \text{if } x_i \geq \lambda \Delta t_i,\ \forall i = 1, \ldots, k,\\ \infty, & \text{otherwise.}\end{cases} \qquad (4.7)
$$

*Proof.* We first claim that $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ satisfies a weak LDP. Once our claim is established, since $I_{\mathbf{t}}$ is a good rate function, and $\mathbb{R}_+^k$ is Polish, $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ is exponentially tight, and consequently, Lemma 1.2.18 of [5] applies, showing that the full LDP is satisfied. Now, to prove the claimed weak LDP, we start with showing that

$$
\underbrace{\limsup_{n\to\infty} \frac{\log \mu_n(A)}{n^\alpha}}_{\triangleq \overline{\mathcal{L}}_A} = \underbrace{\liminf_{n\to\infty} \frac{\log \mu_n(A)}{n^\alpha}}_{\triangleq \underline{\mathcal{L}}_A} \qquad (4.8)
$$

for every $A \in \mathcal{A} \triangleq \{\prod_{i=1}^{k}\left((a_i,b_i)\cap\mathbb{R}_+\right) : a_i < b_i\}$. Let

$$\mathcal{L}_A \triangleq \begin{cases} -\mathcal{B}_0^*\sum_{i=1}^{k}(a_i-\lambda\Delta t_i)_+^{\alpha} & \text{if } b_i \geq \lambda\Delta t_i \text{ for } i=1,\dots,k, \\ -\infty & \text{otherwise.} \end{cases}$$

We will prove (4.8) by showing that $\overline{\mathcal{L}}_A \leq \mathcal{L}_A \leq \underline{\mathcal{L}}_A$. We consider the two cases separately:

case 1. $b_i \geq \lambda\Delta t_i$ for $i=1,\dots,k$;

case 2. $b_i < \lambda\Delta t_i$ for some $i \in \{1,\dots,k\}$.

Let $A = \prod_{i=1}^{k}\left((a_i,b_i)\cap\mathbb{R}_+\right)$ and $a_i < b_i$ for $i=1,\dots,k$. We start with case 1. Since $A \subseteq \prod_{i=1}^{k}[a_i,b_i)$,

$$\overline{\mathcal{L}}_A \leq \limsup_{n\to\infty}\frac{1}{n^{\alpha}}\log\mathbf{P}\left(\sum_{j=1}^{N(nt_1)}W_j \geq na_1,\dots,\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j \geq na_k\right) \leq -\mathcal{B}_0^*\sum_{i=1}^{k}(a_i-\lambda\Delta t_i)^{\alpha} = \mathcal{L}_A, \tag{4.9}$$

where the second inequality is from (4.2). Since $\prod_{i=1}^{k}[a_i+\epsilon,b_i) \subseteq A$ for small enough $\epsilon > 0$,

$$\begin{aligned}
\underline{\mathcal{L}}_A &\geq \liminf_{n\to\infty}\frac{1}{n^{\alpha}}\log\mathbf{P}\left(\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)}W_j,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j\right)\in\prod_{i=1}^{k}[a_i+\epsilon,b_i)\right) \\
&\geq \liminf_{n\to\infty}\frac{1}{n^{\alpha}}\log\left\{\mathbf{P}\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)}W_j > a_1+\epsilon,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j > a_k+\epsilon\right)\right. \\
&\qquad\qquad \left. -\sum_{l=1}^{k}\mathbf{P}\left(\sum_{j=N(nt_{i-1})+1}^{N(nt_i)}W_j \geq na_i\ \forall i\neq l,\ \sum_{j=N(nt_{l-1})+1}^{N(nt_l)}W_j \geq nb_l\right)\right\} \\
&\geq \liminf_{n\to\infty}\frac{1}{n^{\alpha}}\log\mathbf{P}\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)}W_j > a_1+\epsilon,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j > a_k+\epsilon\right) \\
&\quad + \liminf_{n\to\infty}\frac{1}{n^{\alpha}}\log\left(1-\frac{\sum_{l=1}^{k}\mathbf{P}\left(\frac{1}{n}\sum_{j=N(nt_{i-1})+1}^{N(nt_i)}W_j \geq a_i+\epsilon\ \forall i\neq l,\ \frac{1}{n}\sum_{j=N(nt_{l-1})+1}^{N(nt_l)}W_j \geq b_l\right)}{\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)}W_j > a_1+\epsilon,\dots,\frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j > a_k+\epsilon\right)}\right).
\end{aligned} \tag{4.10}$$

