Sample-path large deviations for Lévy processes and random walks with Weibull increments

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Abstract

We study sample-path large deviations for Lévy processes and random walks with heavy-tailed jump-size distributions that are of Weibull type. Our main results include an extended form of an LDP (large deviations principle) in the J_1 topology, and a full LDP in the M_1' topology. The rate function can be represented as the solution of a quasi-variational problem. The sharpness and applicability of these results are illustrated by a counterexample proving nonexistence of a full LDP in the J_1 topology, and an application to the buildup of a large queue length in a queue with multiple servers.

Keywords Sample path large deviations · Lévy processes · random walks · heavy tails

Mathematics Subject Classification 60F10 · 60G17

1 Introduction

In this paper, we develop sample-path large deviations for Lévy processes and random walks, assuming the jump sizes have a semi-exponential distribution. Specifically, let $X(t), t \geq 0$, be a centered Lévy process with positive jumps and Lévy measure ν, assume that $-\log \nu[x, \infty)$ is regularly varying of index $\alpha \in (0, 1)$. Define $\bar{X}_n = \{\bar{X}_n(t), t \in [0, 1]\}$, with $\bar{X}_n(t) = X(nt)/n, t \geq 0$. We are interested in large deviations of $\bar{X}_n$.

The investigation of tail estimates of the one-dimensional distributions of $\bar{X}_n$ (or random walks with heavy-tailed step size distribution) was initiated in Nagaev (1969, 1977). The state of the art of such results is well summarized in Borovkov and Borovkov (2008); Denisov et al. (2008); Embrechts et al. (1997); Foss et al. (2011). In particular, Denisov et al. (2008) describe in detail how fast x needs to grow with n for the asymptotic relation

$$\mathbf{P}(X(n) > x) = n\mathbf{P}(X(1) > x)(1 + o(1)) \tag{1.1}$$

to hold, as $n \to \infty$. If (1.1) is valid, the so-called *principle of one big jump* is said to hold. It turns out that, if x increases linearly with n, this principle holds if $\alpha < 1/2$ and does not hold if $\alpha > 1/2$, and the asymptotic behavior of $\mathbf{P}(X(n) > x)$ becomes more complicated. When studying more general functionals of X it becomes natural to consider logarithmic asymptotics, as is common in large deviations theory, cf. Dembo and Zeitouni (2009); Gantert (1998); Gantert et al. (2014).

The study of large deviations of sample paths of processes with Weibullian increments is quite limited. The only paper we are aware of is Gantert (1998), where the inverse contraction principle is applied to obtain a large deviation principle in the L_1 topology. As noted in Duffy and Sapozhnikov (2008) this topology is not suitable for many applications—ideally one would like to work with the J_1 topology.

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This state of affairs forms one main motivation for this paper. In addition, we are motivated by a concrete application, which is the probability of a large queue length in the $GI/GI/d$ queue, in the case that the job size distribution has a tail of the form $\exp(-L(x)x^\alpha), \alpha \in (0,1)$. The many-server queue with heavy-tailed service times has so far mainly been considered in the case of regularly varying service times, see Foss and Korshunov (2006, 2012). To illustrate the techniques developed in this paper, we show that, for $\gamma \in (0,\infty)$,

$$\lim_{n\to\infty} \frac{-\log \mathbf{P}(Q(\gamma n) > n)}{L(n)n^\alpha} = c^*, \tag{1.2}$$

with c^* the value of the optimization problem

$$\begin{aligned} &\min \sum_{i=1}^{d} x_i^\alpha \quad \text{s.t.} && (1.3)\\ &l\,(s;x) = \lambda s - \sum_{i=1}^{d} (s - x_i)^+ \geq 1 \text{ for some } s \in [0,\gamma]. \\ &x_1, ..., x_d \geq 0\ . \end{aligned}$$

where λ is the arrival rate, and service times are normalized to have unit mean. Note that this problem is equivalent to an L^α-norm minimization problem with $\alpha \in (0,1)$. Such problems also appear in applications such as compressed sensing, and are strongly NP hard in general, see Ge et al. (2011) and references therein. In our particular case, we can analyze this problem exactly, and if $\gamma \geq 1/(\lambda - \lfloor\lambda\rfloor)$, the solution takes the simple form

$$c^* = \min_{l\in\{0,...,\lfloor\lambda\rfloor\}} (d-l) \left(\frac{1}{\lambda - l}\right)^\alpha. \tag{1.4}$$

This simple minimization problem has at most two optimal solutions, which represent the most likely number of big jumps that are responsible for a large queue length to occur, and the most likely buildup of the queue length is through a linear path. For smaller values of γ, asymmetric solutions can occur, leading to a piecewise linear buildup of the queue length; we refer to Section 5 for more details.

Note that the intuition that the solution to (1.3) yields is qualitatively different from the case in which service times have a power law. In the latter case, the optimal number of big jobs equals the minimum number of servers that need to be removed to make the system unstable, which equals $\lceil d - \lambda \rceil$. In the Weibull case, there is a nontrivial trade-off between the *number* of big jobs as well as their *size*, and this trade-off is captured by (1.3) and (1.4). This essentially answers a question posed by Sergey Foss at the Erlang Centennial conference in 2009. For earlier conjectures on this problem we refer to Whitt (2000).

We derive (1.2) by utilizing a tail bound for $Q(t)$, which are derived in Gamarnik and Goldberg (2013). These tail bounds are given in terms of functionals of superpositions of random walks. We show these functionals are (almost) continuous in the M_1' topology, introduced in Puhalskii and Whitt (1997), making this motivating problem fit into our mathematical framework. The J_1 topology is not suitable since the most likely path of the input processes involve jumps at time 0.

Another implication of our results, which will be pursued in detail elsewhere, arises in the large deviations analysis of Markov random walks. More precisely, when studying $\bar{X}_n(t) = \sum_{k=1}^{\lfloor nt\rfloor} f(Y_k)/n$, where $Y_k, k \geq 0$ is a geometrically ergodic Markov chain and $f(\cdot)$ is a given measurable function. Classical large deviations results pioneered by Donsker and Varadhan on this topic (see, for example, Donsker and Varadhan, 1976) and the more recent treatment in Kontoyiannis and Meyn (2005), impose certain Lyapunov-type assumptions involving the underlying function, $f(\cdot)$.

These assumptions are not merely technical requirements, but are needed to a large deviations theory with a linear (in n) speed function (as opposed to sublinear as we obtain here). Even in simple cases (e.g. Blanchet et al., 2011) the case of unbounded $f(\cdot)$ can result in a sublinear large deviations scaling of the type considered here. For Harris chains, this can be seen by splitting $\bar{X}_n(\cdot)$ into cycles. Each term corresponding to a cycle can be seen as the area under a curve generated by $f(Y_\cdot)$. For linear f, this results in a contribution towards $\bar{X}_n(\cdot)$ which often is roughly proportional to the square of the cycle. Hence, the behavior of $\bar{X}_n(1)$ is close to that of a sum of i.i.d. Weibull-type random variables. To summarize, the main results of this paper can be applied to significantly extend the classical Donsker-Varadhan theory to unbounded functionals of Markov chains.

Let us now describe precisely our results. We first develop an extended LDP (large deviations principle) in the J_1 topology, i.e. there exists a rate function $I(\cdot)$ such that

$$\liminf_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^\alpha} \geq -\inf_{x\in A} I(x). \tag{1.5}$$

if A is open, and

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^\alpha} \leq -\lim_{\epsilon\downarrow 0} \inf_{x\in A^\epsilon} I(x). \tag{1.6}$$

if A is closed. Here $A^\epsilon = \{x : d(x,A) \leq \epsilon\}$. The rate function I is given by

$$I(\xi) = \begin{cases} \sum_{t:\xi(t)\neq\xi(t^-)} \big(\xi(t)-\xi(t-)\big)^\alpha & \text{if } \xi \in \mathbb{D}_\infty^\uparrow, \\ \infty, & \textit{otherwise}. \end{cases} \tag{1.7}$$

with $\mathbb{D}_\infty^\uparrow[0,1]$ the subspace of $\mathbb{D}[0,1]$ consisting of non-decreasing pure jump functions vanishing at the origin and continuous at 1. (As usual, $\mathbb{D}[0,1]$ is the space of cadlag functions from $[0,1]$ to $\mathbb{R}$.)

The notion of an extended large deviations principle has been introduced by Borovkov and Mogulskii (2010). We derive this result as follows: we use a suitable representation for the Lévy process in terms of Poisson random measures, allowing us to decompose the process into the contribution generated by the k largest jumps, and the remainder. The contribution generated by the k largest jumps is a step function for which we obtain the large deviations behavior by Bryc's inverse Varadhan lemma (see e.g. Theorem 4.4.13 of Dembo and Zeitouni, 2010). The remainder term is tamed by modifying a concentration bound due to Jelenković and Momčilović (2003).

To combine both estimates we need to consider the ϵ-fattening of the set A, which precludes us from obtaining a full LDP. To show that our approach cannot be improved, we construct a set A that is closed in the Skorokhod J_1 topology for which the large deviation upper bound does not hold; in this sense our extended large deviations principle can be seen as optimal. This is in line with the observation made for the regularly varying Lévy processes and random walks (Rhee et al., 2016), for which the full LDP w.r.t. J_1 topology in classical sense is shown to be unobtainable as well.

We derive several implications of our extended LDP that facilitate its use in applications. First of all, if a Lipschitz functional ϕ of $\bar{X}_n$ is chosen for which the function $I_\phi(y) = \inf_{x:\phi(x)=y} I(x)$ is a good rate function, then $\phi(X_n)$ satisfies an LDP. We illustrate this procedure by considering an example concerning the probability of ruin for an insurance company where large claims are reinsured.

A second implication is a sample path LDP in the M_1' topology. We show that the rate function I is good in this topology, allowing us to conclude $\lim_{\epsilon\downarrow 0}\inf_{x\in A^\epsilon} I(x) = \inf_{x\in A} I(x)$ if A is closed in the M_1' topology. The above-mentioned application to the multiple server queue serves as an application of this result.

We note that both implications constitute two complementary tools, and that the two examples we have chosen can only be dealt with precisely one of them. In particular, the functional in the reinsurance example is not continuous in the M_1' topology, and the most likely paths in the queueing application are discontinuous at time 0, rendering the J_1 or M_1 topologies useless.

This paper is organized as follows. Section 2 introduces notation and presents extended LDP's. These are complemented in Section 3 by LDP's of certain Lipschitz functionals, an LDP in the M_1' topology, and a counterexample. Section 4 is considering an application to boundary crossing probabilities with moderate jumps. The motivating application to queues with multiple servers is presented in Section 5. Additional proofs are presented in Section 6. The appendix develops further details about the M_1' topology that are needed in the body of the paper.

2 Sample path LDPs

In this section, we discuss sample-path large deviations for Lévy processes and random walks. Before presenting the main results, we start with a general result. Let $(\mathbb{S}, d)$ be a metric space, and $\mathcal{T}$ denote the topology induced by d. Let X_n be a sequence of $\mathbb{S}$ valued random variables. Let $A^\epsilon \triangleq \{\xi \in \mathbb{S} : d(\xi,\zeta) \le \epsilon \text{ for some } \zeta \in A\}$ and $A^{-\epsilon} \triangleq \{\xi \in \mathbb{S} : d(\xi,\zeta) \le \epsilon \text{ implies } \zeta \in A\}$. Let I be a non-negative lower semi-continuous function on $\mathbb{S}$, and a_n be a sequence of positive real numbers that tends to infinity as $n \to \infty$. We say that X_n satisfies the LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and the rate function I if

$$-\inf_{x\in A^\circ} I(x) \le \liminf_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \le \limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \le -\inf_{x\in A^-} I(x)$$

for any measurable set A. We say that X_n satisfies the *extended* LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and the rate function I if

$$-\inf_{x\in A^\circ} I(x) \le \liminf_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \le \limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \le -\lim_{\epsilon\to 0}\inf_{x\in A^\epsilon} I(x)$$

for any measurable set A. The next proposition provides the key framework for proving our main results.

Proposition 2.1. *Let I be a rate function. Suppose that for each n, X_n has a sequence of approximations $\{Y_n^k\}_{k=1,\dots}$ such that*

(i) For each k, Y_n^k satisfies the LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and the rate function I_k;

(ii) For each closed set F,
$$\lim_{k\to\infty}\inf_{x\in F} I_k(x) \ge \inf_{x\in F} I(x);$$

(iii) For each $\delta > 0$ and each open set G, there exist $\epsilon > 0$ and $K \ge 0$ such that $k \ge K$ implies
$$\inf_{x\in G^{-\epsilon}} I_k(x) \le \inf_{x\in G} I(x) + \delta;$$

(iv) For every $\epsilon > 0$ it holds that
$$\lim_{k\to\infty}\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(d(X_n, Y_n^k) > \epsilon\right) = -\infty. \tag{2.1}$$

Then, X_n satisfies an extended LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and the rate function I.

The proof of this proposition is provided in Section 6.

2.1 Extended sample-path LDP for Lévy processes

Let X be a Lévy process with a Lévy measure ν with $\nu[x,\infty) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0,1)$ and $L(\cdot)$ is a slowly varying function. We assume that $L(x)x^{\alpha-1}$ is non-increasing for large enough x's. Let $\bar{X}_n(t)$ denote the centered and scaled process:

$$\bar{X}_n(t) \triangleq \frac{1}{n}X(nt) - t\mathbf{E}X(1).$$

Let $\mathbb{D}[0,1]$ denote the Skorokhod space—space of càdlàg functions from $[0,1]$ to $\mathbb{R}$—and $\mathcal{T}_{J_1}$ denote the J_1 Skorokhod topology on $\mathbb{D}[0,1]$. We say that $\xi \in \mathbb{D}[0,1]$ is a pure jump function if $\xi = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i,1]}$ for some x_i's and u_i's such that $x_i \in \mathbb{R}$ and $u_i \in [0,1]$ for each i and u_i's are all distinct. Let $\mathbb{D}_\infty^\uparrow[0,1]$ denote the subspace of $\mathbb{D}[0,1]$ consisting of non-decreasing pure jump functions vanishing at the origin and continuous at 1. For the rest of the paper, if there is no confusion regarding the domain of a function space, we will omit the domain and simply write, for example, $\mathbb{D}_\infty^\uparrow$ instead of $\mathbb{D}_\infty^\uparrow[0,1]$. The next theorem is the main result of this paper.

Theorem 2.1. *$\bar{X}_n$ satisfies the extended large deviation principle in $(\mathbb{D},\mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function*

$$I(\xi) = \begin{cases} \sum_{t:\xi(t)\neq\xi(t-)} \big(\xi(t)-\xi(t-)\big)^\alpha & \text{if } \xi \in \mathbb{D}_\infty^\uparrow, \\ \infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

That is, for any measurable A,

$$-\inf_{\xi\in A^\circ} I(\xi) \le \liminf_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^\alpha} \le \limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^\alpha} \le -\lim_{\epsilon\to 0}\inf_{\xi\in A^\epsilon} I(\xi), \tag{2.3}$$

where $A^\epsilon \triangleq \{\xi \in \mathbb{D} : d_{J_1}(\xi,\zeta) \le \epsilon \text{ for some } \zeta \in A\}$.

Recall that $X_n(\cdot) \triangleq X(n\cdot)$ has Itô representation:

$$X_n(s) = nsa + B(ns) + \int_{x<1} x[\hat{N}([0,ns]\times dx) - ns\nu(dx)] + \int_{x\ge 1} x\hat{N}([0,ns]\times dx), \tag{2.4}$$

with a a drift parameter, B a Brownian motion, and $\hat{N}$ a Poisson random measure with mean measure Leb$\times\nu$ on $[0,n]\times(0,\infty)$; Leb here denotes the Lebesgue measure. We will see that the large deviation behavior is dominated by the last term of (2.4). It turns out to be convenient to consider the following distributional representation of the centered and scaled version of the last term:

$$\bar{Y}_n(\cdot) \triangleq \frac{1}{n}\sum_{l=1}^{N(n\cdot)} (Z_l - \mathbf{E}Z) \stackrel{\mathcal{D}}{=} \frac{1}{n}\int_{x\ge 1} x\hat{N}([0,n\cdot]\times dx) - \frac{1}{n}\hat{N}([0,n\cdot]\times[1,\infty))$$

where $N(t) \triangleq \hat{N}([0,t]\times[1,\infty))$ is a Poisson process with arrival rate ν_1, and Z_i's are i.i.d. copies of Z independent of N and such that $\mathbf{P}(Z \ge t) = \frac{1}{\nu_1}\nu[x\vee 1,\infty)$. To facilitate the proof of Theorem 2.1, we consider a further decomposition of $\bar{Y}_n$ into two pieces, one of which consists of the big increments, and the other one keeps the residual fluctuations. To be more specific, we introduce an extra notation for the rank of the increments. Given $N(n)$, define $\mathbf{S}_{N(n)}$ to be the set of all permutations of $\{1,...,N(n)\}$. Let $R_n : \{1,\dots,N(n)\} \to \{1,\dots,N(n)\}$ be a random permutation of $\{1,\dots,N(n)\}$ sampled uniformly from $\Sigma_n \triangleq \{\sigma \in \mathbf{S}_{N(n)} : Z_{\sigma^{-1}(1)} \ge \cdots \ge Z_{\sigma^{-1}(N(n))}\}$. In words, $R_n(i)$ is the rank of Z_i among $\{Z_1,\dots,Z_{N(n)}\}$

when sorted in decreasing order with the ties broken uniformly randomly. Now, we see that

$$\bar{Y}_n = \underbrace{\frac{1}{n}\sum_{i=1}^{N(nt)} Z_i \mathbb{1}_{\{R_n(i)\le k\}}}_{\triangleq \bar{J}_n^k} + \underbrace{\frac{1}{n}\sum_{i=1}^{N(nt)} \left(Z_i \mathbb{1}_{\{R_n(i)>k\}} - \mathbf{E}Z\right)}_{\triangleq \bar{K}_n^k}.$$

The proof of Theorem 2.1 is easy once the following technical lemmas are in our hands; their proofs are provided in Section 6. Let $\mathbb{D}_{\leqslant k}$ denote the subspace of $\mathbb{D}_\infty^{\uparrow}$ consisting of paths that have less than or equal to k discontinuities and are continuous at 1.

Lemma 2.1. *$\bar{J}_n^k$ satisfy the LDP in $(\mathbb{D}, \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and the rate function*

$$I_k(\xi) = \begin{cases} \sum_{t\in[0,1]} \left(\xi(t) - \xi(t-)\right)^\alpha & \text{if } \xi \in \mathbb{D}_{\leqslant k}, \\ \infty & \text{otherwise.} \end{cases} \tag{2.5}$$

Recall that $A^{-\epsilon} \triangleq \{\xi \in \mathbb{D} : d_{J_1}(\xi, \zeta) \le \epsilon \text{ implies } \zeta \in A\}$.

Lemma 2.2. *For each $\delta > 0$ and each open set G, there exist $\epsilon > 0$ and $K \ge 0$ such that for any $k \ge K$*

$$\inf_{\xi\in G^{-\epsilon}} I_k(\xi) \le \inf_{\xi\in G} I(\xi) + \delta. \tag{2.6}$$

Let $B_{J_1}(\xi, \epsilon)$ be the open ball w.r.t. the J_1 Skorokhod metric centered at ξ with radius ϵ and $B_\epsilon \triangleq B_{J_1}(0, \epsilon)$.