Note that due to the logarithmic asymptotics of Lemma 4.2, for every $l \in \{1,\dots,k\}$,

$$\frac{\mathbf{P}\left(\frac{1}{n}\sum_{j=N(nt_{i-1})+1}^{N(nt_i)}W_j \geq a_i+\epsilon \text{ for } i\in\{1,\dots,k\}\setminus l,\ \frac{1}{n}\sum_{j=N(nt_{j-1})+1}^{N(nt_l)}W_j \geq b_l\right)}{\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)}W_j > a_1+\epsilon,\dots,\ \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j > a_k+\epsilon\right)} \to 0,$$

and hence, the second term of (4.10) disappears. Therefore,

$$\begin{aligned}
\underline{\mathcal{L}}_A &\geq \liminf_{n\to\infty}\frac{\log\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)}W_j > a_1+\epsilon,\dots,\ \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)}W_j > a_k+\epsilon\right)}{n^{\alpha}} \\
&\geq -\mathcal{B}_0^*\sum_{i=1}^{k}(a_i+\epsilon-\lambda\Delta t_i)^{\alpha}.
\end{aligned}$$

Taking $\epsilon \to 0$, we arrive at $\underline{\mathcal{L}}_A \geq \mathcal{L}_A$, which, together with (4.9), proves (4.8) for case 1.

For case 2, note that by Result 1,

$$\overline{\mathcal{L}}_A \leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left( \sum_{j=N(nt_{i-1})+1}^{N(nt_i)} W_j < nb_i \right) = -\infty,$$

and hence, $\overline{\mathcal{L}}_A = \underline{\mathcal{L}}_A = \mathcal{L}_A = -\infty$.

Now note also that

$$I_{\tau_k}(x_1,\ldots,x_k) = -\inf\left\{\mathcal{L}_A : A \ni (x_1,\ldots,x_k)\right\}. \tag{4.11}$$

Since $\mathcal{A}$ is a base of the Euclidean topology, the desired weak LDP follows from (4.8), (4.11), and Theorem 4.1.11 of [5]. $\square$

The following is an immediate Corollary of Lemma 4.3.

Lemma 4.4. *For any given $\mathbf{t} = (t_1,\ldots,t_k)$ such that $0 = t_0 \leq t_1 < \ldots < t_k \leq 1$, the probability measures $(\mu_n)$ of $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\right)$ satisfy an LDP in $\mathbb{R}_+^k$ with speed $n^\alpha$ and with the good rate function, $\tilde{I}_{\mathbf{t}} : \mathbb{R}_+^k \to \mathbb{R}_+$,*

$$\tilde{I}_{\mathbf{t}}(x_1,\ldots,x_k) = \begin{cases} \mathcal{B}_0^* \sum_{i=1}^k (x_i - x_{i-1} - \lambda\Delta t_i)^\alpha & \text{if } x_i - x_{i-1} \geq \lambda\Delta t_i, \text{ for } i = 1,\ldots,k \\ \infty & \textit{otherwise.} \end{cases} \tag{4.12}$$

*Proof.* The proof is an application of the contraction principle. To this end, consider the function $f : \mathbb{R}_+^k \to \mathbb{R}_+^k$, $f(x_1, x_2, \ldots, x_k) = (x_1, x_1 + x_2, \ldots, x_1 + \ldots + x_k)$. Notice that

$$\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\right) = f\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right),$$

where $f$ is a continuous function. That is $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\right)$ satisfies a large deviation principle with the rate function $\tilde{I}_{\mathbf{t}}(y_1,\ldots,y_k) = \inf\{I_{\mathbf{t}}(x) : y = f(x_1,\ldots,x_k)\}$. Since $(y_1,\ldots,y_k) = f(x_1,\ldots,x_k)$, it is immediate that $y_1 \leq y_2 \leq \ldots \leq y_k$. Therefore,

$$\tilde{I}_{\mathbf{t}}(y_1,\ldots,y_k) = \begin{cases} \mathcal{B}_0^* \sum_{i=1}^k (y_i - y_{i-1} - \lambda\Delta t_i)^\alpha, & \text{if } y_{i+1} - y_i \geq \lambda\Delta t_i \text{ for } i = 1,\ldots,k, \\ \infty & \text{otherwise.} \end{cases}$$