Lemma 2.3. *For every $\epsilon > 0$ it holds that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\|\bar{K}_n^k\|_\infty > \epsilon\right) = -\infty. \tag{2.7}$$

Proof of Theorem 2.1. For this proof, we use the following representation of $\bar{X}_n$:

$$\bar{X}_n \stackrel{\mathcal{D}}{=} \bar{Y}_n + \bar{Z}_n = \bar{J}_n^k + \bar{K}_n^k + \bar{R}_n, \tag{2.8}$$

where $\bar{R}_n(s) = \frac{1}{n}B(ns) + \frac{1}{n}\int_{|x|\le 1} x[N([0, ns] \times dx) - ns\nu(dx)] + \frac{1}{n}\hat{N}([0, ns] \times [1, \infty)) - \nu_1 t$. Next, we verify the conditions of Proposition 2.1. Lemma 6.1 confirms that I is a genuine rate function. Lemma 2.1 verifies (i). To see that (ii) is satisfied, note that $I_k(\xi) \ge I(\xi)$ for any $\xi \in \mathbb{D}$. Lemma 2.2 verifies (iii). Since $d_{J_1}(\bar{X}_n, \bar{J}_n^k) \le \|\bar{K}_n^k\|_\infty + \|\bar{R}_n\|_\infty$, Lemma 2.3 and $\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\|\bar{R}_n\|_\infty > \epsilon) = -\infty$ implies (iv). Now, the conclusion of the theorem follows from Proposition 2.1. □

2.2 Extended LDP for random walks

Let $S_k, k \ge 0$, be a mean zero random walk. Set $\bar{S}_n(t) = S_{[nt]}/n, t \ge 0$, and define $\bar{S}_n = \{\bar{S}_n(t), t \in [0, 1]\}$. We assume that $\mathbf{P}(S_1 \ge x) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0, 1)$ and $L(\cdot)$ is a slowly varying function, and again, we assume that $L(x)x^{\alpha-1}$ is non-increasing for large enough x's. The goal is to prove an extended LDP for the scaled random walk $\bar{S}_n$. Recall the rate function I in (2.2).

Theorem 2.2. *$\bar{S}_n$ satisfies the extended large deviation principle in $(\mathbb{D}, \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function I.*

Proof. Let $N(t), t \geq 0$, be an independent unit rate Poisson process. Define the Lévy process $X(t) \triangleq S_{N(t)}, t \geq 0$, and set $\bar{X}_n(t) \triangleq X(nt)/n, t \geq 0$. Then the Lévy measure ν of X is $\nu[x,\infty) = \mathbf{P}(S_1 \geq x)$. We first note that the J_1 distance between $\bar{S}_n$ and $\bar{X}_n$ is bounded by $\sup_{t\in[0,1]} |\lambda_n(t) - t|$ which, in turn, is bounded by $\sup_{t\in[0,1]} |N(tn)/n - t|$. From Etemadi's theorem,

$$\mathbf{P}(\sup_{t\in[0,1]} |N(tn)/n - t|) > \epsilon) \leq 3 \sup_{t\in[0,1]} \mathbf{P}(|N(tn)/n - t|) > \epsilon/3),$$

where $\mathbf{P}(|N(tn)/n - t|) > \epsilon/3)$ vanishes at a geometric rate w.r.t. n uniformly in $t \in [0,1]$. Hence, $\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(d_{J_1}(\bar{S}_n, \bar{X}_n) > \epsilon) = -\infty$. Now we consider the decomposition (2.8) again. Condition (i), (ii), and (iii) of Proposition 2.1 is again verified by Lemma 2.1, Lemma 2.2, and Lemma 2.3. For (iv), note that since $d_{J_1}(\bar{S}_n, \bar{J}_n^k) \leq d_{J_1}(\bar{S}_n, \bar{X}_n) + \|\bar{K}_n^k\|_\infty + \|\bar{R}_n\|_\infty$,

$$\begin{aligned}\limsup_{n\to\infty} \frac{\log \mathbf{P}(d_{J_1}(\bar{S}_n, \bar{J}_n^k) > \epsilon)}{L(n)n^\alpha} &\leq \limsup_{n\to\infty} \frac{\log \left\{\mathbf{P}(d_{J_1}(\bar{S}_n, \bar{X}_n) > \epsilon/3) + \mathbf{P}(\|\bar{K}_n^k\|_\infty > \epsilon/3) + \mathbf{P}(\|\bar{R}_n\|_\infty > \epsilon/3)\right\}}{L(n)n^\alpha} \\ &= -\infty.\end{aligned}$$

Therefore, Proposition 2.1 applies, and the conclusion of the theorem follows. □

2.3 Multi-dimensional processes

Let $X^{(1)}, \ldots, X^{(d)}$ be independent Lévy processes with a Lévy measure ν. As in the previous sections, we assume that ν has Weibull tail distribution with shape parameter α in $(0,1)$ and $L(x)x^{\alpha-1}$ is non-increasing for large enough x's. Let $\bar{X}_n^{(i)}(t)$ denote the centered and scaled processes:

$$\bar{X}_n^{(i)}(t) \triangleq \frac{1}{n} X^{(i)}(nt) - t\mathbf{E}X^{(i)}(1).$$

The next theorem establishes the extended LDP for $(\bar{X}_n^{(1)}, \ldots, \bar{X}_n^{(d)})$.

Theorem 2.3. $\left(\bar{X}_n^{(1)}, \bar{X}_n^{(2)}, \ldots, \bar{X}_n^{(d)}\right)$ *satisfies the extended LDP on* $\left(\prod_{i=1}^d \mathbb{D}\left([0,1], \mathbb{R}_+\right), \prod_{i=1}^d \mathcal{T}_{J_1}\right)$ *with speed* $L(n)n^\alpha$ *and the rate function*

$$I^d(\xi_1, ..., \xi_d) = \begin{cases} \sum_{j=1}^d \sum_{t\in[0,1]} (\xi_j(t) - \xi_j(t-))^\alpha & \text{if } \xi_j \in \mathbb{D}_\infty^\uparrow[0,1] \text{ for each } j = 1, \ldots, d, \\ \infty, & \text{otherwise.} \end{cases} \tag{2.9}$$

For each i, we consider the same distributional decomposition of $\bar{X}_n^{(i)}$ as in Section 2.1:

$$\bar{X}_n^{(i)} \stackrel{\mathcal{D}}{=} \bar{J}_n^{k\,(i)} + \bar{K}_n^{k\,(i)} + \bar{R}_n^{(i)}.$$

The proof of the theorem is immediate as in the one dimensional case, from Proposition 2.1, Lemma 2.3, and the following lemmas that parallel Lemma 2.1 and Lemma 2.2.

Lemma 2.4. $(\bar{J}_n^{k\,(1)}, \ldots, \bar{J}_n^{k\,(d)})$ *satisfy the LDP in* $\left(\prod_{i=1}^d \mathbb{D}, \prod_{i=1}^d \mathcal{T}_{J_1}\right)$ *with speed* $L(n)n^\alpha$ *and the rate function* $I_k^d : \prod_{i=1}^d \mathbb{D} \to [0,\infty]$

$$I_k^d(\xi_1, \ldots, \xi_d) \triangleq \begin{cases} \sum_{i=1}^d \sum_{t\in[0,1]} (\xi_i(t) - \xi_i(t-))^\alpha & \text{if } \xi_i \in \mathbb{D}_{\leqslant k} \text{ for } i = 1, \ldots, d, \\ \infty & \text{otherwise.} \end{cases} \tag{2.10}$$

Lemma 2.5. *For each $\delta > 0$ and each open set G, there exists $\epsilon > 0$ and $K \geq 0$ such that for any $k \geq K$*

$$\inf_{(\xi_1,\ldots,\xi_d)\in G^{-\epsilon}} I_k^d(\xi_1,\ldots,\xi_d) \leq \inf_{(\xi_1,\ldots,\xi_d)\in G} I^d(\xi_1,\ldots,\xi_d) + \delta. \tag{2.11}$$

We conclude this section with the extended LDP for multidimensional random walks. Let $\{S_k^{(i)}, k \geq 0\}$ be a mean zero random walk for each $i = 1,\ldots,d$. Set $\bar{S}_n^{(i)}(t) = S_{[nt]}^{(i)}/n, t \geq 0$, and define $\bar{S}_n^{(i)} = \{\bar{S}_n^{(i)}(t), t \in [0,1]\}$. We assume that $\mathbf{P}(S_1^{(i)} \geq x) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0,1)$ and $L(\cdot)$ is a slowly varying function, and again, we assume that $L(x)x^{\alpha-1}$ is non-increasing for large enough x's. The following theorem can derived from Proposition 2.1 and Theorem 2.3 in the same way as in Section 2.2. Recall the rate function I^d in (2.2).

Theorem 2.4. *$(\bar{S}_n^{(1)},\ldots,\bar{S}_n^{(d)})$ satisfies the extended LDP in $(\prod_{i=1}^d \mathbb{D}, \prod_{i=1}^d \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and the rate function I^d.*

Remark 1. *Note that Theorem 2.3 and Theorem 2.4 can be extended to heterogeneous processes. For example, if the Lévy measure $\nu^{(i)}$ of the process $X^{(i)}$ has Weibull tail distribution $\nu^{(i)}[x,\infty) = \exp(-c_i L(x)x^\alpha)$ where $c_i \in (0,\infty)$ for each $i \leq d_0 < d$, and all the other processes have lighter tails—i.e., $L(x)x^\alpha = o(L_i(x)x^{\alpha_i})$ for $i > d_0$—then it is straightforward to check that $(\bar{X}_n^{(1)},\ldots,\bar{X}_n^{(d)})$ satisfies the extended LDP with the rate function*

$$I^d(\xi_1,\ldots,\xi_d) = \begin{cases} \sum_{j=1}^{d_0} c_j \sum_{t\in[0,1]} \big(\xi_j(t) - \xi_j(t-)\big)^\alpha & \text{if } \xi_j \in \mathbb{D}_\infty^\uparrow[0,1] \text{ for } j = 1,\ldots,d_0 \text{ and } \xi_j \equiv 0 \text{ for } j > d_0, \\ \infty, & \text{otherwise.} \end{cases}$$

Similarly, $(\bar{S}_n^{(1)},\ldots,\bar{S}_n^{(d)})$ satisfies the extended LDP with the same rate function under corresponding condition on the tail distribution of $S_1^{(i)}$'s.

3 Implications and further discussions

3.1 LDP for Lipschitz functions of Lévy processes

Let $\bar{\mathbf{X}}_n$ denote the scaled Lévy processes $(\bar{X}_n^{(1)},\ldots,\bar{X}_n^{(d)})$, and $\bar{\mathbf{S}}_n$ denote the scaled random walks $(\bar{S}_n^{(1)},\ldots,\bar{S}_n^{(d)})$ as defined in Section 2. Recall also the rate function I^d defined in (2.9).

Corollary 3.1. *Let $(\mathbb{S}, d)$ be a metric space and $\phi : \prod_{i=1}^d \mathbb{D} \to \mathbb{S}$ be a Lipschitz continuous mapping w.r.t. the J_1 Skorokhod metric. Set*

$$I'(x) \triangleq \inf_{\phi(\xi)=x} I^d(\xi)$$

and suppose that I' is a good rate function—i.e., $\Psi_{I'}(a) \triangleq \{x \in \mathbb{S} : I'(s) \leq a\}$ is compact for each $a \in [0,\infty)$. Then, $\phi\left(\bar{\mathbf{X}}_n\right)$ and $\phi\left(\bar{\mathbf{S}}_n\right)$ satisfy the large deviation principle in $(\mathbb{S}, d)$ with speed $L(n)n^\alpha$ and the good rate function I'.

Proof. Since the argument for $\phi(\bar{\mathbf{S}}_n)$ is very similar, we only prove the LDP for $\phi(\bar{\mathbf{X}}_n)$. We start with the upper bound. Suppose that the Lipschitz constant ϕ is $\|\phi\|_{\text{Lip}}$. Note that since the J_1 distance is dominated by the supremum distance, $\|\bar{\mathbf{K}}_n^k\|_\infty \leq \epsilon$ and $\|\bar{\mathbf{R}}_n\|_\infty \leq \epsilon$ implies $d_{J_1}\big(\phi(\bar{\mathbf{J}}_n^k), \phi(\bar{\mathbf{X}}_n)\big) \leq 2\epsilon\|\phi\|_{\text{Lip}}$, where $\bar{\mathbf{J}}_n^k \triangleq (\bar{J}_n^{k\,(1)},\ldots,\bar{J}_n^{k\,(d)})$, $\bar{\mathbf{K}}_n^k \triangleq (\bar{K}_n^{k\,(1)},\ldots,\bar{K}_n^{k\,(d)})$, and $\bar{\mathbf{R}}_n \triangleq (\bar{R}_n^{(1)},\ldots,\bar{R}_n^{(d)})$. Therefore, for any closed set F,

$$\begin{aligned}\mathbf{P}\left(\phi(\bar{\mathbf{X}}_n)\in F\right) &= \mathbf{P}\left(\phi\left(\bar{\mathbf{X}}_n\right)\in F, d_{J_1}\left(\phi(\bar{\mathbf{J}}_n^k),\,\phi(\bar{\mathbf{X}}_n)\right)\le 2\epsilon\|\phi\|_{\mathrm{Lip}}\right)+\mathbf{P}\left(d_{J_1}\left(\phi(\bar{\mathbf{J}}_n^k),\,\phi(\bar{\mathbf{X}}_n)\right)>2\epsilon\|\phi\|_{\mathrm{Lip}}\right)\\ &\le \mathbf{P}\left(\phi\left(\bar{\mathbf{J}}_n^k\right)\in F^{2\epsilon\|\phi\|_{\mathrm{Lip}}}\right)+\mathbf{P}\left(d_{J_1}\left(\phi(\bar{\mathbf{J}}_n^k),\,\phi(\bar{\mathbf{X}}_n)\right)>2\epsilon\|\phi\|_{\mathrm{Lip}}\right)\\ &\le \mathbf{P}\left(\bar{\mathbf{J}}_n^k\in\phi^{-1}\left(F^{2\epsilon\|\phi\|_{\mathrm{Lip}}}\right)\right)+\mathbf{P}\left(\|\bar{\mathbf{K}}_n^k\|_\infty>\epsilon\right)+\mathbf{P}\left(\|\bar{\mathbf{R}}_n\|_\infty>\epsilon\right).\end{aligned}$$

Since $\mathbf{P}\left(\|\bar{\mathbf{R}}_n\|_\infty>\epsilon\right)$ decays at an exponential rate and $\mathbf{P}\left(\|\bar{\mathbf{K}}_n^k\|_\infty>\epsilon\right)\le\sum_{i=1}^d\mathbf{P}\left(\|\bar{K}_n^{k\,(i)}\|_\infty>\epsilon\right)$, we get the following bound by applying the principle of the maximum term and Theorem 2.3:

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}\left(\phi\left(\bar{\mathbf{X}}_n\right)\in F\right)\\ &\le\limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\left\{\mathbf{P}\left(\bar{\mathbf{J}}_n^k\in\phi^{-1}\left(F^{2\epsilon\|\phi\|_{\mathrm{Lip}}}\right)\right)+\mathbf{P}\left(\|\bar{\mathbf{K}}_n^k\|_\infty>\epsilon\right)+\mathbf{P}\left(\|\bar{\mathbf{R}}_n\|_\infty>\epsilon\right)\right\}\\ &=\max\left\{\limsup_{n\to\infty}\frac{\log\mathbf{P}\left(\bar{\mathbf{J}}_n^k\in\phi^{-1}\left(F^{2\epsilon\|\phi\|_{\mathrm{Lip}}}\right)\right)}{L(n)n^\alpha},\ \limsup_{n\to\infty}\frac{\log\mathbf{P}\left(\|\bar{\mathbf{K}}_n^k\|_\infty>\epsilon\right)}{L(n)n^\alpha}\right\}\\ &\le\max\left\{-\inf_{(\xi_1,\ldots,\xi_d)\in\phi^{-1}\left(F^{2\epsilon\|\phi\|_{\mathrm{Lip}}}\right)}I_k^d(\xi_1,\ldots,\xi_d),\ \max_{i=1,\ldots,d}\limsup_{n\to\infty}\frac{\log\mathbf{P}\left(\|\bar{K}_n^{k\,(i)}\|_\infty>\epsilon\right)}{L(n)n^\alpha}\right\}\\ &\le\max\left\{-\inf_{(\xi_1,\ldots,\xi_d)\in\phi^{-1}\left(F^{2\epsilon\|\phi\|_{\mathrm{Lip}}}\right)}I^d(\xi_1,\ldots,\xi_d),\ \limsup_{n\to\infty}\frac{\log\mathbf{P}\left(\|\bar{K}_n^{k\,(1)}\|_\infty>\epsilon\right)}{L(n)n^\alpha}\right\}\end{aligned}$$

From Lemma 2.3, we can take $k\to\infty$ to get

$$\limsup_{n\to\infty}\frac{\log\mathbf{P}\left(\phi\left(\bar{\mathbf{X}}_n\right)\in F\right)}{L(n)n^\alpha}\le-\inf_{(\xi_1,\ldots,\xi_d)\in\phi^{-1}\left(F^{2\epsilon\|\phi\|_{\mathrm{Lip}}}\right)}I^d(\xi_1,\ldots,\xi_d)=-\inf_{x\in F^{2\epsilon\|\phi\|_{\mathrm{Lip}}}}I'(x)\tag{3.1}$$

From Lemma 4.1.6 of Dembo and Zeitouni (2010), $\lim_{\epsilon\to 0}\inf_{x\in F^{\epsilon\|\phi\|_{\mathrm{Lip}}}}I'(x)=\inf_{x\in F}I'(x)$. Letting $\epsilon\to 0$ in (3.1), we arrive at the desired large deviation upper bound.

Turning to the lower bound, consider an open set G. Since $\phi^{-1}(G)$ is open, from Theorem 2.3,

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}\left(\phi(\bar{\mathbf{X}}_n)\in G\right)&=\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}\left(\bar{\mathbf{X}}_n\in\phi^{-1}(G)\right)\\ &\ge-\inf_{(\xi_1,\ldots,\xi_d)\in\phi^{-1}(G)}I(\xi)=-\inf_{x\in G}I'(x).\end{aligned}$$

□

3.2 Sample path LDP w.r.t. M_1' topology

In this section, we prove the full LDP for $\bar{X}_n$ and $\bar{S}_n$ w.r.t. the M_1' topology. For the definition of M_1' topology, see Appendix A.

Corollary 3.2. *$\bar{X}_n$ and $\bar{S}_n$ satisfy the LDP in $(\mathbb{D},\mathcal{T}_{M_1'})$ with speed $L(n)n^\alpha$ and the good rate function $I_{M_1'}$.*

$$I_{M_1'}(\xi)\triangleq\begin{cases}\sum_{t\in[0,1]}\left(\xi(t)-\xi(t-)\right)^\alpha & \text{if } \xi \text{ is a non-decreasing pure jump function with } \xi(0)\ge 0,\\ \infty & \text{otherwise.}\end{cases}.$$

Proof. Since the proof for $\bar{S}_n$ is identical, we only provide the proof for $\bar{X}_n$. From Proposition A.3 we know that $I_{M_1'}$ is a good rate function. For the LDP upper bound, suppose that F is a closed set w.r.t. the M_1' topology. Then, it is also closed w.r.t. the J_1 topology. From the upper bound of Theorem 2.1 and the fact that $I_{M_1'}(\xi) \le I(\xi)$ for any $\xi \in \mathbb{D}$,

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in F)}{L(n)n^\alpha} \le -\lim_{\epsilon\to 0}\inf_{\xi\in F^\epsilon} I(\xi) \le -\lim_{\epsilon\to 0}\inf_{\xi\in F^\epsilon} I_{M_1'}(\xi).$$

Turning to the lower bound, suppose that G is an open set w.r.t. the M_1' topology. We claim that

$$\inf_{\xi\in G} I_{M_1'}(\xi) = \inf_{\xi\in G} I(\xi).$$

To show this, we only have to show that the RHS is not strictly larger than the LHS. Suppose that $I_{M_1'}(\xi) < I(\xi)$ for some $\xi \in G$. Since I and $I_{M_1'}$ differ only if the path has a jump at either 0 or 1, this means that ξ is a non-negative pure jump function of the following form:

$$\xi = \sum_{i=1}^\infty z_i \mathbb{1}_{[u_i,1]},$$

where $u_1 = 0$, $u_2 = 1$, u_i's are all distinct in $(0,1)$ for $i \ge 3$ and $z_i \ge 0$ for all i's. Note that one can pick an arbitrarily small ϵ so that $\sum_{i\in\{n:u_n<\epsilon\}} z_i < \epsilon$, $\sum_{i\in\{n:u_n>1-\epsilon\}} z_i < \epsilon$, $\epsilon \ne u_i$ for all $i \ge 2$, and $1-\epsilon \ne u_i$ for all $i \ge 2$. For such ϵ's, if we set

$$\xi_\epsilon \triangleq z_1\mathbb{1}_{[\epsilon,1]} + z_2\mathbb{1}_{[1-\epsilon,1]} + \sum_{i=3}^\infty z_i\mathbb{1}_{[u_i,1]},$$

then $d_{M_1'}(\xi,\xi_\epsilon) \le \epsilon$ while $I(\xi_\epsilon) = I_{M_1'}(\xi)$. That is, we can find an arbitrarily close element ξ_ϵ from ξ w.r.t. the M_1' metric by pushing the jump times at 0 and 1 slightly to the inside of $(0,1)$; at such an element, I assumes the same value as $I_{M_1'}(\xi)$. Since G is open w.r.t. M_1', one can choose ϵ small enough so that $\xi_\epsilon \in G$. This proves the claim. Now, the desired LDP lower bound is immediate from the LDP lower bound in Theorem 2.1 since G is also an open set in J_1 topology. □

3.3 Nonexistence of large deviation principle in the J_1 topology

Consider a compound Poisson process whose jump distribution is Weibull with the shape parameter 1/2. More specifically, $\bar{X}_n(t) \triangleq \frac{1}{n}\sum_{i=1}^{N(nt)} Z_i - t$ with $\mathbf{P}(Z_i \ge x) \sim \exp(-x^\alpha)$, $\mathbf{E}Z_i = 1$, and $\alpha = 1/2$. If $\bar{X}_n$ satisfies a full LDP w.r.t. the J_1 topology, the rate function that controls the LDP (with speed n^α) associated with $\bar{X}_n$ should be of the same form as the one that controls the extended LDP:

$$I(\xi) = \begin{cases} \sum_{t\in[0,1]} \big(\xi(t)-\xi(t-)\big)^\alpha & \text{if } \xi \in \mathbb{D}_\infty^\uparrow, \\ \infty & \text{otherwise.}\end{cases}$$

To show that such a LDP is fundamentally impossible, we will construct a closed set A for which

$$\limsup_{n\to\infty}\frac{\log \mathbf{P}(\bar{X}_n\in A)}{n^\alpha} > -\inf_{\xi\in A} I(\xi). \tag{3.2}$$

Let

$$\varphi_{s,t}(\xi) \triangleq \lim_{\epsilon\to 0}\sup_{0\vee(s-\epsilon)\le u\le v\le 1\wedge(t+\epsilon)} (\xi(v)-\xi(u)).$$

Let $A_{c;s,t} \triangleq \{\xi : \varphi_{s,t}(\xi) \geq c\}$ be (roughly speaking) the set of paths which increase at least by c between time s and t. Then $A_{c;s,t}$ is a closed set for each c, s, and t.

Let

$$A_m \triangleq \Big(A_{1;\frac{m+1}{m+2},\frac{m+1}{m+2}+mh_m}\Big) \cap \Big(A_{1;\frac{m}{m+2},\frac{m}{m+2}+mh_m}\Big) \cap \Big(\bigcap_{j=0}^{m-1} A_{\frac{1}{m^2};\frac{j}{m+2},\frac{j}{m+2}+mh_m}\Big)$$

where $h_m = \frac{1}{(m+1)(m+2)}$, and let

$$A \triangleq \bigcup_{m=1}^{\infty} A_m.$$

To see that A is closed, we first claim that $\zeta \in \mathbb{D} \setminus A$ implies the existence of $\epsilon > 0$ and $N \geq 0$ such that $B(\zeta;\epsilon) \cap A_m = \emptyset$ for all $m \geq N$. To prove this claim, suppose not. It is straightforward to check that for each n, there has to be $s_n, t_n \in [1-1/n, 1)$ such that $s_n \leq t_n$ and $\zeta(t_n) - \zeta(s_n) \geq 1/2$, which in turn implies that ζ must possess infinite number of increases of size at least $1/2$ in $[1-\delta, 1)$ for any $\delta > 0$. This implies that ζ cannot possess a left limit, which is contradictory to the assumption that $\zeta \in \mathbb{D} \setminus A$. On the other hand, since each A_m is closed, $\bigcup_{i=1}^{N} A_i$ is also closed, and hence, there exists $\epsilon' > 0$ such that $B(\zeta;\epsilon') \cap A_m = \emptyset$ for $m = 1, \ldots, N$. Now, from the construction of ϵ and ϵ', $B(\zeta, \epsilon \vee \epsilon') \cap A = \emptyset$, proving that A is closed.

Next, we show that A satisfies (3.2). First note that if ξ is a pure jump function that belongs to A_m, ξ has to possess m upward jumps of size $1/m^2$ and 2 upward jumps of size 1, and hence,

$$\inf_{\xi \in A} I(\xi) \geq \inf_m \Big(1^{1/2} + 1^{1/2} + m(1/m^2)^{1/2}\Big) = 3. \tag{3.3}$$

On the other hand, letting $\Delta\xi(t) \triangleq \xi(t) - \xi(t-)$,

$$\begin{aligned}
&\mathbf{P}(\bar{X}_{(n+1)(n+2)} \in A_n) \\
&\geq \prod_{j=0}^{n-1} \mathbf{P}\bigg(\sup_{t\in[0,1]} \Big\{\bar{X}_{(n+1)(n+2)}\Big(\tfrac{(n+1)j+nt}{(n+1)(n+2)}\Big) - \bar{X}_{(n+1)(n+2)}\Big(\tfrac{(n+1)j}{(n+1)(n+2)}\Big)\Big\} \geq \frac{1}{n^2}\bigg) \\
&\qquad \cdot \mathbf{P}\bigg(\sup_{t\in(0,1]} \Big\{\Delta\bar{X}_{(n+1)(n+2)}\Big(\tfrac{(n+1)n+nt}{(n+1)(n+2)}\Big)\Big\} \geq 1\bigg) \cdot \mathbf{P}\bigg(\sup_{t\in(0,1]} \Big\{\Delta\bar{X}_{(n+1)(n+2)}\Big(\tfrac{(n+1)(n+1)+nt}{(n+1)(n+2)}\Big)\Big\} \geq 1\bigg) \\
&= \mathbf{P}\bigg(\sup_{t\in[0,1]} \Big\{\bar{X}_{(n+1)(n+2)}\Big(\tfrac{nt}{(n+1)(n+2)}\Big)\Big\} \geq \frac{1}{n^2}\bigg)^n \cdot \mathbf{P}\bigg(\sup_{t\in[0,1]} \Big\{\Delta\bar{X}_{(n+1)(n+2)}\Big(\tfrac{nt}{(n+1)(n+2)}\Big)\Big\} \geq 1\bigg)^2 \\
&= \mathbf{P}\bigg(\sup_{t\in[0,1]} \Big\{X(nt)\Big\} \geq \frac{(n+1)(n+2)}{n^2}\bigg)^n \cdot \mathbf{P}\bigg(\sup_{t\in[0,1]} \Big\{\Delta X(nt)\Big\} \geq (n+1)(n+2)\bigg)^2 \\
&\geq \mathbf{P}\bigg(\sup_{t\in[0,1]} \Big\{X(nt)\Big\} \geq 6\bigg)^n \cdot \mathbf{P}\bigg(\sup_{t\in[0,1]} \Big\{\Delta X(nt)\Big\} \geq (n+1)(n+2)\bigg)^2,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{n^\alpha} \geq \limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_{(n+1)(n+2)} \in A_n)}{\big((n+1)(n+2)\big)^\alpha} \\
&\geq \limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{X(nt)\Big\} \geq 6\Big)^n}{\big((n+1)(n+2)\big)^\alpha} + 2\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{\Delta X(nt)\Big\} \geq (n+1)(n+2)\Big)}{\big((n+1)(n+2)\big)^\alpha} \\
&= \text{(I)} + \text{(II)}.
\end{aligned} \tag{3.4}$$

Letting $p_n \triangleq \mathbf{P}\Big(\sup_{t\in[0,n]}\Big\{X(t)\Big\} < 6\Big)$,

$$\text{(I)} = \limsup_{n\to\infty} \frac{\log(1-p_n)^n}{\big((n+1)(n+2)\big)^\alpha} = \limsup_{n\to\infty} \frac{np_n \log(1-p_n)^{1/p_n}}{\big((n+1)(n+2)\big)^\alpha} = \limsup_{n\to\infty} \frac{-np_n}{\big((n+1)(n+2)\big)^\alpha} = 0 \tag{3.5}$$

since $p_n \to 0$ as $n \to \infty$.

For the second term, from the generic inequality $1 - e^{-x} \geq x(1-x)$,

$$\begin{aligned}
\text{(II)} &= 2\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{\Delta X(nt)\Big\} \geq (n+1)(n+2)\Big)}{\big((n+1)(n+2)\big)^\alpha} \\
&= 2\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(Q_n^{\leftarrow}(\Gamma_1) \geq (n+1)(n+2)\Big)}{\big((n+1)(n+2)\big)^\alpha} = 2\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\Gamma_1 \leq Q_n((n+1)(n+2))\Big)}{\big((n+1)(n+2)\big)^\alpha} \\
&= 2\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\Gamma_1 \leq n\exp(-((n+1)(n+2))^\alpha)\Big)}{\big((n+1)(n+2)\big)^\alpha} \\
&= 2\limsup_{n\to\infty} \frac{\log\Big\{1 - \exp\Big(-n\exp\big(-((n+1)(n+2))^\alpha\big)\Big)\Big\}}{\big((n+1)(n+2)\big)^\alpha} \\
&\geq 2\limsup_{n\to\infty} \frac{\log\Big\{\Big(n\exp\big(-((n+1)(n+2))^\alpha\big)\Big)\Big[1 - \Big(n\exp\big(-((n+1)(n+2))^\alpha\big)\Big)\Big]\Big\}}{\big((n+1)(n+2)\big)^\alpha} \\
&= -2.
\end{aligned} \tag{3.6}$$

From (3.4), (3.5), (3.6),

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{n^\alpha} \geq -2. \tag{3.7}$$

This along with (3.3),

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{n^\alpha} \geq -2 > -3 \geq -\inf_{\xi\in A} I(\xi),$$

which means that A indeed is a counter example for the desired LDP.

Note that a simpler counterexample can be constructed using the peculiarity of J_1 topology at the boundary of the domain; that is, jumps (of size 1, say) at time 0 are bounded away from the jumps at arbitrarily close jump times. For example, if $\bar{Y}_n(t) \triangleq \frac{1}{n}\sum_{i=1}^{N(nt)} Z_i + t$ where the right tail of Z is Weibull and $\mathbf{E}Z = -1$, then the set $B = \{x : x(t) \geq t/2 \text{ for all } t \in [0,1]\}$ gives a counterexample for the LDP. Note that M_1' topology we used in Section 3.2 is a treatment for the same peculiarity of M_1 topology at time 0. However, it should be clear from the above counterexample $\bar{X}_n$ and A that the LDP is fundamentally impossible w.r.t. J_1-like topologies—i.e., the ones that do not allow merging two or more jumps to approximate a single jump at any time—and hence, there is no hope for the full LDP in the case of J_1 topology.

4 Boundary crossing with moderate jumps

In this section, we illustrate the value of Corollary 3.1 We consider level crossing probabilities of Lévy processes where the jump sizes are conditioned to be moderate. Specifically, we apply Corollary 3.1 in order to provide large-deviations estimates for

$$\mathbf{P}\left(\sup_{t\in[0,1]} \bar{X}_n(t) \geq c,\ \sup_{t\in[0,1]} \left|\bar{X}_n(t) - \bar{X}_n(t-)\right| \leq b\right). \tag{4.1}$$

We emphasize that this type of rare events are difficult to analyze with the tools developed previously. In particular, the sample path LDP proved in Gantert (1998) is w.r.t. the L_1 topology. Since the closure of the sets in (4.1) w.r.t. the L_1 topology is the entire space, the LDP upper bound would not provide any information.

Functionals like (4.1) appear in actuarial models, in case excessively large insurance claims are reinsured and therefore do not play a role in the ruin of an insurance company. Asmussen and Pihlsgård (2005), for example, studied various estimates of infinite-time ruin probabilities with analytic methods. Rhee et al. (2016) studied the finite-time ruin probability, using probabilistic techniques in case of regularly varying Lévy measures and confirmed that the conventional wisdom "the principle of a single big jump" can be extended to "the principle of the minimal number of big jumps" in such a context. Here we show that a similar result—with subtle differences—can be obtained in case the Lévy measure has a Weibull tail.

Define the function $\phi : \mathbb{D} \to \mathbb{R}^2$ as

$$\phi(\xi) = (\phi_1(\xi), \phi_2(\xi)) \triangleq \left(\sup_{t\in[0,1]} \xi(t),\ \sup_{t\in[0,1]} |\xi(t) - \xi(t-)|\right).$$

In order to apply Corollary 3.1, we will validate that ϕ is Lipschitz continuous and that $I'(x,y) \triangleq \inf_{\{\xi\in\mathbb{D}:\phi(\xi)=(x,y)\}} I(\xi)$ is a good rate function.

For the Lipschitz continuity of ϕ, we claim that each component of ϕ is Lipschitz continuous. We first examine ϕ_1. Let $\xi, \zeta \in \mathbb{D}$ and suppose w.l.o.g. that $\sup_{t\in[0,1]} \xi(t) > \sup_{t\in[0,1]} \zeta(t)$. For an arbitrary non-decreasing homeomorphism $\lambda : [0,1] \to [0,1]$,

$$\begin{aligned} |\phi_1(\xi) - \phi_1(\zeta)| &= |\sup_{t\in[0,1]} \xi(t) - \sup_{t\in[0,1]} \zeta(t)| = |\sup_{t\in[0,1]} \xi(t) - \sup_{t\in[0,1]} \zeta\circ\lambda(t)| \\ &\leq \sup_{t\in[0,1]} |\xi(t) - \zeta\circ\lambda(t)| \leq \sup_{t\in[0,1]} |\xi(t) - \zeta\circ\lambda(t)| \vee |\lambda(t) - t|. \end{aligned} \tag{4.2}$$

Taking the infimum over λ, we conclude that

$$|\phi_1(\xi) - \phi_1(\zeta)| \leq \inf_{\lambda\in\Lambda} \sup_{t\in[0,1]} \{|\xi(t) - \zeta(\lambda(t))| \vee |\lambda(t) - t|\} = d_{J_1}(\xi,\zeta).$$

Therefore, ϕ_1 is Lipschitz with the Lipschitz constant 1.

Now, in order to prove that $\phi_2(\xi)$ is Lipschitz, fix two distinct paths $\xi, \zeta \in \mathbb{D}$ and assume w.l.o.g. that $\phi_2(\zeta) > \phi_2(\xi)$. Let $c \triangleq \phi_2(\zeta) - \phi_2(\xi) > 0$, and let $t^* \in [0,1]$ be the maximum jump time of ξ, i.e., $\phi_2(\xi) = |\xi(t^*) - \xi(t^*-)|$. For any $\epsilon > 0$ there exists λ^* so that

$$\begin{aligned} d_{J_1}(\xi,\zeta) &\triangleq \inf_{\lambda\in\Lambda}\{\|\xi - \zeta\circ\lambda\|_\infty \vee \|\lambda - e\|_\infty\} \geq \|\xi - \zeta\circ\lambda^*\|_\infty \vee \|\lambda^* - e\|_\infty - \epsilon. \\ &\geq |\xi(t^*) - \zeta\circ\lambda^*(t^*)| \vee |\xi(t^*-) - \zeta\circ\lambda^*(t^*-)| - \epsilon. \end{aligned} \tag{4.3}$$

From the general inequality $|a-b| \vee |c-d| \geq \frac{1}{2}(|a-c| - |b-d|)$,

$$\begin{aligned} |\xi(t^*) - \zeta\circ\lambda^*(t^*)| \vee |\xi(t_{1-}) - \zeta\circ\lambda^*(t^*-)| &\geq \frac{1}{2}\big(|\xi(t^*) - \xi(t^*-)| - |\zeta\circ\lambda^*(t^*) - \zeta\circ\lambda^*(t^*-)|\big) \\ &= \frac{1}{2}\big(\phi_2(\xi) - \phi_2(\zeta)\big) = c/2. \end{aligned} \tag{4.4}$$

In view of (4.3) and (4.4), $d_{J_1}(\xi,\zeta) \geq \frac{1}{2}|\phi(\xi) - \phi(\zeta)| - \epsilon$. Since ϵ is arbitrary, we get the desired Lipschitz bound with Lipschitz constant 2. Therefore, $|\phi(\xi) - \phi(\zeta)| = |\phi_1(\xi) - \phi_1(\zeta)| \vee |\phi_2(\xi) - \phi_2(\zeta)| \leq 2d_{J_1}(\xi,\zeta)$ and hence, ϕ is Lipschitz with Lipschitz constant 2.

Now, we claim that I' is of the following form:

$$I'(c,b) = \begin{cases} \lfloor \frac{c}{b} \rfloor b^\alpha + \big(c - \lfloor \frac{c}{b} \rfloor b\big)^\alpha & \text{if } 0 < b \leq c, \\ 0 & \text{if } b = c = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{4.5}$$

Note first that (4.5) is obvious except for the first case, and hence, we will assume that $0 < b \leq c$ from now on. Note also that $I'(c,b) = \inf\{I(\xi) : \xi \in \mathbb{D}_\infty^\uparrow,\ \phi(\xi) = (c,b)\}$ since $I(\xi) = \infty$ if $\xi \notin \mathbb{D}_\infty^\uparrow$. Set $\mathcal{C} \triangleq \{\xi \in \mathbb{D}_\infty^\uparrow, (c,b) = \phi(\xi)\}$ and remember that any $\xi \in \mathbb{D}_\infty^\uparrow$ admits the following representation:

$$\xi = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i,1]}, \tag{4.6}$$

where u_i's are distinct in $(0,1)$ and x_i's are non-negative and sorted in a decreasing order. Consider a step function $\xi_0 \in \mathcal{C}$, with $\lfloor \frac{c}{b} \rfloor$ jumps of size b and one jump of size $c - \lfloor \frac{c}{b} \rfloor b$, so that $\xi_0 = \sum_{i=1}^{\lfloor \frac{c}{b} \rfloor} b\mathbb{1}_{[u_i,1]} + (c - \lfloor \frac{c}{b} \rfloor)\mathbb{1}_{[u_{\lfloor \frac{c}{b} \rfloor + 1},1]}$. Clearly, $\phi(\xi_0) = (c,b)$ and $I(\xi_0) = \lfloor \frac{c}{b} \rfloor b^\alpha + \big(c - \lfloor \frac{c}{b} \rfloor b\big)^\alpha$. Since $\xi_0 \in \mathcal{C}$, the infimum of I over $\mathcal{C}$ should be at most $I(\xi_0)$ i.e., $I(\xi_0) \geq I'(c,b)$.