$\square$

Now, for a path $\xi \in \mathbb{D}$ let

$$I_\alpha(\xi) = \begin{cases} \sum_{t:\xi(t)\neq\xi(t-)} (\xi(t) - \xi(t-))^\alpha, & \text{for } \xi \in \mathbb{D}^{(\lambda)}[0,1], \\ \infty & \text{otherwise.} \end{cases}$$

Since $\bar{\mathcal{Z}}_n$ satisfies a finite dimensional LDP, the Dawnson and Gärtner projective limit theorem implies that $\bar{\mathcal{Z}}_n$ obeys a sample path LDP in $\mathbb{D}[0,1]$ endowed with the pointwise convergence topology. The next lemma verifies that the rate function associated with the LDP of $\bar{\mathcal{Z}}_n$, is indeed $I_\alpha$.

LEMMA 4.5. *Let* $\mathbf{T} = \{(t_1, \ldots, t_k) \in [0,1]^k : k \geq 1\}$ *be the collection of all ordered finite subsets of* $[0,1]$. *Then*

$$\sup_{\mathbf{t}\in\mathbf{T}} \tilde{I}_{\mathbf{t}}(\xi) = I_\alpha(\xi).$$

*Proof.* This proof is essentially identical to the proof of Lemma 4 of [10] and hence omitted. □

We derive the sample path LDP for the stochastic process $\bar{\mathcal{Z}}_n$ w.r.t. the pointwise convergence topology, which we denote with $\mathcal{W}$. Recall that $\mathbb{D}^{(\lambda)}[0,1]$ denotes the subspace of increasing piecewise linear jump functions with slope $\lambda$.

LEMMA 4.6. *The stochastic process* $\bar{\mathcal{Z}}_n$ *satisfies a large deviation principle in* $(\mathbb{D}[0,1], \mathcal{W})$, *with speed* $n^\alpha$ *and good rate function* $I_Z : \mathbb{D} \to \mathbb{R}_+$ *where*

$$I_Z(\xi) = \begin{cases} \mathcal{B}_0^* \sum_{t:\xi(t)\neq\xi(t-)} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}^{(\lambda)}[0,1], \\ \infty & \textit{otherwise}. \end{cases} \tag{4.13}$$

*Proof.* The proof is an immediate consequence of the Dawnson and Gärtner's projective limit theorem, (Theorem 4.6.1 of [5]), and Lemma 4.5. □

Next, we establish the sample path LDP for the stochastic process $\bar{\mathcal{Z}}_n$ in $(\mathbb{D}, \mathcal{T}_{M_1'})$.

LEMMA 4.7. *The stochastic process* $\bar{\mathcal{Z}}_n$ *satisfies a large deviation principle in* $\mathbb{D}[0,1]$ *w.r.t.* $M_1'$ *topology with speed* $n^\alpha$ *and the good rate function* $I_Z$.

*Proof.* For the upper bound, consider a set $K_M \triangleq \{\xi \in \mathbb{D} :\ \xi \text{ is nondcreasing},\ \xi(0) \geq 0,\ \|\xi\|_\infty \leq M\}$. Let $F$ be a closed set in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$. Then,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n \in F\right) &\leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log\left\{\mathbf{P}\left(\bar{\mathcal{Z}}_n \in F \cap K_M\right) + \mathbf{P}\left(\bar{\mathcal{Z}}_n \in K_M^c\right)\right\} \\ &\leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log\left\{\mathbf{P}\left(\bar{\mathcal{Z}}_n \in F \cap K_M\right) + \mathbf{P}\left(\sum_{j=1}^{N(nt)} W_j \geq M\right)\right\}.\end{aligned}$$

From Proposition $A$.2 of [1], one can check that point-wise convergence in $K_M$ implies the convergence w.r.t. the $M_1'$ topology, and $K_M$ (and hence $F \cap K_M$ as well) is closed w.r.t. $\mathcal{T}_{M_1'}$. Suppose that $\xi$ is in the closure of $F \cap K_M$ w.r.t. $\mathcal{W}$. Then, because of the above mentioned properties of $K_M$, there exists a sequence of paths $\{\xi_n\}$ in $F \cap K_M$ such that $\xi_n \to \xi$ w.r.t. $\mathcal{T}_{M_1'}$, which, in turn, implies that $\xi \in F \cap K_M$. That is, $F \cap K_M$ is closed in $\mathcal{W}$ as well. Now, applying the sample path LDP w.r.t. $\mathcal{W}$ we proved in the above lemma, and then picking $M$ large enough,

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n \in F\right) \leq \max\left\{-\inf_{\xi\in F\cap K_M} I_Z(\xi),\ -\mathcal{B}_0^* M\right\} = -\inf_{\xi\in F\cap K_M} I_Z(\xi) \leq -\inf_{\xi\in F} I_Z(\xi).$$

Moving on to the lower bound, let $G$ be an open set in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$. We assume that $I(G) < \infty$ since we have nothing to show otherwise. Fix an arbitrary $\xi \in G \cap \mathbb{D}^{(\lambda)}[0,1]$, and let $k$ be such that an open ball of radius $\frac{1+\lambda}{k}$ around $\xi$ is inside of $G$; that is, $B_{M_1'}(\xi; \frac{1+\lambda}{k}) \triangleq \{\zeta \in \mathbb{D}[0,1] : d_{M_1'}(\xi, \zeta) <$

$\frac{1+\lambda}{k}\} \subseteq G$. Note that since $\xi \in \mathbb{D}^{(\lambda)}[0,1]$ and $\bar{\mathcal{Z}}_n$ is non-decreasing, $\{|\bar{\mathcal{Z}}_n(i/k) - \xi(i/k)| < 1/k,$ for $i = 0,\dots,k\} \subseteq \{\bar{\mathcal{Z}}_n \in B_{M_1'}(\xi; \frac{1+\lambda}{k})\}$. Therefore, in view of Lemma 4.4,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n \in G\right) &\geq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n \in B_{M_1'}\Big(\xi, \frac{1+\lambda}{k}\Big)\right) \\
&\geq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(|\bar{\mathcal{Z}}_n(i/k) - \xi(i/k)| < 1/k, \text{ for } i = 0,\dots,k\right) \\
&= - \inf_{(y_1,\dots,y_k)\in \prod_{i=1}^k (\xi(i/k)-1/k, \xi(i/k)+1/k)} \tilde{I}_{\tau_k}(y_1,\dots,y_k) \\
&\geq -\mathcal{B}_0^{*(p)} \sum_{i=1}^k \Big(\xi(i/k) - \xi((i-1)/k) - \lambda/k\Big)^\alpha \\
&\geq -\mathcal{B}_0^{*(p)} \sum_{t:\xi(t)\neq\xi(t-)} \left(\xi(t) - \xi(t-)\right)^\alpha = -I_{\mathcal{Z}}(\xi).
\end{aligned}$$

Since $\xi$ was an arbitrary element of $G \cap \mathbb{D}^{(\lambda)}[0,1]$, we arrive at the desired large deviation lower bound concluding the proof:

$$-\inf_{\xi\in G} I_{\mathcal{Z}}(\xi) = -\inf_{\xi \in G\cap\mathbb{D}^{(\lambda)}[0,1]} I_{\mathcal{Z}}(\xi) \leq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n \in G\right).$$

□

Our next lemma shows that $\bar{\mathcal{Z}}_n + \bar{S}_n$ is exponentially equivalent to $\bar{Y}_n$.

LEMMA 4.8. *$\bar{Y}_n$ and $\bar{\mathcal{Z}}_n + \bar{S}_n$ are exponentially equivalent in* $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$.

*Proof.* Fix an $\epsilon > 0$, and define $D_n(\epsilon) = \{N(n)/n \geq 1/\mathbf{E}\tau - \epsilon\}$. Due to the construction of $\bar{Y}_n$, $\bar{\mathcal{Z}}_n$, and $\bar{S}_n$, we have that for any $\delta > 0$,

$$\{d_{M_1'}(\bar{Y}_n, \bar{\mathcal{Z}}_n + \bar{S}_n) \geq \delta\} \subseteq \{(n - T_{N(n)})/n \geq \delta\} \cup \{\exists j \leq N(n) : \tau_j \geq n\delta\}. \tag{4.14}$$