To prove that ξ_0 is the minimizer of I over $\mathcal{C}$, we will show that $I(\xi) \geq I(\xi_0)$ for any $\xi = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i,1]} \in \mathcal{C}$ by constructing ξ' such that $I(\xi) \geq I(\xi')$ while $I(\xi') = I(\xi_0)$. There has to be an integer k such that $x'_k \triangleq \sum_{i=k}^\infty x_i \leq b$. Let $\xi^1 \triangleq \sum_{i=1}^k x_i^1 \mathbb{1}_{[u_i,1]}$ where x_i^1 is the i^{th} largest element of $\{x_1, \ldots, x_{k-1}, x'_k\}$. Then, $\xi^1 \in \mathcal{C}$ and $I(\xi^1) \leq I(\xi)$ due to the sub-additivity of $x \mapsto x^\alpha$. Now, given $\xi^j = \sum_{i=1}^k x_i^j \mathbb{1}_{[u_i,1]}$, we construct ξ^{j+1} as follows. Find the first l such that $x_l^j < b$. If $x_l^j = 0$ or $x_{l+1}^j = 0$, set $\xi^{j+1} \triangleq \xi^j$. Otherwise, find the first m such that $x_{m+1}^j = 0$ and merge the l^{th} jump and the m^{th} jump. More specifically, set $x_l^{j+1} \triangleq x_l^j + x_m^j \wedge (b - x_l^j)$, $x_m^{j+1} \triangleq x_m^j - x_m^j \wedge (b - x_l^j)$, $x_i^{j+1} \triangleq x_i^j$ for $i \neq l, m$, and $\xi^{j+1} \triangleq \sum_{i=1}^k x_i^{j+1} \mathbb{1}_{[u_i,1]}$. Note that $x_l^{j+1} + x_m^{j+1} = x_l^j + x_m^j$ while either $x_l^{j+1} = b$ or $x_m^{j+1} = 0$. That is, compared to ξ^j, ξ^{j+1} has either one less jump or one more jump with size b, while the total sum of the jump sizes and the maximum jump size remain the same. From this observation and the concavity of $x \mapsto x^\alpha$, it is straightforward to check that $I(\xi^{j+1}) \leq I(\xi^j)$. By iterating this procedure k times, we arrive at $\xi' \triangleq \xi^k$ such that all the jump sizes of ξ' are b, or there is only one jump whose size is not b. From this, we see that ξ^k has to coincide with ξ_0. We conclude that $I(\xi) \geq I(\xi^1) \geq \cdots \geq I(\xi^k) = I(\xi') = I(\xi_0)$, proving that ξ_0 is indeed a minimizer.

Now we check that $\Psi_{I'}(\gamma) \triangleq \{(c,b) : I'(c,b) \le \gamma\}$ is compact for each $\gamma \in [0,\infty)$ so that I' is a good rate function. It is clear that $\Psi_{I'}(\gamma)$ is bounded. To see that $\Psi_{I'}(\gamma)$ is closed, suppose that $(c_1, b_1) \notin \Psi_{I'}(\gamma)$. In case $0 < b_1 < c_1$, note that I' can be written as $I'(c,b) = b^\alpha\Big\{\big(c/b - \lfloor c/b \rfloor\big)^\alpha + \lfloor c/b \rfloor\Big\}$, from which it is easy to see that I' is continuous at such (c_1, b_1)'s. Therefore, one can find an open ball around (c_1, b_1) in such a way that it doesn't intersect with $\Psi_{I'}(\gamma)$. By considering the cases $c_1 < b_1$, $b_1 = 0$, $b_1 = c_1$ separately, the rest of the cases are straightforward to check. We thus conclude that I' is a good rate function. Now we can apply Corollary 3.1. Note that

$$\inf_{(x,y)\in[c,\infty)\times[0,b]} I'(x,y) = \inf_{(x,y)\in(c,\infty)\times[0,b)} I'(x,y) = I'(c,b),$$

and hence,

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\sup_{t\in[0,1]} \bar{X}_n(t) \ge c, \sup_{t\in[0,1]} \big|\bar{X}_n(t) - \bar{X}_n(t-)\big| \le b\Big)}{L(n)n^\alpha} = \begin{cases} \left\lfloor \frac{c}{b} \right\rfloor b^\alpha + \left(c - \left\lfloor \frac{c}{b} \right\rfloor b\right)^\alpha & \text{if } 0 < b \le c, \\ 0 & \text{if } b = c = 0, \\ \infty & \text{otherwise.} \end{cases}$$

From the expression of the rate function, it can be inferred that the most likely way level c is reached is due to $\left\lfloor \frac{c}{b} \right\rfloor$ jumps of size b and one jump of size $\left(c - \left\lfloor \frac{c}{b} \right\rfloor b\right)$. If we compare this with the insights obtained from the case of truncated regularly varying tails in Rhee et al. (2016), we see that the total number of jumps is identical, but the size of the jumps are deterministic and non-identical, while in the regularly varying case, they are identically distributed with Pareto distribution.

5 Multiple Server Queue

Let Q denote the queue length process of the GI/GI/d queueing system with d servers, i.i.d. inter-arrival times with generic inter-arrival time A, and i.i.d. service times with generic service time S; we refer to Gamarnik and Goldberg (2013) for a detailed model description. Our goal in this section is to identify the limit behavior of $\mathbf{P}\big(Q(\gamma n) > n\big)$ as $n \to \infty$ in terms of the distributions of A and S, assuming $P(S > x) = \exp\{-L(x)x^\alpha\}, \alpha \in (0,1)$. Set $\lambda = 1/E[A]$ and $\mu = 1/E[S]$. Let M be the renewal process associated with A. That is,

$$M(t) = \inf\{s : A(s) > t\},$$

and $A(t) \triangleq A_1 + A_2 + \cdots + A_{\lfloor t \rfloor}$ where $A_1, A_2, \ldots$ are iid copies of A. Similarly, let $N^{(i)}$ be a renewal process associated with S for each $i = 1, \ldots, d$. Let $\bar{M}_n$ and $\bar{N}_n^{(i)}$ be scaled processes of M and $N^{(i)}$ in $\mathbb{D}[0,\gamma]$. That is, $\bar{M}_n(t) = M(nt)/n$ and $\bar{N}_n^{(i)}(t) = N^{(i)}(nt)/n$ for $t \ge 0$. Recall Theorem 3 of Gamarnik and Goldberg (2013), which implies

$$\mathbf{P}\big(Q(\gamma n) > n\big) \le \mathbf{P}\left(\sup_{0\le s\le\gamma} \Big(\bar{M}_n(s) - \sum_{i=1}^d \bar{N}_n^{(i)}(s)\Big) \ge 1\right) \tag{5.1}$$

for each $\gamma > 0$. It should be noted that in Gamarnik and Goldberg (2013), A_1 and $S_1^{(i)}$ are defined to have the residual distribution to make M and $N^{(i)}$ equilibrium renewal processes, but such assumptions are not necessary for (5.1) itself. In view of this, a natural way to proceed is to establish LDPs for $\bar{M}_n$ and $\bar{N}_n^{(i)}$'s. Note, however, that $\bar{M}_n$ and $\bar{N}_n^{(i)}$'s depend on random number of A_j's and $S_j^{(i)}$'s, and hence may depend on arbitrarily large number of A_j's and $S_j^{(i)}$'s with strictly positive probabilities. This does not exactly correspond to the large deviations framework we developed in the earlier sections. To accommodate such a context, we introduce the following maps. Fix $\gamma > 0$. Define for $\mu > 0$, $\Psi_\mu : \mathbb{D}[0, \gamma/\mu] \to \mathbb{D}[0, \gamma]$ be

$$\Psi_\mu(\xi)(t) \triangleq \sup_{s\in[0,t]} \xi(s),$$

and for each μ define a map $\Phi_\mu : \mathbb{D}[0, \gamma/\mu] \to \mathbb{D}[0, \gamma]$ as

$$\Phi_\mu(\xi)(t) \triangleq \varphi_\mu(\xi)(t) \wedge \psi_\mu(\xi)(t),$$

where

$$\varphi_\mu(\xi)(t) \triangleq \inf\{s \in [0, \gamma/\mu] : \xi(s) > t\} \qquad \text{and} \qquad \psi_\mu(\xi)(t) \triangleq \frac{1}{\mu}\Big(\gamma + \big[t - \Psi(\xi)(\gamma/\mu)\big]_+\Big).$$

Here we denoted $\max\{x, 0\}$ with $[x]_+$. In words, between the origin and the supremum of ξ, $\Phi_\mu(\xi)(s)$ is the first passage time of ξ crossing the level s; from there to the final point γ, $\Phi_\mu(\xi)$ increases linearly from γ/μ at rate $1/\mu$ (instead of jumping to ∞ and staying there). Define $\bar{A}_n \in \mathbb{D}[0, \gamma/\mathbf{E}A]$ as $\bar{A}_n(t) \triangleq A(nt)/n$ for $t \in [0, \gamma/\mathbf{E}A]$ and $\bar{S}_n^{(i)} \in \mathbb{D}[0, \gamma/\mathbf{E}S]$ as $\bar{S}_n^{(i)}(t) \triangleq S^{(i)}(nt)/n = \frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor} S_j^{(i)}$ for $t \in [0, \gamma/\mathbf{E}S]$. We will show that

- $\bar{A}_n$ and $\bar{S}_n^{(i)}$ satisfy certain LDPs (Proposition 5.1);
- $\Phi_{\mathbf{E}A}(\cdot)$ and $\Phi_{\mathbf{E}S}(\cdot)$ are continuous functions, and hence, $\Phi_{\mathbf{E}A}(\bar{A}_n)$ and $\Phi_{\mathbf{E}S}(\bar{S}_n^{(i)})$ satisfy the LDPs deduced by a contraction principle (Proposition 5.3, 5.4);
- $\bar{M}_n$ and $\bar{N}_n^{(i)}$ are equivalent to $\Phi_{\mathbf{E}A}(\bar{A}_n)$ and $\Phi_{\mathbf{E}S}(\bar{S}_n^{(i)})$, respectively, in terms of their large deviations (Proposition 5.2); so $\bar{M}_n$ and $\bar{N}_n^{(i)}$ satisfy the same LDPs (Proposition 5.4);
- and hence, the log asymptotics of $\mathbf{P}(Q(\gamma n) > n)$ can be bounded by the solution of a quasi-variational problem characterized by the rate functions of such LDPs (Proposition 5.5);

and then solve the quasi-variational problem to establish the asymptotic bound.

Let $\mathbb{D}_p^{\uparrow}[0, \gamma/\mu]$ be the subspace of $\mathbb{D}[0, \gamma/\mu]$ consisting of non-decreasing pure jump functions that assume non-negative values at the origin, and define $\zeta_\mu \in \mathbb{D}[0, \gamma/\mu]$ as $\zeta_\mu(t) \triangleq \mu t$. Let $\mathbb{D}^\mu[0, \gamma/\mu] \triangleq \zeta_\mu + \mathbb{D}_p^{\uparrow}[0, \gamma/\mu]$.

Proposition 5.1. *$\bar{A}_n$ satisfies the LDP on $\big(\mathbb{D}[0, \gamma/\mathbf{E}A], d_{M_1'}\big)$ with speed $L(n)n^\alpha$ and rate function*

$$I_0(\xi) = \begin{cases} 0 & \text{if } \xi = \zeta_{\mathbf{E}A}, \\ \infty & \text{otherwise,} \end{cases}$$

and $\bar{S}_n^{(i)}$ satisfies the LDP on $\big(\mathbb{D}[0, \gamma/\mathbf{E}S], d_{M_1'}\big)$ with speed $L(n)n^\alpha$ and the rate function

$$I_i(\xi) = \begin{cases} \sum_{t\in[0,\gamma/\mathbf{E}S]} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}^{\mathbf{E}S}[0, \gamma/\mathbf{E}S], \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Firstly, note that $\frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor} (A_j - \mathbf{E}A \cdot t)$ satisfies the LDP with the rate function

$$I_A(\xi) = \begin{cases} 0 & \text{if } \xi = 0, \\ \infty & \text{otherwise,} \end{cases}$$

whereas $\frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor}\left(S_j^{(i)} - \mathbf{E}S\cdot t\right)$ satisfies the LDP with the rate function

$$I_{S^{(i)}}(\xi) = \begin{cases} \sum_{t\in[0,\gamma/\mathbf{E}S]}(\xi(t)-\xi(t-))^\alpha & \text{if } \xi\in\mathbb{D}_p^{\uparrow}[0,\gamma/\mathbf{E}S], \\ \infty & \text{otherwise.}\end{cases}$$

Consider the map $\Upsilon_\mu : \left(\mathbb{D}[0,\gamma/\mu], \mathcal{T}_{M_1'}\right) \to \left(\mathbb{D}[0,\gamma/\mu], \mathcal{T}_{M_1'}\right)$ where $\Upsilon_\mu(\xi) \triangleq \xi + \zeta_\mu$. We prove that $\Upsilon_{\mathbf{E}A}$ is a continuous function w.r.t. the M_1' topology. Suppose that, $\xi_n \to \xi$ in $\mathbb{D}[0,\gamma/\mathbf{E}A]$ w.r.t. the M_1' topology. As a result, there exist parameterizations $(u_n(s), t_n(s))$ of ξ_n and $(u(s), t(s))$ of ξ so that,

$$\sup_{t\le\gamma/\mathbf{E}A}\{|u_n(s)-u(s)| + |t_n(s)-t(s)|\} \to 0 \qquad \text{as } n\to\infty.$$

This implies that $\max\{\sup_{t\le\gamma/\mathbf{E}A}|u_n(s)-u(s)|, \sup_{t\le\gamma/\mathbf{E}A}|t_n(s)-t(s)|\} \to 0$ as $n\to\infty$. Observe that, if $(u(s), t(s))$ is a parameterization for ξ, then $(u(s)+\mathbf{E}A\cdot t(s), t(s))$ is a parameterization for $\Upsilon_{\mathbf{E}A}(\xi)$. Consequently,

$$\begin{aligned} &\sup_{t\le\gamma/\mathbf{E}A}\{|u_n(s)+\mathbf{E}A\cdot t_n(s) - u(s) - \mathbf{E}A\cdot t(s)| + |t_n(s)-t(s)|\} \\ &\quad\le \sup_{t\le\gamma/\mathbf{E}A}\{|u_n(s)-u(s)|\} + \sup_{t\le\gamma/\mathbf{E}A}\{(\mathbf{E}A+1)|t_n(s)-t(s)|\} \to 0.\end{aligned}$$

Thus, $\Upsilon_{\mathbf{E}A}(\xi_n)\to\Upsilon_{\mathbf{E}A}(\xi)$ in the M_1' topology, proving that the map is continuous. The same argument holds for $\Upsilon_{\mathbf{E}S}$. By the contraction principle, $\bar{A}_n$ obeys the LDP with the rate function $I_0(\zeta)\triangleq\inf\{I_A(\xi) : \xi\in\mathbb{D}[0,\gamma/\mathbf{E}A],\, \zeta = \Upsilon_{\mathbf{E}A}(\xi)\}$. Observe that $I_A(\xi)=\infty$ for $\xi\neq 0$ therefore,

$$I_0(\zeta) = \begin{cases} 0 & \text{if } \zeta = \Upsilon_{\mathbf{E}A}(0) = \zeta_{\mathbf{E}A}, \\ \infty & \text{otherwise.}\end{cases}$$

Similarly, $I_{S^{(i)}}(\xi)=\infty$ when ξ is not a non-decreasing pure jump function. Note that $\xi\in\mathbb{D}_s^{\uparrow}$ implies that $\zeta = \Upsilon_{\mathbf{E}S}(\xi)$ belongs to $\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]$. Taking into account the form of $I_{S^{(i)}}$ and that $I_i(\zeta)\triangleq\inf\{I_{S^{(i)}}(\xi) : \xi\in\mathbb{D}[0,\gamma/\mathbf{E}S], \zeta = \Upsilon_{\mathbf{E}S}(\xi)\}$, we conclude

$$I_i(\zeta) = \begin{cases} \sum_{t\in[0,\gamma/\mathbf{E}S]}(\zeta(t)-\zeta(t-))^\alpha & \text{for } \zeta\in\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S], \\ \infty & \text{otherwise.}\end{cases}$$

□

Proposition 5.2. *$\bar{M}_n$ and $\Phi_{\mathbf{E}A}(\bar{A}_n)$ are exponentially equivalent in $\left(\mathbb{D}[0,\gamma],\, \mathcal{T}_{M_1'}\right)$. $\bar{N}_n^{(i)}$ and $\Phi_{\mathbf{E}S}(\bar{S}_n^{(i)})$ are exponentially equivalent in $\left(\mathbb{D}[0,\gamma],\, \mathcal{T}_{M_1'}\right)$ for each $i=1,\ldots,d$.*

Proof. We first claim that $d_{M_1'}(\bar{M}_n, \Phi_{\mathbf{E}A}(\bar{A}_n)) \ge \epsilon$ implies either

$$\gamma - \Psi(\bar{A}_n)(\gamma/\mathbf{E}A) \ge \frac{\mathbf{E}A}{2}\epsilon \qquad \text{or} \qquad \sup_{t\in[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A),\,\gamma]} \bar{M}_n(t) - \gamma/\mathbf{E}A \ge \epsilon/2.$$

To see this, suppose not. That is,

$$\gamma - \Psi(\bar{A}_n)(\gamma/\mathbf{E}A) < \frac{\mathbf{E}A}{2}\epsilon \qquad \text{and} \qquad \sup_{t\in[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A),\,\gamma]} \bar{M}_n(t) - \gamma/\mathbf{E}A < \epsilon/2. \tag{5.2}$$

By the construction of $\bar{A}_n$ and $\bar{M}_n$ we see that $\bar{M}_n(t) \geq \gamma/\mathbf{E}A$ for $t \geq \Psi(\bar{A}_n)(\gamma/\mathbf{E}A)$. Therefore, the second condition of (5.2) implies

$$\sup_{t\in[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A),\,\gamma]} |\bar{M}_n(t) - \gamma/\mathbf{E}A| < \epsilon/2.$$

On the other hand, since the slope of $\Phi_{\mathbf{E}A}(\bar{A}_n)$ is $1/\mathbf{E}A$ on $[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A), \gamma]$, the first condition of (5.2) implies that

$$\sup_{t\in[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A),\,\gamma]} |\Phi_{\mathbf{E}A}(\bar{A}_n)(t) - \gamma/\mathbf{E}A| < \epsilon/2,$$

and hence,

$$\sup_{t\in[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A),\,\gamma]} |\Phi_{\mathbf{E}A}(\bar{A}_n)(t) - \bar{M}_n(t)| < \epsilon. \tag{5.3}$$

Note also that by the construction of $\Phi_{\mathbf{E}A}$, $\bar{M}_n(t)$ and $\Phi_{\mathbf{E}A}(\bar{A}_n)(t)$ coincide on $t \in [0, \Psi(\bar{A}_n)(\gamma/\mathbf{E}A))$. From this along with (5.3), we see that

$$\sup_{t\in[0,\gamma]} |\Phi_{\mathbf{E}A}(\bar{A}_n)(t) - \bar{M}_n(t)| < \epsilon,$$

which implies that $d_{M_1'}(\Phi_{\mathbf{E}A}(\bar{A}_n)(t), \bar{M}_n(t)) < \epsilon$. The claim is proved. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(d_{M_1'}(\bar{M}_n, \Phi_{\mathbf{E}A}(\bar{A}_n)) \geq \epsilon\Big)}{L(n)n^\alpha} \\
&\leq \limsup_{n\to\infty} \frac{\log \Big\{\mathbf{P}\Big(\gamma - \Psi(\bar{A}_n)(\gamma/\mathbf{E}A) \geq \frac{\mathbf{E}A}{2}\epsilon\Big) + \mathbf{P}\Big(\sup_{t\in[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A),\gamma]} \bar{M}_n(t) - \gamma/\mathbf{E}A \geq \epsilon/2\Big)\Big\}}{L(n)n^\alpha} \\
&\leq \limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\gamma - \Psi(\bar{A}_n)(\gamma/\mathbf{E}A) \geq \frac{\mathbf{E}A}{2}\epsilon\Big)}{L(n)n^\alpha} \vee \limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\sup_{t\in[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A),\gamma]} \bar{M}_n(t) - \gamma/\mathbf{E}A \geq \epsilon/2\Big)}{L(n)n^\alpha},
\end{aligned}$$

and we are done for the exponential equivalence between $\bar{M}_n$ and $\Phi_{\mathbf{E}A}(\bar{A}_n)$ if we prove that

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\gamma - \Psi(\bar{A}_n)(\gamma/\mathbf{E}A) \geq \frac{\mathbf{E}A}{2}\epsilon\Big)}{L(n)n^\alpha} = -\infty \tag{5.4}$$

and

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\sup_{t\in[\Psi(\bar{A}_n)(\gamma/\mathbf{E}A),\gamma]} \bar{M}_n(t) - \gamma/\mathbf{E}A \geq \epsilon/2\Big)}{L(n)n^\alpha} = -\infty. \tag{5.5}$$

For (5.4), note that $\Psi(\bar{A}_n)(\gamma/\mathbf{E}A) \leq \gamma - \mathbf{E}A\epsilon/2$ implies that $d_{M_1'}(\bar{A}_n, \zeta_{\mathbf{E}A}) \geq \mathbf{E}A\epsilon/2$, and hence,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(\gamma - \Psi(\bar{A}_n)(\gamma/\mathbf{E}A) \geq \frac{\mathbf{E}A}{2}\epsilon\Big)}{L(n)n^\alpha} &\leq \limsup_{n\to\infty} \frac{\log \mathbf{P}\Big(d_{M_1'}(\bar{A}_n, \zeta_{\mathbf{E}A}) \geq \frac{\mathbf{E}A}{2}\epsilon\Big)}{L(n)n^\alpha} \\
&\leq -\inf_{\xi\in B_{M_1'}(\zeta_{\mathbf{E}A};\,\mathbf{E}A/2)^c} I_0(\xi) \leq -\infty,
\end{aligned}$$

where the second inequality is due to the LDP upper bound for $\bar{A}_n$ in Proposition 5.1. For (5.5), we arrive at the same conclusion by considering the LDP for $A(n\cdot)/n$ on $\mathbb{D}[0, \gamma/\mathbf{E}S + \epsilon/2]$. This concludes the proof for the exponential equivalence between $\bar{M}_n$ and $\Phi_{\mathbf{E}A}(\bar{A}_n)$. The exponential equivalence between $\bar{N}_n^{(i)}$ and $\Phi_{\mathbf{E}S}(\bar{S}_n^{(i)})$ is essentially identical with slight differences, and hence, omitted. □

Let $\mathcal{D}_{\Phi_\mu} \triangleq \{\xi \in \mathbb{D}[0,\gamma/\mu] : \Phi_\mu(\xi)(\gamma) - \Phi_\mu(\xi)(\gamma-) > 0\}$.