To bound the probability of the first set, note that $\mathbf{P}\left((n - T_{N(n)})/n > \delta\right) = \mathbf{P}\left(T_{N(n)} \leq n(1-\delta)\right) = \mathbf{P}\left(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)\right) + \mathbf{P}\left(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)^c\right)$, and hence,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left((n - T_{N(n)})/n \geq \delta\right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \left\{\mathbf{P}\left(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)\right) + \mathbf{P}\left(D_n(\epsilon)^c\right)\right\} \\
&= \max\left\{\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)\right), \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(D_n(\epsilon)^c\right)\right\}.
\end{aligned} \tag{4.15}$$

Let $\epsilon < \delta/(2\mathbf{E}\tau)$, then,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(T_{N(n)} \leq n(1-\delta), D_n(\epsilon)\right) &\leq \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(T_{\lfloor n(\frac{1}{\mathbf{E}\tau} - \epsilon)\rfloor} \leq n(1-\delta), D_n(\epsilon)\right) \\
&= \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(N(n(1-\delta)) \geq \left\lfloor n\left(\frac{1}{\mathbf{E}\tau} - \epsilon\right)\right\rfloor, D_n(\epsilon)\right) \\
&= -\infty.
\end{aligned}$$

Using the definition of a renewal process and Cramérs theorem we obtain

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(D_n(\epsilon)^c\right) = -\infty. \tag{4.16}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\big((n - T_{N(n)})/n > \delta\big) = -\infty. \tag{4.17}$$

Moving on to the bound for the probability of the second term in (4.14), for a any $\epsilon > 0$,

$$\begin{aligned}
\mathbf{P}\left(\{\exists j \le N(n) : \tau_j \ge n\delta\}\right) &= \mathbf{P}\left(\exists j \le N(n) : \tau_j \ge n\delta, N(n)/n \le 1/\mathbf{E}\tau + \epsilon\right) + \mathbf{P}\left(N(n)/n > 1/\mathbf{E}\tau + \epsilon\right) \\
&\le \mathbf{P}\left(\exists j \le \lceil n/\mathbf{E}(\tau) + n\epsilon\rceil : \tau_j \ge n\delta\right) + \mathbf{P}\left(N(n)/n > 1/\mathbf{E}\tau + \epsilon\right) \\
&\le \lceil n/\mathbf{E}(\tau) + n\epsilon\rceil\, \mathbf{P}\left(\tau_1 \ge n\delta\right) + \mathbf{P}\left(N(n)/n > 1/\mathbf{E}\tau + \epsilon\right).
\end{aligned}$$

Since $\mathbf{P}\left(\tau_1 \ge n\delta\right)$ and $\mathbf{P}\left(N(n)/n > 1/\mathbf{E}\tau + \epsilon\right)$ decays at exponential rates,

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\{\exists j \le N(n) : \tau_j \ge n\delta\}\right) = -\infty.$$

This along with (4.17) and (4.14) proves the desired exponential equivalence. □

Now, we have all the necessary components to prove Theorem 2.1.

*Proof of Theorem 2.1.* The preceding sequence of lemmas has resulted in LDPs of $\bar{\mathcal{Z}}_n$ (Lemma 4.7) and $\bar{S}_n$ (Lemma 4.1). Since $\bar{\mathcal{Z}}_n$ and $\bar{S}_n$ are independent, $(\bar{\mathcal{Z}}_n, \bar{S}_n)$ satisfies an LDP in $\prod_{i=1}^2 \mathbb{D}[0,1]$ with the rate function $I_{\mathcal{Z},S}(\zeta,\xi) = I_{\mathcal{Z}}(\zeta) + I_S(\xi)$; see, for example, Theorem 4.14 of [9].

Let $\phi : \prod_{i=1}^2 \mathbb{D}[0,1] \to \mathbb{D}[0,1]$ denote the addition function $\phi(\xi,\zeta) = \xi + \zeta$. Since $\phi$ is continuous on $(\xi,\zeta)$ as far as $\xi$ and $\zeta$ do not share a jump time with opposite directions, $\phi$ is continuous on the effective domain of $I_{\mathcal{Z},S}$. Let $I_{\mathbf{W}}(\zeta) \triangleq \inf\left\{I_{\mathcal{Z},S}(\xi_1,\xi_2) : \zeta = \xi_1 + \xi_2,\ \xi_1 \in \mathbb{D}^{(\lambda)}[0,1],\, \xi_2 \in \mathbb{D}^{\leqslant 1}[0,1]\right\}$, and note that it is straightforward to check that $I_W = I_Y$. By the extended contraction principle—see p.367 of [16]—we conclude that $\bar{\mathcal{Z}}_n + \bar{S}_n$ satisfies the sample path LDP with the rate function $I_Y$.

We now prove the large deviation upper bound. Let $F$ be a closed set w.r.t. the $M_1'$ topology, and let $F_\epsilon \triangleq \{\xi \in \mathbb{D}[0,1] : d_{M_1'}(\xi, F) \le \epsilon\}$. Then,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{Y}_n \in F\right) \\
&= \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \left\{\mathbf{P}\left(\bar{Y}_n \in F,\ d_{M_1'}(\bar{Y}_n, \bar{\mathcal{Z}}_n + \bar{S}_n) \le \epsilon\right) + \mathbf{P}\Big(d_{M_1'}(\bar{Y}_n, \bar{\mathcal{Z}}_n + \bar{S}_n) > \epsilon\Big)\right\} \\
&\le \limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n + \bar{S}_n \in F_\epsilon\right) \\
&\le -\inf_{\xi \in F_\epsilon} I_Y(\zeta)
\end{aligned}$$

where the first inequality is due to Lemma 4.8. Note that $\lim_{\epsilon\to 0} \inf_{\xi\in F_\epsilon} I_Y(\zeta) = \inf_{\xi\in F} I_Y(\zeta)$ since $I_W$ is good w.r.t. $\mathcal{T}_{M_1'}$. The desired large deviation upper bound follows by taking $\epsilon \to 0$.

For the lower bound, let $G$ be an open set in $\mathcal{T}_{M_1'}$. We assume that $\inf_{\xi\in G} I_Y(\xi) < \infty$ since the lower bound is trivial otherwise. For any given $\epsilon > 0$, pick $\zeta \in G$ such that $I(\zeta) \leq \inf_{\xi\in G} I_Y(\xi) + \epsilon$. Let $\delta > 0$ be such that $B_{M_1'}(\zeta, 2\delta) \in G$. Then, we know from Lemma 4.8, $\mathbf{P}\left(d(\bar{Y}_n, \bar{\mathcal{Z}}_n + \bar{S}_n) < \delta\right) / \mathbf{P}\left(\bar{\mathcal{Z}}_n + \bar{S}_n \in B_{M_1'}(\zeta,\delta)\right) \to 0$, and hence,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{Y}_n \in G\right) \\
&\geq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n + \bar{S}_n \in B_{M_1'}(\zeta,\delta),\ d(\bar{Y}_n, \bar{\mathcal{Z}}_n + \bar{S}_n) < \delta\right) \\
&\geq \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n + \bar{S}_n \in B_{M_1'}(\zeta,\delta)\right) \left\{1 - \frac{\mathbf{P}\left(d(\bar{Y}_n, \bar{\mathcal{Z}}_n + \bar{S}_n) < \delta\right)}{\mathbf{P}\left(\bar{\mathcal{Z}}_n + \bar{S}_n \in B_{M_1'}(\zeta,\delta)\right)}\right\} \\
&= \liminf_{n\to\infty} \frac{1}{n^\alpha} \log \mathbf{P}\left(\bar{\mathcal{Z}}_n + \bar{S}_n \in B_{M_1'}(\zeta,\delta)\right) \\
&\geq -\inf_{\xi \in B_{M_1'}(\zeta,\delta)} I_{\mathbf{Y}}(\xi) \geq -I_Y(\zeta) \geq -\inf_{\xi\in G} I_Y(\xi) - \epsilon.
\end{aligned}$$

Taking $\epsilon \to 0$, we arrive at the desired lower bound. □

## Appendix A: Results on the theory of Markov chains.

Let $X_n$ be a geometrically ergodic Markov chain on the state space $S$, which includes an element 0, and invariant distribution $\pi$, such that $\pi(\{0\}) = \pi(0) \neq 0$. Let $X_n^*$ be the time-reversed stationary version of the original Markov chain $X_n$. Recall that for a two-sided stationary version of the chain $(X_n : -\infty < n < \infty)$, we have that $(X_n^* : -\infty < n < \infty)$ satisfies the equality in distribution $(X_n, ..., X_{n+m}) = (X_{n+m}^*, ..., X_n^*)$ for any $-\infty < n < \infty$ and $m \geq 0$. Since $\pi(0) \neq 0$, the following lemma follows directly applying this distributional identity. In fact, the the identity can be seen to hold path-wise since the we can define $X_n = X_{-n}^*$, assuming that $X_0$ follows $\pi$.