Proposition 5.3. *For each $\mu \in \mathbb{R}$, $\Phi_\mu : (\mathbb{D}[0,\gamma/\mu], \mathcal{T}_{M_1'}) \to (\mathbb{D}[0,\gamma], \mathcal{T}_{M_1'})$ is continuous on $\mathcal{D}^{\mathsf{c}}_{\Phi_\mu}$.*

Proof. Note that $\Phi_\mu = \Phi_\mu \circ \Psi$ and Ψ is continuous, so we only need to check the continuity of Φ_μ over the range of Ψ, in particular, non-decreasing functions. Let ξ be a non-decreasing function in $\mathbb{D}[0,\gamma/\mu]$. We consider two cases separately: $\Phi_\mu(\xi)(\gamma) > \gamma/\mu$ and $\Phi_\mu(\xi)(\gamma) \leq \gamma/\mu$.

We start with the case $\Phi_\mu(\xi)(\gamma) > \gamma/\mu$. Pick $\epsilon > 0$ such that $\Phi_\mu(\xi) > \gamma/\mu + 2\epsilon$ and $\xi(\gamma/\mu) + 2\epsilon < \gamma$. For such an ϵ, it is straightforward to check that $d_{M_1'}(\zeta,\xi) < \epsilon$ implies $\Phi_\mu(\zeta)(\gamma) > \mu/\gamma$ and ζ never exceeds γ on $[0,\gamma/\mu]$. Therefore, the parametrizations of $\Phi_\mu(\xi)$ and $\Phi_\mu(\zeta)$ consist of the parametrizations—with the roles of space and time interchanged—of the original ξ and ζ concatenated with the linear part coming from ψ_μ. More specifically, suppose that $(x,t) \in \Gamma(\xi)$ and $(y,r) \in \Gamma(\zeta)$ are parametrizations of ξ and ζ. If we define on $s \in [0,1]$,

$$x'(s) \triangleq \begin{cases} t(2s) & \text{if } s \leq 1/2 \\ \frac{1}{\mu}\big(t'(s) - \Psi(\xi)(\gamma/\mu) + \gamma\big) & \text{if } s > 1/2 \end{cases}, \quad t'(s) \triangleq \begin{cases} x(2s) & \text{if } s \leq 1/2 \\ \big(\gamma - \Psi(\xi)(\gamma/\mu)\big)(2s-1) + \Psi(\xi)(\gamma/\mu) & \text{if } s > 1/2 \end{cases}$$

and

$$y'(s) \triangleq \begin{cases} r(2s) & \text{if } s \leq 1/2 \\ \frac{1}{\mu}\big(r'(s) - \Psi(\zeta)(\gamma/\mu) + \gamma\big) & \text{if } s > 1/2 \end{cases}, \quad r'(s) \triangleq \begin{cases} y(2s) & \text{if } s \leq 1/2 \\ \big(\gamma - \Psi(\zeta)(\gamma/\mu)\big)(2s-1) + \Psi(\zeta)(\gamma/\mu) & \text{if } s > 1/2 \end{cases},$$

then $(x',t') \in \Gamma(\Phi_\mu(\xi))$, $(y',r') \in \Gamma(\Phi_\mu(\zeta))$. Noting that

$$\begin{aligned} &\|x'-y'\|_\infty + \|t'-r'\|_\infty \\ &= \sup_{s\in[0,1/2]} |t(2s)-r(2s)| \vee \sup_{s\in(1/2,1]} |x'(s)-y'(s)| + \sup_{s\in[0,1/2]} |x(2s)-y(2s)| \vee \sup_{s\in(1/2,1]} |t'(s)-r'(s)| \\ &= \|t-r\|_\infty \vee \mu^{-1}|\Psi(\zeta)(\gamma/\mu) - \Psi(\xi)(\gamma/\mu)| + \|x-y\|_\infty \vee |\Psi(\zeta)(\gamma/\mu) - \Psi(\xi)(\gamma/\mu)| \\ &\leq \mu^{-1}\|t-r\|_\infty \vee \|x-y\|_\infty + \|x-y\|_\infty \leq (1+\mu^{-1})(\|x-y\|_\infty + \|t-r\|_\infty), \end{aligned}$$

and taking the infimum over all possible parametrizations, we conclude that $d_{M_1'}(\Phi_\mu(\xi),\Phi_\mu(\zeta)) \leq (1+\mu^{-1})d_{M_1'}(\xi,\zeta) \leq (1+\mu^{-1})\epsilon$, and hence, Φ_μ is continuous at ξ.

Turning to the case $\Phi_\mu(\xi)(\gamma) \leq \gamma/\mu$, let $\epsilon > 0$ be given. Due to the assumption that $\Phi_\mu(\xi)$ is continuous at γ, there has to be a $\delta > 0$ such that $\varphi_\mu(\xi)(\gamma) + \epsilon < \varphi_\mu(\xi)(\gamma-\delta) \leq \varphi_\mu(\xi)(\gamma+\delta) \leq \varphi_\mu(\xi)(\gamma) + \epsilon$. We will prove that if $d_{M_1'}(\xi,\zeta) < \delta \wedge \epsilon$, then $d_{M_1'}(\Phi_\mu(\xi),\Phi_\mu(\zeta)) \leq 8\epsilon$. Since the case where $\Phi_\mu(\zeta)(\gamma) \geq \mu/\gamma$ is similar to the above argument, we focus on the case $\Phi_\mu(\zeta)(\gamma) < \mu/\gamma$; that is, ζ also crosses level γ before γ/μ. Let $(x,t) \in \Gamma(\xi)$ and $(y,r) \in \Gamma(\zeta)$ be such that $\|x-y\|_\infty + \|t-r\|_\infty < \delta$. Let $s_x \triangleq \inf\{s \geq 0 : x(s) > \gamma\}$ and $s_y \triangleq \inf\{s \geq 0 : y(s) > \gamma\}$. Then it is straightforward to check $t(s_x) = \varphi_\mu(\xi)(\gamma)$ and $r(s_y) = \varphi_\mu(\zeta)(\gamma)$. Of course, $x(s_x) = \gamma$ and $y(s_y) = \gamma$. If we set $x'(s) \triangleq t(s \wedge s_x)$, $t'(s) \triangleq x(s \wedge s_x)$, and $y'(s) \triangleq r(s \wedge s_y)$, $r'(s) \triangleq y(s \wedge s_y)$, then

$$\begin{aligned} \|x'-y'\|_\infty &\leq \|t-r\|_\infty + \sup_{s\in[s_x\wedge s_y, s_x\vee s_y]} \big\{|t(s_x)-r(s)| \vee |t(s)-r(s_y)|\big\} \\ &\leq \|t-r\|_\infty + \sup_{s\in[s_x\wedge s_y, s_x\vee s_y]} \big\{\big(|t(s_x)-t(s)| + |t(s)-r(s)|\big) \vee \big(|t(s)-t(s_y)| + |t(s_y)-r(s_y)|\big)\big\} \\ &\leq \|t-r\|_\infty + \big(|t(s_x)-t(s_y)| + \|t-r\|_\infty\big) \vee \big(|t(s_y)-t(s_x)| + \|t-r\|_\infty\big) \\ &\leq 2\|t-r\|_\infty + 2|t(s_x)-t(s_y)|. \end{aligned}$$

Now we argue that $t(s_x) - \epsilon \leq t(s_y) \leq t(s_x) + \epsilon$. To see this, note first that $x(s_y) < x(s_x) + \delta = \gamma + \delta$, and hence,

$$t(s_y) \leq \varphi_\mu(\xi)(x(s_y)) \leq \varphi_\mu(\xi)(\gamma+\delta) \leq \varphi_\mu(\xi)(\gamma) + \epsilon = t(s_x) + \epsilon.$$

On the other hand,

$$t(s_x) - \epsilon = \varphi_\mu(\xi)(\gamma) - \epsilon \le \varphi_\mu(\gamma - \delta) \le t(s_y),$$

where the last inequality is from $\xi(t(s_y)) \ge x(s_y) > x(s_x) - \delta = \gamma - \delta$ and the definition of φ_μ. Therefore, $\|x' - y'\|_\infty \le 2\delta + 2\epsilon < 4\epsilon$. Now we are left with showing that $\|t' - r'\|_\infty$ can be bounded in terms of ϵ.

$$\begin{aligned}
\|t' - r'\|_\infty &\le \|x - y\|_\infty + \sup_{s \in [s_x \wedge s_y, s_x \vee s_y]} \{|x(s_x) - y(s)| \vee |x(s) - y(s_y)|\} \\
&\le \|x - y\|_\infty + \sup_{s \in [s_x \wedge s_y, s_x \vee s_y]} \big\{ \big(|x(s_x) - x(s)| + |x(s) - y(s)|\big) \vee \big(|x(s) - x(s_y)| + |x(s_y) - y(s_y)|\big) \big\} \\
&\le \|x - y\|_\infty + \big(|t(s_x) - t(s_y)| + \|x - y\|_\infty\big) \vee \big(|x(s_x) - x(s_y)| + \|x - y\|_\infty\big) \\
&\le 2\|x - y\|_\infty + 2|x(s_x) - x(s_y)| = 2\|x - y\|_\infty + 2|y(s_y) - x(s_y)| \le 4\|x - y\|_\infty < 4\epsilon.
\end{aligned}$$

Therefore, $d_{M_1'}\big(\Phi_\mu(\xi), \Phi_\mu(\zeta)\big) \le \|x' - y'\|_\infty + \|t' - r'\|_\infty < 8\epsilon$. □

Let $\check{\mathbb{C}}^\mu[0,\gamma] \triangleq \{\zeta \in \mathbb{C}[0,\gamma] : \zeta = \varphi_\mu(\xi)$ for some $\xi \in \mathbb{D}^\mu[0,\gamma/\mu]\}$ where $\mathbb{C}[0,\gamma]$ is the subspace of $\mathbb{D}[0,\gamma]$ consisting of continuous paths, and define $\tau_s(\xi) \triangleq \max\Big\{0, \sup\{t \in [0,\gamma] : \xi(t) = s\} - \inf\{t \in [0,\gamma] : \xi(t) = s\}\Big\}$.

Proposition 5.4. *$\Phi_{\mathbf{E}A}(\bar{A}_n)$ and $\bar{M}_n$ satisfy the LDP with speed $L(n)n^\alpha$ and the rate function*

$$I_0'(\xi) \triangleq \begin{cases} 0 & \text{if } \xi(t) = t/\mathbf{E}A, \\ \infty & \text{otherwise,} \end{cases}$$

and for $i = 1, ..., d$, $\Phi_{\mathbf{E}S}(\bar{S}_n^{(i)})$ and $\bar{N}_n^{(i)}$ satisfy the LDP with speed $L(n)n^\alpha$ and the rate function

$$I_i'(\xi) \triangleq \begin{cases} \sum_{s \in [0,\gamma/\mathbf{E}S]} \tau_s(\xi)^\alpha & \text{if } \xi \in \check{\mathbb{C}}^{1/\mathbf{E}S}[0,\gamma], \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let $\hat{I}_0'(\zeta) \triangleq \inf\{I_0(\xi) : \xi \in \mathbb{D}[0,\gamma/\mathbf{E}A],\ \zeta = \Phi_{\mathbf{E}A}(\xi)\}$ and $\hat{I}_i'(\zeta) \triangleq \inf\{I_i(\xi) : \xi \in \mathbb{D}[0,\gamma/\mathbf{E}S],\ \zeta = \Phi_{\mathbf{E}S}(\xi)\}$ for $i = 1, \dots, d$. From Proposition 5.1, 5.2, 5.3, and the extended contraction principle (p.367 of Puhalskii and Whitt, 1997, Theorem 2.1 of Puhalskii, 1995), it is enough to show that $I_i' = \hat{I}_i'$ for $i = 0, \dots, d$.

Starting with $i = 0$, note that $I_0(\xi) = \infty$ if $\xi \ne \zeta_{\mathbf{E}A}$, and hence,

$$\hat{I}_0'(\zeta) = \inf\{I_0(\xi) : \xi \in \mathbb{D}[0,\gamma/\mathbf{E}A],\ \zeta = \Phi_{\mathbf{E}A}(\xi),\ \xi = \zeta_{\mathbf{E}A}\} = \begin{cases} 0 & \text{if } \zeta = \Phi_{\mathbf{E}A}(\zeta_{\mathbf{E}A}) \\ \infty & \text{o.w.} \end{cases}. \tag{5.6}$$

Also, since $\Psi(\zeta_{\mathbf{E}A})(\gamma/\mathbf{E}A) = \gamma$, $\psi_{\mathbf{E}A}(\zeta_{\mathbf{E}A})(t) = \frac{1}{\mathbf{E}A}(\gamma + [t-\gamma]_+) = \gamma/\mathbf{E}A$. Therefore, $\Phi_{\mathbf{E}A}(\zeta_{\mathbf{E}A})(t) = \inf\{s \in [0,\gamma/\mathbf{E}A] : s > t/\mathbf{E}A\} \wedge (\gamma/\mathbf{E}A) = (t/\mathbf{E}A) \wedge (\gamma/\mathbf{E}A) = t/\mathbf{E}A$ for $t \in [0,\gamma]$. With (5.6), this implies $I_0' = \hat{I}_0'$.

Turning to $i = 1, \dots, d$, note first that since $I_i(\xi) = \infty$ for any $\xi \notin \mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]$,

$$\hat{I}_i'(\zeta) = \inf\{I_i(\xi) : \xi \in \mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S],\ \zeta = \Phi_{\mathbf{E}S}(\xi)\}.$$

Note also that $\Phi_{\mathbf{E}S}$ can be simplified on $\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]$: it is easy to check that if $\xi \in \mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]$, $\psi_{\mathbf{E}S}(\xi)(t) = \gamma/\mathbf{E}S$ and $\varphi_{\mathbf{E}S}(\xi)(t) \le \gamma/\mathbf{E}S$ for $t \in [0,\gamma]$. Therefore, $\Phi_{\mathbf{E}S}(\xi) = \varphi_{\mathbf{E}S}(\xi)$, and hence,

$$\hat{I}_i'(\zeta) = \inf\{I_i(\xi) : \xi \in \mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S],\ \zeta = \varphi_{\mathbf{E}S}(\xi)\}.$$

Now if we define $\varrho_{\mathbf{E}S} : \mathbb{D}[0,\gamma/\mathbf{E}S] \to \mathbb{D}[0,\gamma/\mathbf{E}S]$ as

$$\varrho_{\mathbf{E}S}(\xi)(t) \triangleq \begin{cases} \xi(t) & t \in [0, \varphi_{\mathbf{E}S}(\xi)(\gamma)) \\ \gamma + (t - \varphi_{\mathbf{E}S}(\xi)(\gamma))\mathbf{E}S & t \in [\varphi_{\mathbf{E}S}(\xi)(\gamma), \gamma/\mathbf{E}S] \end{cases},$$

then it is straightforward to check that $I_i(\xi) \geq I_i(\varrho_{\mathbf{E}S}(\xi))$ and $\varphi_{\mathbf{E}S}(\xi) = \varphi_{\mathbf{E}S}(\varrho_{\mathbf{E}S}(\xi))$ whenever $\xi \in \mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]$. Moreover, $\varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]) \subseteq \mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]$. From these observations, we see that

$$\hat{I}_i'(\zeta) = \inf\{I_i(\xi) : \xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]),\, \zeta = \varphi_{\mathbf{E}S}(\xi)\}. \tag{5.7}$$

Note that $\xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S])$ and $\zeta = \varphi_{\mathbf{E}S}(\xi)$ implies that $\zeta \in \check{\mathbb{C}}^{1/\mathbf{E}S}[0,\gamma]$. Therefore, in case $\zeta \notin \check{\mathbb{C}}^{1/\mathbf{E}S}[0,\gamma]$, no $\xi \in \mathbb{D}[0,\gamma/\mathbf{E}S]$ satisfies the two conditions simultaneously, and hence,

$$\hat{I}_i'(\zeta) = \inf \emptyset = \infty = I_i'(\zeta). \tag{5.8}$$

Now we prove that $\hat{I}_i'(\zeta) = I_i'(\zeta)$ for $\zeta \in \check{\mathbb{C}}^{1/\mathbf{E}S}[0,\gamma]$. We claim that if $\xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S])$,

$$\tau_s(\varphi_{\mathbf{E}S}(\xi)) = \xi(s) - \xi(s-)$$

for all $s \in [0,\gamma/\mathbf{E}S]$. The proof of this claim will be provided at the end of the proof of the current proposition. Using this claim,

$$\begin{aligned}
\hat{I}_i(\zeta) &= \inf\left\{\textstyle\sum_{s\in[0,\gamma/\mathbf{E}S]}(\xi(s)-\xi(s-))^\alpha : \xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]),\, \zeta = \varphi_{\mathbf{E}S}(\xi)\right\} \\
&= \inf\left\{\textstyle\sum_{s\in[0,\gamma/\mathbf{E}S]}\tau_s(\varphi_{\mathbf{E}S}(\xi))^\alpha : \xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]),\, \zeta = \varphi_{\mathbf{E}S}(\xi)\right\} \\
&= \inf\left\{\textstyle\sum_{s\in[0,\gamma/\mathbf{E}S]}\tau_s(\zeta)^\alpha : \xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]),\, \zeta = \varphi_{\mathbf{E}S}(\xi)\right\}.
\end{aligned}$$

Note also that $\zeta \in \check{\mathbb{C}}^{1/\mathbf{E}S}[0,\gamma]$ implies the existence of ξ such that $\zeta = \varphi_{\mathbf{E}S}(\xi)$ and $\xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S])$. To see why, note that there exists $\xi' \in \mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]$ such that $\zeta = \varphi_{\mathbf{E}S}(\xi')$ due to the definition of $\check{\mathbb{C}}^{1/\mathbf{E}S}[0,\gamma]$. Let $\xi \triangleq \varrho_{\mathbf{E}S}(\xi')$. Then, $\zeta = \varphi_{\mathbf{E}S}(\xi)$ and $\xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S])$. From this observation, we see that

$$\left\{\textstyle\sum_{s\in[0,\gamma/\mathbf{E}S]}\tau_s(\zeta)^\alpha : \xi \in \varrho_{\mathbf{E}S}(\mathbb{D}^{\mathbf{E}S}[0,\gamma/\mathbf{E}S]),\, \zeta = \varphi_{\mathbf{E}S}(\xi)\right\} = \left\{\textstyle\sum_{s\in[0,\gamma/\mathbf{E}S]}\tau_s(\zeta)^\alpha\right\},$$

and hence,

$$\hat{I}_i(\zeta) = \sum_{s\in[0,\gamma/\mathbf{E}S]}\tau_s(\zeta)^\alpha = I_i'(\zeta) \tag{5.9}$$

for $\zeta \in \check{\mathbb{C}}^{1/\mathbf{E}S}[0,\gamma]$. From (5.8) and (5.9), we conclude that $I_i' = \hat{I}_i$ for $i = 1,\ldots,d$.