Lemma A.1. *Let $X_n^*$ be the time reversed chain of $X_n$. It holds that*

$$\mathbf{P}_0\left(X_i \in A_i : 1 \leq i \leq n\right) = \frac{1}{\pi(0)} \mathbf{P}_\pi\left(X_i^* \in A_{n-i} : 1 \leq i \leq n, X_n^* = 0\right) \tag{A.1}$$

$$\mathbf{E}_0\left[f(0, X_1, ..., X_n)\right] = \frac{1}{\pi(0)} \mathbf{E}_\pi\left[f(0, X_{n-1}^*, ..., X_0^*) I(X_n^* = 0)\right] \tag{A.2}$$

Using the previous result, we can now establish the following lemma.

Lemma A.2. *We have*

$$\mathbf{P}_\pi\left(\sum_{k=0}^{T^*} (X_k^*)^p \geq x, X_n^* = 0\right) \leq (n+1)\mathbf{P}_\pi\left(\sum_{k=0}^{T} X_k^p \geq x\right). \tag{A.3}$$

*In addition, let $n_0$ be such that $\inf_{k\geq n_0} \mathbf{P}_0(X_k = 0) \geq \pi(0)/2$. Define $T = \inf\{n \in \{1, \ldots\} : X_n = 0\}$ and $T^* = \inf\{n \in \{1, \ldots\} : X_n^* = 0\}$, and suppose that $\mathbf{P}_\pi(T > n) = \mathcal{O}(e^{-cn})$ for some $c > 0$. Then,*

$$\mathbf{P}_\pi\left(\sum_{k=0}^{T^*} (X_k^*)^p \geq x, X_n^* = 0\right) \geq (\pi(0)^2/2)\mathbf{P}_0\left(\sum_{k=0}^{T} X_k^p \geq x\right) - \mathcal{O}(e^{-cn}) \tag{A.4}$$

*Proof.* We first derive the upper bound, by noting that

$$\begin{aligned}
\mathbf{P}_\pi\left(\sum_{k=0}^{T^*}(X_k^*)^p \geq x, X_n^*=0\right) &= \sum_{m=0}^{n}\mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X_k^*)^p \geq x, T^*>m-1, X_m^*=0, X_n^*=0\right)\\
&\leq \sum_{m=0}^{n}\mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X_{k-(m-1)}^*)^p \geq x, X_{-(m-1)}^*>0,\ldots,X_0^*>0\right)\\
&= \sum_{m=0}^{n}\mathbf{P}_\pi\left(\sum_{k=0}^{m-1}X_{m-1-k}^p \geq x, X_{m-1}>0,\ldots,X_0>0\right)\\
&= \sum_{m=0}^{n}\mathbf{P}_\pi\left(\sum_{k=0}^{m-1}X_k^p \geq x, T>m-1\right) \leq \sum_{m=0}^{n}\mathbf{P}_\pi\left(\sum_{k=0}^{T}X_k^p \geq x, T>m-1\right)\\
&\leq (n+1)\mathbf{P}_\pi\left(\sum_{k=0}^{T}X_k^p \geq x\right).
\end{aligned}$$

For the lower bound, first write

$$\begin{aligned}
\mathbf{P}_\pi\left(\sum_{k=0}^{T^*}(X_k^*)^p \geq x, X_n^*=0\right) &= \sum_{m=1}^{n}\mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X_k^*)^p \geq x, T^*=m, X_n^*=0\right)\\
&= \sum_{m=1}^{n}\mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X_k^*)^p \geq x, T^*=m\right)\mathbf{P}_0\left(X_{n-m}=0\right)
\end{aligned}$$

Apply Lemma A.1 [by using $g(y_1,...,y_n)=I(\sum_i y_i^p>x, y_i>0.i<n)$] to observe that

$$\begin{aligned}
\mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X_k^*)^p \geq x, T^*=m\right) &= \mathbf{P}_\pi\left(\sum_{k=0}^{m-1}(X_k^*)^p \geq x, X_i^*>0, i=1,\ldots,m-1, X_m^*=0\right)\\
&= \pi(0)\mathbf{P}_0\left(\sum_{k=1}^{m}X_k^p \geq x, X_i^*>0, i=1,\ldots,m-1,\right)\\
&= \pi(0)\mathbf{P}_0\left(\sum_{k=1}^{m}X_k^p \geq x, T\geq m\right)\\
&\geq \pi(0)\mathbf{P}_0\left(\sum_{k=1}^{m}X_k^p \geq x, T=m\right)\\
&= \pi(0)\mathbf{P}_0\left(\sum_{k=0}^{T}X_k^p \geq x, T=m\right)
\end{aligned}$$

Consequently, for every fixed $n_0$ such that $\inf_{k\geq n_0}\mathbf{P}_0(X_k=0)\geq \pi(0)/2$,

$$\begin{aligned}
\mathbf{P}_\pi\left(\sum_{k=0}^{T^*}(X_k^*)^p \geq x, X_n^*=0\right) &\geq \pi(0)\sum_{m=0}^{n-n_0}\mathbf{P}_0\left(\sum_{k=0}^{T}X_k^p \geq x, T=m\right)\mathbf{P}_0\left(X_{n-m}=0\right)\\
&\geq \pi(0)\mathbf{P}_0\left(\sum_{k=0}^{T}X_k^p \geq x, T\leq n-n_0\right)\inf_{k\geq n_0}\mathbf{P}_0(X_k=0).\\
&\geq (\pi(0)^2/2)\mathbf{P}_0\left(\sum_{k=0}^{T}X_k^p \geq x\right) - \mathcal{O}(e^{-cn}).
\end{aligned}$$

□

## Appendix B: LDP results.

We review some LDP results that have appeared in the literature. A straightforward adaptation of Corollary 3.2 in [1] to our context is

Result 1. Let $K_n$ be a random walk such that $K_0 = 0$ and $\mathbf{P}(K_1 \geq x) = e^{-L(x)x^{\alpha}}$ for $\alpha \in (0,1)$, and suppose that $L(x)x^{\alpha-1}$ is eventually decreasing. Then, $\bar{K}_n$ satisfies the LDP in $(\mathbb{D}[0,T], \mathcal{T}_{M_1'})$ with speed $L(n)n^{\alpha}$ and rate function $I_{M_1'} : \mathbb{D}[0,T] \to [0,\infty]$,

$$I_{M_1'}(\xi) \triangleq \begin{cases} \sum_{t\in[0,1]} (\xi(t) - \xi(t-))^{\alpha} & \text{if } \xi \in \mathbb{D}^{(\mathbf{E}S_1)}[0,T] \text{ with } \xi(0) \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

The following result, by [15], provides the logarithmic asymptotics for the steady state distribution of the reflected random walk.

Result 2 ([15]). For the steady state distribution $(\pi)$ of the reflected random walk, it holds that,

$$\lim_{n\to\infty} \frac{\log \pi([n,\infty))}{n} = -\beta.$$

Finally, we mention a recent sample path LDP for random walks, developed in [18] with light-tailed increments that we use in this paper. Now, let $\{U_i\}_{i\geq 1}$ be i.i.d. random variables and define $\bar{K}_n = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} U_i,\ t \in [0,1]$.

Result 3. Let $U_1$ satisfy **Assumptions** 2.1 and 2.2. Define

$$I_K(\xi) \triangleq \begin{cases} \int_0^1 \Lambda^*(\dot{\xi}^{(a)}(s))ds + \theta_+(\xi^{(u)}(1)) + \theta_-|\xi^{(l)}(1)| & \text{if } \xi \in \mathbb{BV}[0,1] \text{ and } \ \xi(0) = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{B.1}$$

(i) ([3, 4]) $\bar{K}_n$ satisfies a large deviations lower bound in the $\mathcal{M}_1$ topology with rate function $I_K$.

(ii) ([18]) Let $\phi$ be a real-valued function on $\mathbb{D}[0,1]$ which is uniformly continuous in the $\mathcal{M}_1$ topology on the level sets $\{\xi : I_K(\xi) \leq \alpha\}$. Then $\phi(\bar{K}_n)$ satisfies an LDP with rate function $J_\phi(u) = \inf_{\xi:\phi(\xi)=u} I_K(\xi)$.


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