Now we are done if we prove the claim. We consider the cases $s > \varphi_{\mathbf{E}S}(\xi)(\gamma)$ and $s \leq \varphi_{\mathbf{E}S}(\xi)(\gamma)$ separately. First, suppose that $s > \varphi_{\mathbf{E}S}(\xi)(\gamma)$. Since $\varphi_{\mathbf{E}S}(\xi)$ is non-decreasing, this means that $\varphi_{\mathbf{E}S}(\xi)(t) < s$ for all $t \in [0,\gamma]$, and hence, $\{t \in [0,\gamma] : \varphi_{\mathbf{E}S}(t) = s\} = \emptyset$. Therefore,

$$\tau_s(\varphi_{\mathbf{E}S}(\xi)) = 0 \vee \big(\sup\{t\in[0,\gamma] : \varphi_{\mathbf{E}S}(t) = s\} - \inf\{t\in[0,\gamma] : \varphi_{\mathbf{E}S}(t) = s\}\big) = 0 \vee (-\infty - \infty) = 0.$$

On the other hand, since ξ is continuous on $[\varphi_{\mathbf{E}S}(\xi)(\gamma), \gamma/\mathbf{E}S]$ by its construction,

$$\xi(s) - \xi(s-) = 0.$$

Therefore,

$$\tau_s(\varphi_{\mathbf{E}S}(\xi)) = 0 = \xi(s) - \xi(s-)$$

for $s > \varphi_{\mathbf{E}S}(\xi)(\gamma)$.

Now we turn to the case $s \le \varphi_{\mathbf{E}S}(\xi)(\gamma)$. Since $\varphi_{\mathbf{E}S}(\xi)$ is continuous, this implies that there exists $u \in [0,\gamma]$ such that $\varphi_{\mathbf{E}S}(\xi)(u) = s$. From the definition of $\varphi_{\mathbf{E}S}(\xi)(u)$, it is straightforward to check that

$$u \in [\xi(s-), \xi(s)] \qquad \Longleftrightarrow \qquad s = \varphi_{\mathbf{E}S}(\xi)(u). \tag{5.10}$$

Note that $[\xi(s-), \xi(s)] \subseteq [0,\gamma]$ for $s \le \varphi_{\mathbf{E}S}(\xi)(\gamma)$ due to the construction of ξ. Therefore, the above equivalence (5.10) implies that $[\xi(s-), \xi(s)] = \{u \in [0,\gamma] : \varphi_{\mathbf{E}S}(\xi)(u) = s\}$, which in turn implies that $\xi(s-) = \inf\{u \in [0,\gamma] : \varphi_{\mathbf{E}S}(\xi)(u) = s\}$ and $\xi(s) = \sup\{u \in [0,\gamma] : \varphi_{\mathbf{E}S}(\xi)(u) = s\}$. We conclude that

$$\tau_s(\varphi_{\mathbf{E}S}(\xi)) = \xi(s) - \xi(s-)$$

for $s \le \varphi_{\mathbf{E}S}(\xi)(\gamma)$. □

Now we are ready to characterize an asymptotic bound for $\mathbf{P}\big(Q(\gamma n) > n\big)$. Recall that $\tau_s(\xi) \triangleq \max\Big\{0, \sup\{t \in [0,\gamma] : \xi(t) = s\} - \inf\{t \in [0,\gamma] : \xi(t) = s\}\Big\}$.

Proposition 5.5.

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\big(Q(\gamma n) > n\big) \le -c^*$$

where c^* is the solution of the following quasi-variational problem:

$$\begin{aligned} \inf_{\xi_1,\dots,\xi_d} \quad & \sum_{i=1}^{d} \sum_{s\in[0,\gamma/\mathbf{E}S]} \tau_s(\xi_i)^\alpha \\ \textit{subject to} \quad & \sup_{0\le s\le\gamma} \Big(\frac{s}{\mathbf{E}A} - \sum_{i=1}^{d} \xi_i(s)\Big) \ge 1; \\ & \xi_i \in \check{\mathbb{C}}^{1/\mathbf{E}S}[0,\gamma] \quad \textit{for } i = 1,\dots,d. \end{aligned} \tag{5.11}$$

Proof. Note first that for any $\epsilon > 0$,

$$\begin{aligned} &\mathbf{P}\left(\sup_{0\le s\le\gamma} \Big(\bar{M}_n(s) - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s)\Big) \ge 1\right) \\ &= \mathbf{P}\left(\sup_{0\le s\le\gamma} \Big(\bar{M}_n(s) - \frac{s}{\mathbf{E}A}\Big) + \sup_{0\le s\le\gamma} \Big(\frac{s}{\mathbf{E}A} - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s)\Big) \ge 1\right) \\ &\le \mathbf{P}\left(\sup_{0\le s\le\gamma} \Big(\bar{M}_n(s) - \frac{s}{\mathbf{E}A}\Big) \ge \epsilon\right) + \mathbf{P}\left(\sup_{0\le s\le\gamma} \Big(\frac{s}{\mathbf{E}A} - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s)\Big) \ge 1-\epsilon\right). \end{aligned}$$

Since $\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\sup_{0\le s\le\gamma} \Big(\bar{M}_n(s) - \frac{s}{\mathbf{E}A}\Big) \ge \epsilon\right) = -\infty$,

$$\limsup_{n\to\infty} \frac{\mathbf{P}\left(\sup_{0\le s\le\gamma} \Big(\bar{M}_n(s) - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s)\Big) \ge 1\right)}{L(n)n^\alpha} = \limsup_{n\to\infty} \frac{\mathbf{P}\left(\sup_{0\le s\le\gamma} \Big(\frac{s}{\mathbf{E}A} - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s)\Big) \ge 1-\epsilon\right)}{L(n)n^\alpha}.$$

To bound the right hand side, we proceed to deriving an LDP for $\sup_{0\le s\le\gamma}\left(\frac{s}{\mathbf{E}A}-\sum_{i=1}^{d}\bar N_n^{(i)}(s)\right)$. Due to Proposition 5.4, $(\bar N_n^{(i)},\dots,\bar N_n^{(d)})$ satisfy the LDP in $\prod_{i=1}^{d}\mathbb{D}[0,\gamma]$ (w.r.t. the d-fold product topology of $\mathcal{T}_{M_1'}$) with speed $L(n)n^{\alpha}$ and rate function

$$I'(\xi_1,\dots,\xi_d)\triangleq\sum_{i=1}^{d}I_i'(\xi_i).$$

Let $\mathbb{D}^{\uparrow}[0,\gamma]$ denote the subspace of $\mathbb{D}[0,\gamma]$ consisting of non-decreasing functions. Since $\bar N_n^{(i)}\in\mathbb{D}^{\uparrow}[0,\gamma]$ with probability 1 for each $i=1,\dots,d$, we can apply Lemma 4.1.5 (b) of Dembo and Zeitouni (2010) to deduce the same LDP for $(\bar N_n^{(i)},\dots,\bar N_n^{(d)})$ in $\prod_{i=1}^{d}\mathbb{D}^{\uparrow}[0,\gamma]$. If we define $f:\prod_{i=1}^{d}\mathbb{D}^{\uparrow}[0,\gamma]\to\mathbb{D}[0,\gamma]$ as

$$f(\bar N_n^{(1)},\dots,\bar N_n^{(d)})\triangleq\xi_{\mathbf{E}A}-\sum_{i=1}^{d}\bar N_n^{(i)}$$

where $\xi_{\mathbf{E}A}(t)\triangleq t/\mathbf{E}A$, then f is continuous since all the jumps are in one direction, and hence, we can apply the contraction principle to deduce the LDP for $\xi_{\mathbf{E}A}-\sum_{i=1}^{d}\bar N_n^{(i)}$, which is controlled by the rate function

$$I''(\zeta)\triangleq\inf_{\{(\xi_1,\dots,\xi_d):\,\xi_{\mathbf{E}A}-\sum_{i=1}^{d}\xi_i=\zeta\}}I'(\xi_0,\dots,\xi_d).$$

Now, applying the contraction principle again with the supremum functional to $\xi_{\mathbf{E}A}-\sum_{i=1}^{d}\bar N_n^{(i)}$, we get the LDP for $\sup_{t\in[0,\gamma]}\left(\xi_{\mathbf{E}A}-\sum_{i=1}^{d}\bar N_n^{(i)}(t)\right)$, which is controlled by the rate function

$$I'''(x)=\inf_{\{\zeta:\,\sup_{t\in[0,\gamma]}\zeta(t)=x\}}I''(\zeta)=\inf_{\left\{(\xi_1,\dots,\xi_d):\,\sup_{t\in[0,\gamma]}\left(\xi_{\mathbf{E}A}(t)-\sum_{i=1}^{d}\xi_i(t)\right)=x\right\}}I'(\xi_0,\dots,\xi_d).$$

The conclusion of the proposition follows from considering the upper bound of this LDP for the closed set $[1,\infty)$ and taking $\epsilon\to0$. □

To show that the large deviations upper bound is tight, and to obtain more insight in the way the rare event $\{Q(\gamma n)>n\}$ occurs, we now simply the expression of c^* given in Proposition 5.5. To ease notation, we assume from now on that $E[S]=\mu^{-1}=1$.

Proposition 5.6. *If $\gamma<1/\lambda$, $c^*=\infty$. If $\gamma\ge1/\lambda$, c^* can be computed via*

$$\begin{aligned}&\min\sum_{i=1}^{d}x_i^{\alpha}\quad s.t.\\&l\,(s;x)=\lambda s-\sum_{i=1}^{d}(s-x_i)^{+}\ge1\ \text{for some } s\in[0,\gamma].\\&x_1,\dots,x_d\ge0\,,\end{aligned}\tag{5.12}$$

which in turn equals

$$\begin{aligned}&\min\{\inf_{0<k\le\lfloor\lambda\rfloor:\gamma<1/(\lambda-k)}\left\{(d-k)\,\gamma^{\alpha}+(1-\gamma\lambda+\gamma k)^{\alpha}\,(k-\lfloor\lambda\rfloor\wedge\lfloor\lambda-1/\gamma\rfloor)^{1-\alpha}\right\}\\&\min_{l=0}^{\lfloor\lambda\rfloor\wedge\lfloor\lambda-1/\gamma\rfloor}\left\{(d-l)\left(\frac{1}{\lambda-l}\right)^{\alpha}\right\}\}.\end{aligned}\tag{5.13}$$

Proof. We want to show that c^* is equal to

$$\begin{aligned}
&\inf \sum_{i=1}^{d} \sum_{s\in[0,\gamma]} \tau_s\left(\zeta_i\right)^{\alpha} \qquad (5.14)\\
&\text{s.t.}\\
&\sup_{0\le s\le\gamma}\left\{\lambda s-\sum_{i=1}^{d}\zeta_i\left(s\right)\right\}\ge 1\\
&\zeta_i=\varphi_\mu\left(\xi_i\right),\ \xi_i\in\mathbb{D}^{\mu}[0,\gamma/\mu] \text{ for } i\in 1,...,d\ .
\end{aligned}$$

After a simple transformation, we might assume that $\mu = 1$. For simplicity in the exposition we will assume the existence of an optimizer. The argument that we present can be carried out with $\bar{\varepsilon}$-optimizers. In the end, the representation that we will provide will show the existence of an optimizer. First, we will argue that without loss of generality we may assume that if $(\zeta_1, ..., \zeta_d)$ is an optimal solution then the corresponding functions $\xi_1, ..., \xi_d$ have at most one jump which occurs at time zero. To see this suppose that $(\zeta_1, ..., \zeta_d)$ is an optimal solution and consider the corresponding functions $(\xi_1, ..., \xi_d)$ such that $\zeta_i = \varphi_\mu(\xi_i)$. By feasibility, we must have that at least one of the ξ_i's exhibit at least one jump in $[0, \gamma]$. Assume that ξ_i exhibits two or more jumps and select two jump times, say $0 \le u_0 < u_1 \le \gamma$, with corresponding jump sizes x_0 and x_1, respectively. Let

$$\bar{\xi}_i\left(\cdot\right)=\xi_i\left(\cdot\right)-x_1\mathbb{I}_{[u_1,\gamma]}\left(\cdot\right)+x_1\mathbb{I}_{[u_0,\gamma]}\left(\cdot\right);$$

in simple words, $\bar{\xi}_i(\cdot)$ is obtained by merging the jump at time u_1 with the jump at time u_0. It is immediate (since $x_0, x_1 > 0$) that for each t

$$\bar{\xi}_i\left(t\right)\ge\xi_i\left(t\right)$$

and, therefore, letting $\bar{\zeta}_i = \varphi_\mu\left(\bar{\xi}_i\right)$ we obtain (directly from the definition of the functional $\bar{\zeta}_i$ as a generalized inverse) that for every s

$$\bar{\zeta}_i\left(s\right)\le\zeta_i\left(s\right).$$

Therefore, we conclude that the collection $\zeta_1, ..., \bar{\zeta}_i, ...\zeta_d$ is feasible. Moreover, since

$$\sum_{s\in[0,\gamma]}\tau_s\left(\bar{\zeta}_i\right)^{\alpha}=\sum_{s\in[0,\gamma]}\tau_s\left(\zeta_i\right)^{\alpha}+\left(x_0+x_1\right)^{\beta}-x_0^{\beta}-x_1^{\beta}$$

and, by strict concavity,

$$\left(x_0+x_1\right)^{\beta}<x_0^{\beta}+x_1^{\beta},$$

we conclude that $\zeta_1, ..., \bar{\zeta}_i, ..., \zeta_d$ improves the objective function, thus violating the optimality of $\zeta_1, ..., \zeta_d$. So, we may assume that $\xi_i(\cdot)$ has a single jump of size $x_i > 0$ at some time u_i and therefore

$$\zeta_i\left(s\right)=\min\left(s,u_i\right)+\left(s-x_i-u_i\right)^{+}. \qquad (5.15)$$

Now, define $t = \inf\{s \in [0, \gamma] : \lambda s - \sum_{i=1}^{d} \zeta_i(s) \ge 1\}$, then

$$\lambda t-\sum_{i=1}^{d}\zeta_i\left(t\right)=1 \qquad (5.16)$$

and we must have that $t \ge x_i + u_i$; otherwise, if $x_i + u_i > t$ then we might reduce the value of the objective function while preserving feasibility (this can be seen from the form of $\zeta_i(\cdot)$), thus contradicting optimality. Now, suppose that $u_i > 0$, choose $\varepsilon \in (0, \min(u_i, x_i))$ and define

$$\bar{\xi}_i\left(s\right)=\xi_i\left(s\right)-x_i\mathbb{I}_{[u_i,\gamma]}\left(s\right)+x_i\mathbb{I}_{[u_i-\varepsilon,\gamma]}\left(s\right).$$

In simple words, we just moved the first jump slightly earlier (by an amount ε). Once again, let $\bar{\zeta}_i = \varphi_\mu\left(\bar{\xi}_i\right)$, and we have that

$$\bar{\zeta}_i\left(s\right) = \min\left(s, u_i - \varepsilon\right) + \left(s - x_i - u_i + \varepsilon\right)^+ \leq \zeta_i\left(s\right).$$

Therefore, we preserve feasibility without altering the objective function. As a consequence, we may assume that $u_i = 0$ and using expression (5.15) we then obtain that (5.11) takes the form

$$\min \sum_{i=1}^{d} x_i^\alpha \quad \text{s.t.} \tag{5.17}$$
$$l\left(s; x\right) = \lambda s - \sum_{i=1}^{d}(s - x_i)^+ \geq b \text{ for some } s \in [0, \gamma],$$
$$x_1, ..., x_d \geq 0\ .$$

Let $x = (x_1, ..., x_d)$ be any optimal solution, we may assume without loss of generality that $0 \leq x_1 \leq ... \leq x_d$. We claim that x satisfies the following features. First, $x_d \leq \gamma$, this is immediate from the fact that we are minimizing over the x_i's and if $x_d > \gamma$ we can reduce the value of x_d without affecting the feasibility of x, thereby improving the value of (5.17). The same reasoning allows us to conclude that $\inf\{s : l\left(s; x\right) \geq 1\} = x_d$. Consequently, letting $x_i = a_1 + ... + a_i$, (5.17) is equivalent to

$$\min \sum_{i=1}^{m}\left(\sum_{j=1}^{i} a_j\right)^\alpha \quad \text{s.t.}$$
$$\lambda\left(a_1 + ... + a_d\right) - \sum_{i=1}^{d}(a_1 + ... + a_d - \sum_{j=1}^{i} a_1) \geq 1$$
$$a_1 + ... + a_d \leq \gamma\ ,\ a_1, ..., a_d \geq 0.$$

This problem can be simplified to

$$\min \sum_{i=1}^{m}\left(\sum_{j=1}^{i} a_j\right)^\alpha \quad \text{s.t.}$$
$$\lambda a_1 + \left(\lambda - 1\right) a_2 + ... + \left(\lambda - d + 1\right) a_d \geq 1$$
$$a_1 + ... + a_d \leq \gamma\ ,\ a_1, ..., a_d \geq 0.$$

In turn, we know that $0 < \lambda < d$, then it suffices to consider

$$\min \sum_{i=1}^{\lfloor\lambda\rfloor}\left(\sum_{j=1}^{i} a_j\right)^\alpha + \left(d - \lfloor\lambda\rfloor\right)\left(\sum_{j=1}^{\lfloor\lambda\rfloor+1} a_j\right)^\alpha \quad \text{s.t.} \tag{5.18}$$
$$\lambda a_1 + \left(\lambda - 1\right) a_2 + ... + \left(\lambda - \lfloor\lambda\rfloor\right) a_{\lfloor\lambda\rfloor+1} = 1 \tag{5.19}$$
$$a_1 + ... + a_{\lfloor\lambda\rfloor+1} \leq \gamma\ , \tag{5.20}$$
$$a_1, ..., a_{\lfloor\lambda\rfloor+1} \geq 0\ , \tag{5.21}$$

because $(\lambda - m) < 0$ implies $a_{\lambda-m+1} = 0$ (otherwise we can reduce the value of the objective function).

We first consider the case $\lambda > \lfloor\lambda\rfloor$. Moreover, observe that if $\gamma \geq 1/(\lambda - \lfloor\lambda\rfloor))$ then any solution satisfying (5.19) and (5.21) automatically satisfies (5.20), so we can ignore the constraint (5.20) if assume that $\gamma \geq 1/(\lambda - \lfloor\lambda\rfloor)$. If λ is an integer we will simply conclude that $a_{\lfloor\lambda\rfloor+1} = 0$ and if we only assume $\gamma > 1/\lambda$ we will need to evaluate certain extreme points, as we shall explain later.

Now, the objective function is clearly concave and lower bounded inside the feasible region, which in turn is a compact polyhedron. Therefore, the optimizer is achieved at some extreme point in the feasible region (see Rockafellar (1970)). Under our simplifying assumptions, we only need to characterize the extreme points of (5.19), (5.21), which are given by $a_i = 1/(\lambda - i + 1)$ for $i = 1, ..., \lfloor\lambda\rfloor + 1$.

So, the solution, assuming that $\gamma \geq 1/(\lambda - \lfloor\lambda\rfloor)$, is given by

$$\begin{aligned} & \min\{da_1^\alpha, (d-1)\, a_2^\alpha, ..., (d - \lfloor\lambda\rfloor)\, a_{\lfloor\lambda\rfloor+1}^\alpha\} \\ = & \min_{i=1}^{\lfloor\lambda\rfloor+1} \left\{ (d - i + 1) \left(\frac{1}{\lambda - i + 1} \right)^\alpha \right\}. \end{aligned}$$

In the general case, that is, assuming $\gamma > \lambda^{-1}$ and also allowing the possibility that $\lambda = \lfloor\lambda\rfloor$, our goal is to show that the additional extreme points which arise by considering the inclusion of (5.20) might potentially give rise to solutions in which large service requirements are not equal across all the servers. We wish to identify the extreme points of (5.19), (5.20), (5.21) which we represent as

$$\begin{aligned} & \lambda a_1 + (\lambda - 1)\, a_2 + ... + (\lambda - \lfloor\lambda\rfloor)\, a_{\lfloor\lambda\rfloor+1} = 1 \ , \\ & a_0 + a_1 + ... + a_{\lfloor\lambda\rfloor+1} = \gamma \ , \\ & a_0, a_1, ..., a_{\lfloor\lambda\rfloor+1} \geq 0 \ . \end{aligned}$$

Note the introduction of the slack variable $a_0 \geq 0$. From elementary results in polyhedral combinatorics, we know that extreme points correspond to basic feasible solutions. Choosing $a_{i+1} = 1/(\lambda - i)$ and $a_0 = \gamma - a_{i+1}$ recover basic solutions which correspond to the extreme points identified earlier, when we ignored (5.20). If $\lambda = \lfloor\lambda\rfloor$ we must have, as indicated earlier, that $a_{\lfloor\lambda\rfloor+1} = 0$; so we can safely assume that $\lambda - i > 0$. We observe that $\gamma \geq 1/(\lambda - i)$ implies that $a_{i+1} = 1/(\lambda - i)$ and $a_j = 0$ for $j \neq i + 1$ is a basic feasible solution for the full system (i.e. including (5.20)). Additional basic solutions (not necessarily feasible) are obtained by solving (assuming that $0 \leq l < k < \lambda$)

$$\begin{aligned} 1 &= (\lambda - k)\, a_{k+1} + (\lambda - l)\, a_{l+1}, \\ \gamma &= a_{k+1} + a_{l+1}. \end{aligned}$$

This system of equations always has a unique solution because the equations are linearly independent if $l \neq k$. The previous pair of equations imply that

$$\lambda\gamma - 1 = ka_{k+1} + la_{l+1}.$$

Therefore, we obtain the solution $(\bar{a}_{k+1}, \bar{a}_{l+1})$ is given by

$$\begin{aligned} (k - l)\, \bar{a}_{k+1} &= (\lambda - l)\, \gamma - 1, \\ (k - l)\, \bar{a}_{l+1} &= 1 - \gamma\, (\lambda - k)\, . \end{aligned}$$

So, for the solution to be both basic and feasible we must have that $1/(\lambda - l) \leq \gamma \leq 1/(\lambda - k)$ (with strict inequality holding on one side).

If we evaluate the solution $a_{k+1} = \bar{a}_{k+1}$, $a_{l+1} = \bar{a}_{l+1}$, $a_{i+1} = 0$ for $i \notin \{k, l\}$ in the objective function we obtain

$$\begin{aligned} & \sum_{i=1}^{\lfloor\lambda\rfloor} \left(\sum_{j=1}^{i} a_j \right)^\alpha + (d - \lfloor\lambda\rfloor) \left(\sum_{j=1}^{\lfloor\lambda\rfloor+1} a_j \right)^\alpha \\ = \ & \bar{a}_{l+1}^\alpha\, (k - l) + (\lfloor\lambda\rfloor - k)\, (\bar{a}_{k+1} + \bar{a}_{l+1})^\alpha + (d - \lfloor\lambda\rfloor)\, (\bar{a}_{k+1} + \bar{a}_{l+1})^\alpha \\ = \ & \bar{a}_{l+1}^\alpha\, (k - l) + (d - k)\, (\bar{a}_{k+1} + \bar{a}_{l+1})^\alpha\, . \end{aligned}$$

Note that in the case $\gamma = 1/(\lambda - k)$ we have that $a_{k+1} = 1/(\lambda - k)$ and $a_i = 0$ for $i \neq k+1$ is a feasible extreme point with better performance than the solution involving $\bar{a}_{k+1}$ and $\bar{a}_{l+1}$,

$$\bar{a}_{l+1}^{\alpha}(k-l) + (d-k)(\bar{a}_{k+1} + \bar{a}_{l+1})^{\alpha} > (d-k)\, a_{k+1}^{\alpha}.$$

Consequently, we may consider only cases $1/(\lambda - l) \leq \gamma < 1/(\lambda - k)$ and we conclude that the general solution is given by

$$\min\{ \min_{0<k\leq\lfloor\lambda\rfloor:\gamma<1/(\lambda-k)} \left\{ (d-k)\gamma^{\alpha} + (1-\gamma(\lambda-k))^{\alpha} \min_{0\leq l<\lfloor\lambda\rfloor:1/(\lambda-l)\leq\gamma} \left(\frac{1}{k-l}\right)^{\alpha}(k-l) \right\},$$
$$\min_{l=0}^{\lfloor\lambda\rfloor\wedge\lfloor\lambda-1/\gamma\rfloor} \left\{ (d-l)\left(\frac{1}{\lambda-l}\right)^{\alpha} \right\}\}.$$

Simplifying, we obtain (5.13). □

We conclude with some comments that are meant to provide some physical insight, and highlight differences with the case of regularly varying job sizes.

If $\gamma < 1/\lambda$, no finite number of large jobs suffice, and we conjecture that the large deviations behavior is driven by a combination of light and heavy tailed phenomena in which the light tailed dynamics involve pushing the arrival rate by exponential tilting to the critical value $1/\gamma$, followed by the heavy-tailed contribution evaluated as we explain in the following development.

If $\gamma > 1/\lambda$ the following features are contrasting with the case of regularly varying service-time tails:

1. The large deviations behavior is not driven by the smallest number of jumps which drives the queueing system to instability (i.e. $\lceil d - \lambda \rceil$). In other words, in the Weibull setting, it might be cheaper to block more servers.

2. The amount by which the servers are blocked may not be the same among all of the servers which are blocked.

To illustrate the first point, assume $\gamma > b/(\lambda - \lfloor\lambda\rfloor)$, in which case

$$\lfloor\lambda\rfloor \leq \lfloor\lambda - b/\gamma\rfloor,$$

and the optimal solution of c^* reduces to

$$\min_{l=0}^{\lfloor\lambda\rfloor} \left\{ (d-l)\left(\frac{b}{\lambda-l}\right)^{\alpha} \right\}.$$

Let us use l^* to denote an optimizer for the previous expression; intuitively, $d - l^*$ represents the optimal number of servers to be blocked (observe that $d - \lfloor\lambda\rfloor = \lceil d - \lambda \rceil$ corresponds to the number of servers blocked in the regularly varying case). Note that if we define

$$f(t) = (d-t)(\lambda-t)^{-\alpha},$$

for $t \in [0, \lfloor\lambda\rfloor]$, then the derivative $\dot{f}(\cdot)$ satisfies

$$\dot{f}(t) = \alpha(d-t)(\lambda-t)^{-\alpha-1} - (\lambda-t)^{-\alpha}.$$

Hence,

$$\dot{f}(t) < 0 \Longleftrightarrow t < \frac{(\lambda - \alpha d)}{(1-\alpha)}$$

and

$$\dot{f}(t) > 0 \Longleftrightarrow t > \frac{(\lambda - \alpha d)}{(1-\alpha)},$$

with $\dot{f}(t) = 0$ if and only if $t = (\lambda - \alpha d)/(1-\alpha)$. This observation allows to conclude that whenever $\gamma > b/(\lambda - \lfloor \lambda \rfloor)$ we can distinguish two cases. The first one occurs if

$$\lfloor \lambda \rfloor \leq \frac{(\lambda - \alpha d)}{(1-\alpha)},$$

in which case $l^* = \lfloor \lambda \rfloor$ (this case is qualitatively consistent with the way in which large deviations occur in the regularly varying case). On the other hand, if

$$\lfloor \lambda \rfloor > \frac{(\lambda - \alpha d)}{(1-\alpha)},$$

then we must have that l^*

$$l^* = \left\lfloor \frac{(\lambda - \alpha d)}{(1-\alpha)} \right\rfloor \text{ or } l^* = \left\lceil \frac{(\lambda - \alpha d)}{(1-\alpha)} \right\rceil,$$

this case is the one which we highlighted in feature i) in which we may obtain $d - l^* > \lceil d - \lambda \rceil$ and therefore more servers are blocked relative to the large deviations behavior observed in the regularly varying case. Still, however, the blocked servers are symmetric in the sense that they are treated in exactly the same way.

In contrast, the second feature indicates that the most likely path to overflow may be obtained by blocking not only a specific amount to drive the system to instability, but also by blocking the corresponding servers by different loads in the large deviations scaling. To appreciate this we must assume that $\lambda^{-1} < \gamma \leq 1/(\lambda - \lfloor \lambda \rfloor)$.

In this case, the contribution of the infimum in (5.13) becomes relevant. In order to see that we can obtain mixed solutions, it suffices to consider the case $d = 2$, and $1 < \lambda < 2$ and

$$1/\lambda < \gamma < 1/(\lambda - 1).$$

Moreover, select $\gamma = 1/(\lambda-1) - \delta$ and $\lambda = 2 - \delta^3$ for $\delta > 0$ sufficiently small, then

$$\gamma^\alpha + (1 - \gamma(\lambda - 1))^\alpha = 1 - \delta\alpha + \delta^\alpha + o\left(\delta^2\right) \leq 2^{1-\alpha},$$

concluding that

$$\gamma^\alpha + (1 - \gamma(\lambda - 1))^\alpha < 2\left(\frac{1}{\lambda}\right)^\alpha$$

for δ small enough and therefore we can have mixed solutions.

For example, consider the case $d = 2$, $\lambda = 1.49$, $\alpha = 0.1$ and $\gamma = \frac{1}{\lambda-1} - 0.1$. For these values, $\gamma_1^\alpha + (1 - \gamma_1(\lambda - 1))^\alpha < 2\left(\frac{1}{\lambda}\right)^\alpha$, and the most likely scenario leading to a large queue length is two big jobs arriving at the beginning and blocking both servers with different loads. On the other hand, if $\gamma = \frac{1}{\lambda-1}$ the most likely scenario is a single big job blocking one server. These two scenarios are illustrated in Figure 1.

We conclude this section with a sketch of the proof for the matching lower bound in case $\gamma > 1/\lambda$. Considering the obvious coupling between Q and $(M, N^{(1)}, \cdots, N^{(d)})$, one can see that $M(s) - \sum_{i=1}^d N^{(i)}(s)$ can be interpreted as (a lower bound of) the length of an imaginary queue at time s where the servers can start working on the jobs that have not arrived yet. Therefore, $\mathbf{P}(Q((a+s)n) > bn) \geq \mathbf{P}(Q((a+s)n) > bn | Q(a) = 0) \geq \mathbf{P}(\bar{M}_n(s) - \sum_{i=1}^d \bar{N}_n^{(i)}(s) > b)$ for any $a, b \geq 0$. Let s^* be the level crossing time of the

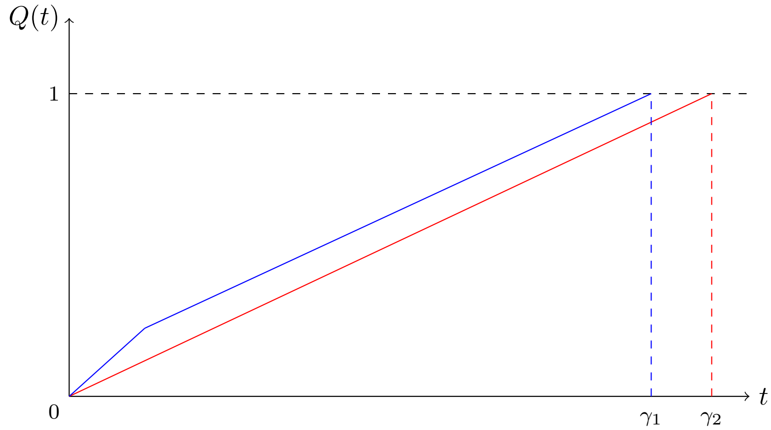

Figure 1: Most likely path for the queue build-up upto times $\gamma_1 = \frac{1}{\lambda-1} - 0.1$ and $\gamma_2 = \frac{1}{\lambda-1}$ where the number of servers is $d = 2$, the arrival rate is $\lambda = 1.49$, and the Weibull shape parameter of the service time is $\alpha = 0.1$.

optimal solution of (5.11). Then, for any $\epsilon > 0$,

$$\begin{aligned}\mathbf{P}(Q(\gamma n) > n) &\geq \mathbf{P}\Big(\bar{M}_n(s^*) - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s^*) > b\Big)\\ &\geq \mathbf{P}\Big(\bar{M}_n(s^*) - s^*/\mathbf{E}A > -\epsilon \ \text{ and } \ s^*/\mathbf{E}A - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s^*) > b+\epsilon\Big)\\ &\geq \mathbf{P}\Big(s^*/\mathbf{E}A - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s^*) > b+\epsilon\Big) - \mathbf{P}\Big(\bar{M}_n(s^*) - s^*/\mathbf{E}A \leq -\epsilon\Big)\end{aligned}$$

Since $\mathbf{P}(\bar{M}_n(s^*) - s^*/\mathbf{E}A \leq -\epsilon)$ decays exponentially fast w.r.t. n,

$$\liminf_{n\to\infty} \frac{\log \mathbf{P}(Q(\gamma n) > n)}{L(n)n^\alpha} \geq \liminf_{n\to\infty} \frac{\log \mathbf{P}(s^*/\mathbf{E}A - \sum_{i=1}^{d} \bar{N}_n^{(i)}(s^*) > b+\epsilon)}{L(n)n^\alpha} \geq -\inf_{(\xi_1,\ldots,\xi_d)\in A^\circ} I'(\xi_1,\ldots,\xi_d)$$

where $A = \{(\xi_1,\ldots,\xi_d) : s^*/\mathbf{E}A - \sum_{i=1}^{d} \xi(s^*) > b+\epsilon\}$. Note that the optimizer $(\xi_1^*,\ldots,\xi_d^*)$ of (5.11) satisfies $s^*/\mathbf{E}A - \sum_{i=1}^{d} \xi(s^*) \geq b$. Consider $(\xi_1',\ldots,\xi_d')$ obtained by increasing one of the job size of $(\xi_1^*,\ldots,\xi_d^*)$ by $\delta > 0$. One can always find a small enough such δ since $\gamma > 1/\lambda$. Note that there exists $\epsilon > 0$ such that $s'/\mathbf{E}A - \sum_{i=1}^{d} \xi(s') > b+\epsilon$. Therefore,

$$\liminf_{n\to\infty} \frac{\log \mathbf{P}(Q(\gamma n) > n)}{L(n)n^\alpha} \geq -I'(\xi_1',\ldots,\xi_d') \geq -c^* - \delta^\alpha$$

where the second inequality is from the subadditivity of $x \mapsto x^\alpha$. Since δ can be chosen arbitrarily small, letting $\delta \to 0$, we arrive at the matching lower bound.

6 Proofs

6.1 Lower semi-continuity of I and I^d

Recall the definition of I in (2.2) and I^d in (2.9).

Lemma 6.1. *I and I^d are lower semi-continuous, and hence, rate functions.*

Proof. We start with I. To show that the sub-level sets $\Psi_I(\gamma)$ are closed for each $\gamma < \infty$, let ξ be any given path that does not belong to $\Psi_I(\gamma)$. We will show that there exists an $\epsilon > 0$ such that $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \epsilon$. Note that $\Psi_I(\gamma)^c = (A \cap B \cap C \cap D)^c = (A^c) \cup (A \cap B^c) \cup (A \cap B \cap C^c) \cup (A \cap B \cap C \cap D^c)$ where

$$A = \{\xi \in \mathbb{D} : \xi(0) = 0 \text{ and } \xi(1) = \xi(1-)\}, \qquad B = \{\xi \in \mathbb{D} : \xi \text{ is non-decreasing}\},$$
$$C = \{\xi \in \mathbb{D} : \xi \text{ is a pure jump function}\}, \qquad D = \{\xi \in \mathbb{D} : \textstyle\sum_{t\in[0,1]}(\xi(t) - \xi(t-))^\alpha \leq \gamma\}.$$

For $\xi \in A^c$, we will show that $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \delta$ where $\delta = \frac{1}{2}\max\{|\xi(0)|, |\xi(1) - \xi(1-)|\}$. Suppose not so that there exists $\zeta \in \Psi_I(\gamma)$ such that $d_{J_1}(\xi, \zeta) < \delta$. Then $|\zeta(0)| \geq |\xi(0)| - 2\delta$ and $|\zeta(1) - \zeta(1-)| > |\xi(1) - \xi(1-)| - 2\delta$. That is, $\max\{|\zeta(0)|, |\zeta(1) - \zeta(1-)|\} > \max\{|\xi(0)| - 2\delta, |\xi(1) - \xi(1-)| - 2\delta\} = 0$. Therefore, $\zeta \in A^c$, and hence, $I(\zeta) = \infty$, which contradicts to that $\zeta \in \Psi_I(\gamma)$.

If $\xi \in A \cap B^c$, there are $T_s < T_t$ such that $c \triangleq \xi(T_s) - \xi(T_t) > 0$. We claim that $d_{J_1}(\xi, \zeta) \geq c$ if $\zeta \in \Psi_I(\gamma)$. Suppose that this is not the case and there exists $\zeta \in \Psi_I(\gamma)$ such that $d_{J_1}(\xi, \zeta) < c/2$. Let λ be a non-decreasing homeomorphism $\lambda : [0,1] \to [0,1]$ such that $\|\zeta \circ \lambda - \xi\|_\infty < c/2$, in particular, $\zeta \circ \lambda(T_s) > \xi(T_s) - c/2$ and $\zeta \circ \lambda(T_t) < \xi(T_t) + c/2$. Subtracting the latter inequality from the former, we get $\zeta \circ \lambda(T_s) - \zeta \circ \lambda(T_t) > \xi(T_s) - \xi(T_t) - c = 0$. That is, ζ is not non-decreasing, which is contradictory to the assumption $\zeta \in \Psi_I(\gamma)$. Therefore, the claim has to be the case.

If $\xi \in A \cap B \cap C^c$, there exists an interval $[T_s, T_t]$ so that ξ is continuous and $c \triangleq \xi(T_t) - \xi(T_s) > 0$. Pick δ small enough so that $(c - 2\delta)(2\delta)^{\alpha-1} > \gamma$. We will show that $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \delta$. Suppose that $\zeta \in \Psi_I(\gamma)$ and $d_{J_1}(\zeta, \xi) < \delta$, and let λ be a non-decreasing homeomorphism such that $\|\zeta \circ \lambda - \xi\|_\infty < \delta$. Note that this implies that each of the jump sizes of $\zeta \circ \lambda$ in $[T_s, T_t]$ has to be less than 2δ. On the other hand, $\zeta \circ \lambda(T_t) \geq \xi(T_t) - \delta$ and $\zeta \circ \lambda(T_s) \leq \xi(T_s) + \delta$, which in turn implies that $\zeta \circ \lambda(T_t) - \zeta \circ \lambda(T_s) \geq c - 2\delta$. Since $\zeta \circ \lambda$ is a non-decreasing pure jump function,

$$\begin{aligned}
c - 2\delta \leq \zeta \circ \lambda(T_t) - \zeta \circ \lambda(T_s) &= \sum_{t\in(T_s,T_t]} \big(\zeta \circ \lambda(t) - \zeta \circ \lambda(t-)\big) \\
&= \sum_{t\in(T_s,T_t]} \big(\zeta \circ \lambda(t) - \zeta \circ \lambda(t-)\big)^\alpha \big(\zeta \circ \lambda(t) - \zeta \circ \lambda(t-)\big)^{1-\alpha} \leq \sum_{t\in(T_s,T_t]} \big(\zeta \circ \lambda(t) - \zeta \circ \lambda(t-)\big)^\alpha (2\delta)^{1-\alpha}.
\end{aligned}$$

That is, $\sum_{t\in(T_s,T_t]} \big(\zeta \circ \lambda(t) - \zeta \circ \lambda(t-)\big)^\alpha \geq (2\delta)^{\alpha-1}(c - 2\delta) > \gamma$, which is contradictory to our assumption that $\zeta \in \Psi_I(\gamma)$. Therefore, $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \delta$.

Finally, let $\xi \in A \cap B \cap C \cap D^c$. This implies that ξ admits the following representation: $\xi = \sum_{i=1}^\infty x_i \mathbb{1}_{[u_i,1]}$ where u_i's are all distinct in $(0,1)$ and $\sum_{i=1}^\infty x_i^\alpha > \gamma$. Choose k large enough and δ small enough so that $\sum_{i=1}^k (x_i - 2\delta)^\alpha > \gamma$. We will show that $d_{J_1}(\xi, \Psi_I(\gamma)) \geq \delta$. Suppose that this is not the case. That is, there exists $\zeta \in \Psi_I(\gamma)$ so that $d_{J_1}(\xi, \zeta) < \delta$. Let λ be a non-decreasing homeomorphism such that $\|\zeta \circ \lambda - \xi\|_\infty < \delta$. Thus for each $i \in \{1, \ldots, k\}$, $\zeta \circ \lambda(u_i) - \zeta \circ \lambda(u_i-) \geq \xi(u_i) - \xi(u_i-) - 2\delta = x_i - 2\delta$, and hence,

$$I(\zeta) = \sum_{t\in[0,1]} (\zeta \circ \lambda(t_i) - \zeta \circ \lambda(t_i-))^\alpha \geq \sum_{i=1}^k (\zeta \circ \lambda(u_i) - \zeta \circ \lambda(u_i-)) \geq \sum_{i=1}^k (x_i - 2\delta)^\alpha > \gamma,$$

which contradicts to the assumption that $\zeta \in \Psi_I(\gamma)$. □

6.2 Proof of Proposition 2.1

Proof of Proposition 2.1. We start with the extended large deviation upper bound. For any measurable set A,

$$\begin{aligned}\mathbf{P}\left(X_n \in A\right) &= \mathbf{P}\left(X_n \in A,\, d(X_n, Y_n^k) \le \epsilon\right) + \mathbf{P}\left(X_n \in A,\, d(X_n, Y_n^k) > \epsilon\right) \\ &\le \underbrace{\mathbf{P}\big(Y_n^k \in A^\epsilon\big)}_{\triangleq(\mathrm{I})} + \underbrace{\mathbf{P}\big(d(X_n, Y_n^k) > \epsilon\big)}_{\triangleq(\mathrm{II})}.\end{aligned} \tag{6.1}$$

From the principle of the largest term and (i),

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{a_n} \le \max\left\{-\inf_{x\in A^\epsilon} I_k(x),\, \limsup_{n\to\infty} \frac{1}{a_n}\log \mathbf{P}\big(d(X_n, Y_n^k) > \epsilon\big)\right\}.$$

Now letting $k \to \infty$ and then $\epsilon \to 0$, (ii) and (iv) lead to

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\bar{X}_n \in A\right) \le -\lim_{\epsilon\to 0} \inf_{x\in A^\epsilon} I(x),$$

which is the upper bound of the extended LDP.

Turning to the lower bound, note that the lower bound is trivial if $\inf_{x\in A^\circ} I(x) = \infty$. Therefore, we focus on the case $\inf_{x\in A^\circ} I(x) < \infty$. Consider an arbitrary but fixed $\delta \in (0, 1)$. In view of (iii) and (iv), one can pick $\epsilon > 0$ and $k \ge 1$ in such a way that

$$-\inf_{x\in A^\circ} I(x) \le -\inf_{x\in A^{-\epsilon}} I_k(x) + \delta \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{\log \mathbf{P}\big(d(X_n, Y_n^k) > \epsilon\big)}{a_n} \le -\inf_{x\in A^\circ} I(x) - 1. \tag{6.2}$$

Hence

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}\big(d(X_n, Y_n^k) > \epsilon\big)}{a_n} \le -\inf_{x\in A^{-\epsilon}} I_k(x) + \delta - 1. \tag{6.3}$$

We first claim that $\frac{\mathbf{P}\left(d(X_n, Y_n^k) > \epsilon\right)}{\mathbf{P}\left(Y_n^k \in A^{-\epsilon}\right)} \to 0$ as $n \to \infty$. To prove the claim, we observe that

$$\begin{aligned}\frac{\mathbf{P}\big(d(X_n, Y_n^k) > \epsilon\big)}{\mathbf{P}\big(Y_n^k \in A^{-\epsilon}\big)} &= \frac{\exp\left(\log \mathbf{P}\big(d(X_n, Y_n^k) > \epsilon\big)\right)}{\exp\left(\log \mathbf{P}\big(Y_n^k \in A^{-\epsilon}\big)\right)} \\ &= \left\{\exp\left(\frac{\log \mathbf{P}\big(d(X_n, Y_n^k) > \epsilon\big)}{a_n} - \frac{\log \mathbf{P}\big(Y_n^k \in A^{-\epsilon}\big)}{a_n}\right)\right\}^{a_n}.\end{aligned} \tag{6.4}$$

From the lower bound of the LDP for Y_n^k,

$$\limsup_{n\to\infty} -\frac{\log \mathbf{P}\big(Y_n^k \in A^{-\epsilon}\big)}{a_n} \le \inf_{\xi\in A^{-\epsilon}} I_k(\xi).$$

This along with (6.3) implies that $\limsup_{n\to\infty} \exp\left(\frac{\log \mathbf{P}\left(d(X_n, Y_n^k) > \epsilon\right)}{a_n} - \frac{\log \mathbf{P}\left(Y_n^k \in A^{-\epsilon}\right)}{a_n}\right) \le e^{\delta - 1} < 1$, which in turn proves the claim in view of (6.4). Using the claim and the first inequality of (6.2),

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{a_n}\log\mathbf{P}\left(X_n\in A\right) &\geq \liminf_{n\to\infty}\frac{1}{a_n}\log\mathbf{P}\left(Y_n^k\in A^{-\epsilon}, d(X_n,Y_n^k)\leq\epsilon\right)\\ &\geq \liminf_{n\to\infty}\frac{1}{a_n}\log\Big(\mathbf{P}\left(Y_n^k\in A^{-\epsilon}\right)-\mathbf{P}(d(X_n,Y_n^k)>\epsilon)\Big)\\ &= \liminf_{n\to\infty}\frac{1}{a_n}\log\left(\mathbf{P}\big(Y_n^k\in A^{-\epsilon}\big)\left(1-\frac{\mathbf{P}\big(d(X_n,Y_n^k)>\epsilon\big)}{\mathbf{P}\big(Y_n^k\in A^{-\epsilon}\big)}\right)\right)\\ &= \liminf_{n\to\infty}\frac{1}{a_n}\log\mathbf{P}\left(Y_n^k\in A^{-\epsilon}\right)\geq-\inf_{x\in A^{-\epsilon}}I_k(x)\geq-\inf_{x\in A}I(x)-\delta.\end{aligned}$$

Since δ was arbitrary in $(0,1)$, the lower bound is proved by letting $\delta\to 0$. □

6.3 Proof of Lemma 2.1

We prove Lemma 2.1 in several steps. Before we proceed, we introduce some notation and recall a distributional representation of the compound Poisson processes Y_n. It is straightforward to check that

$$\int_{x\geq 1} xN([0,n\cdot]\times dx)\stackrel{\mathcal{D}}{=}\sum_{l=1}^{\tilde{N}_n}Q_n^{\leftarrow}(\Gamma_l)\mathbb{1}_{[U_l,1]}(\cdot),$$

where $\Gamma_l=E_1+E_2+...+E_l$; E_i's are i.i.d. and standard exponential random variables; U_l's are i.i.d. and uniform variables in $[0,1]$; $\tilde{N}_n=N_n\big([0,1]\times[1,\infty)\big)$; $N_n=\sum_{l=1}^{\infty}\delta_{(U_l,Q_n^{\leftarrow}(\Gamma_l))}$, where $\delta_{(x,y)}$ is the Dirac measure concentrated on (x,y); $Q_n(x)\triangleq n\nu[x,\infty)$, and $Q_n^{\leftarrow}(y)\triangleq\inf\{s>0: n\nu[s,\infty)<y\}$. It should be noted that $\tilde{N}_n$ is the number of Γ_l's such that $\Gamma_l\leq n\nu_1$, where $\nu_1\triangleq\nu[1,\infty)$, and hence, $\tilde{N}_n\sim$ Poisson$(n\nu_1)$. From this, we observe that $\bar{J}_n^k$ has another distributional representation:

$$\bar{J}_n^k\stackrel{\mathcal{D}}{=}\underbrace{\frac{1}{n}\sum_{i=1}^{k}Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}}_{\triangleq\hat{J}_n^{\leqslant k}}-\underbrace{\frac{1}{n}\mathbb{1}\{\tilde{N}_n<k\}\sum_{i=\tilde{N}_n+1}^{k}Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}}_{\triangleq\check{J}_n^{\leqslant k}}.$$

Roughly speaking, $(Q_n^{\leftarrow}(\Gamma_1)/n,\dots,Q_n^{\leftarrow}(\Gamma_k)/n)$ represents the k largest jump sizes of $\bar{Y}_n$, and $\hat{J}_n^{\leqslant k}$ can be regarded as the process obtained by keeping only k largest jumps of $\bar{Y}_n$ while disregarding the rest. Lemma 6.2 and Corollary 6.1 prove an LDP for $(Q_n^{\leftarrow}(\Gamma_1)/n,\dots,Q_n^{\leftarrow}(\Gamma_k)/n,U_1,\dots,U_k)$. Consequently, Lemma 6.3 yields a sample path LDP for $\hat{J}_n^{\leqslant k}$. Finally, Lemma 2.1 is proved by showing that $\bar{J}_n^k$ satisfies the same LDP as the one satisfied by $\hat{J}_n^{\leqslant k}$.

Lemma 6.2. $\Big(Q_n^{\leftarrow}(\Gamma_1)/n,Q_n^{\leftarrow}(\Gamma_2)/n,....,Q_n^{\leftarrow}(\Gamma_k)/n\Big)$ *satisfies a large deviation principle in* $\mathbb{R}_+^k$ *with normalization* $L(n)n^\alpha$*, and with good rate function*

$$\check{I}_k(x_1,...x_k)=\begin{cases}\sum_{i=1}^k x_i^\alpha & \text{if } x_1\geq x_2\geq\cdots\geq x_k\geq 0\\ \infty, & \text{o.w.}\end{cases}. \tag{6.5}$$

Proof. It is straightforward to check that $\check{I}_k$ is a good rate function. For each $f\in\mathcal{C}_b(\mathbb{R}_+^k)$, let

$$\Lambda_f^*\triangleq\lim_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\left(\mathbf{E}e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(\Gamma_1)/n,\,Q_n^{\leftarrow}(\Gamma_2)/n,\dots,\,Q_n^{\leftarrow}(\Gamma_k)/n\right)}\right). \tag{6.6}$$

Applying Bryc's inverse Varadhan lemma (see e.g. Theorem 4.4.13 of Dembo and Zeitouni, 2010), we can show that $(Q_n^{\leftarrow}(\Gamma_1)/n, \dots, Q_n^{\leftarrow}(\Gamma_k)/n)$ satisfies a large deviation principle with speed $L(n)n^\alpha$ and good rate function $\check{I}_k(x)$ if

$$\Lambda_f^* = \sup_{x\in\mathbb{R}_+^k} \{f(x) - \check{I}_k(x)\} \tag{6.7}$$

for every $f \in \mathcal{C}_b(\mathbb{R}_+^k)$.

To prove (6.7), fix $f \in C_b(\mathbb{R}_+^k)$ and let M be a constant such that $|f(x)| \le M$ for all $x \in \mathbb{R}_+^k$. We first claim that the supremum of $\Lambda_f \triangleq f - \check{I}_k$ is attained. Pick a constant R so that $R^\alpha > 2M$ and let $A_R \triangleq \{(x_1, \dots, x_k) \in \mathbb{R}_+^k : R \ge x_1 \ge \cdots \ge x_k\}$. Since Λ_f is continuous on A_R, which is compact, there exists a maximizer $\hat{x} \triangleq (\hat{x}_1, \dots, \hat{x}_k)$ of Λ_f on A_R. It turns out that $\hat{x}$ is a global maximizer of Λ_f. To see this, note that on A_R,

$$\sup_{x\in A_R} \{f(x) - \check{I}_k(x)\} \ge \inf_{x\in A_R} f(x) - \inf_{x\in A_R} \check{I}_k(x) = \inf_{x\in A_R} \{f(x)\} \ge -M.$$

while

$$\sup_{x\in\mathbb{R}_+^k\setminus A_R} \{f(x) - \check{I}_k(x)\} < \sup_{x\in\mathbb{R}_+^k\setminus A_R} \{f(x) - 2M\} \le -M$$

since $\check{I}_k(x_1, \dots, x_k) > 2M$ on $\mathbb{R}_+^k \setminus A_R$. Therefore, $\sup_{x\in\mathbb{R}_+^k} \Lambda_f(x) = \sup_{x\in A_R} \Lambda_f(x)$, showing that $\hat{x}$ is indeed a global maximizer. Now, it is enough to prove that

$$\Lambda_f(\hat{x}) \le \liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \Upsilon_f(n) \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \Upsilon_f(n) \le \Lambda_f(\hat{x}), \tag{6.8}$$

where

$$\Upsilon_f(n) \triangleq \int_{\mathbb{R}_+^k} e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(y_1)/n, \dots, Q_n^{\leftarrow}(y_1+\cdots+y_k)/n\right)} e^{-\sum_{i=1}^k y_i}\, dy_1 \dots dy_k.$$

We start with the lower bound—i.e., the first inequality of (6.8). Fix an arbitrary $\epsilon > 0$. Since Λ_f is continuous on $A_\infty \triangleq \{(x_1, \dots, x_k) \in \mathbb{R}_+^k : x_1 \ge \cdots \ge x_k\}$, there exists $\delta > 0$ such that $x \in B(\hat{x}; 2\sqrt{k}\delta) \cap A_\infty$ implies $\Lambda_f(x) \ge \Lambda_f(\hat{x}) - \epsilon$. Since $\prod_{j=1}^k [\hat{x}_j + \delta, \hat{x}_j + 2\delta] \subseteq B(\hat{x}; 2\sqrt{k}\delta)$ and $Q_n^{\leftarrow}(\cdot)$ is non-increasing, $Q_n^{\leftarrow}\big(\sum_{i=1}^j y_i\big)/n \in [\hat{x}_j + \delta, \hat{x}_j + 2\delta]$ for all $j = 1, \dots, k$ implies

$$\Lambda_f\left(Q_n^{\leftarrow}(y_1)/n, ..., Q_n^{\leftarrow}(y_1 + \cdots + y_k)/n\right) \ge \Lambda_f(\hat{x}) - \epsilon. \tag{6.9}$$

That is, if we define $D_n^j (= D_n^{y_1,\dots,y_{j-1}})$ as

$$D_n^j \triangleq \{y_j \in \mathbb{R}_+ : Q_n^{\leftarrow}\big(\textstyle\sum_{i=1}^j y_i\big)/n \in [\hat{x}_j + \delta, \hat{x}_j + 2\delta]\},$$

then (6.9) holds for $(y_1,\dots,y_k)$'s such that $y_j\in D_n^j$ for $j=1,\dots,k$. Therefore,

$$\begin{aligned}
\Upsilon_f(n) &= \int_{\mathbb{R}_+^k} e^{L(n)n^\alpha\Lambda_f\left(Q_n^\leftarrow(y_1)/n,\dots,Q_n^\leftarrow(y_1+\dots+y_k)/n\right)+L(n)\sum_{i=1}^k Q_n^\leftarrow(\sum_{j=1}^i y_j)^\alpha-\sum_{i=1}^k y_i}\,dy_1\dots dy_k\\
&\ge \int_{D_n^1}\cdots\int_{D_n^k} e^{L(n)n^\alpha\Lambda_f\left(Q_n^\leftarrow(y_1)/n,\dots,Q_n^\leftarrow(y_1+\dots+y_k)/n\right)+L(n)\sum_{i=1}^k Q_n^\leftarrow(\sum_{j=1}^i y_j)^\alpha-\sum_{i=1}^k y_i}dy_k\dots dy_1\\
&\ge \int_{D_n^1}\cdots\int_{D_n^k} e^{L(n)n^\alpha\left(\Lambda_f(\hat x_1,\dots,\hat x_k)-\epsilon\right)}e^{L(n)\sum_{i=1}^k Q_n^\leftarrow(\sum_{j=1}^i y_j)^\alpha-\sum_{i=1}^k y_i}dy_k\dots dy_1\\
&\ge \int_{D_n^1}\cdots\int_{D_n^k} e^{L(n)n^\alpha\left(\Lambda_f(\hat x_1,\dots,\hat x_k)-\epsilon\right)}e^{L(n)\sum_{i=1}^k\left(n(\hat x_i+\delta)\right)^\alpha-\sum_{i=1}^k y_i}dy_k\dots dy_1\\
&= \underbrace{e^{L(n)n^\alpha\left(\Lambda_f(\hat x_1,\dots,\hat x_k)-\epsilon\right)}}_{\triangleq(\mathrm{I})_n}\ \underbrace{e^{L(n)\sum_{i=1}^k\left(n(\hat x_i+\delta)\right)^\alpha}}_{\triangleq(\mathrm{II})_n}\underbrace{\int_{D_n^1}\cdots\int_{D_n^k} e^{-\sum_{i=1}^k y_i}dy_k\dots dy_1}_{\triangleq(\mathrm{III})_n}
\end{aligned}\tag{6.10}$$

where the first equality is obtained by adding and subtracting $L(n)\sum_{i=1}^k Q_n^\leftarrow(\sum_{j=1}^i y_j)^\alpha$ to the exponent of the integrand. Note that by the construction of D_n^j's,

$$Q_n\big(n(\hat x_j+2\delta)\big)\le y_1+\dots+y_j\le Q_n\big(n(\hat x_j+\delta)\big)$$

on the domain of the integral in $(\mathrm{III})_n$, and hence,

$$(\mathrm{III})_n\ge e^{-Q_n(n(\hat x_k+\delta))}\prod_{i=1}^k\Big(Q_n\big(n(\hat x_i+\delta)\big)-Q_n\big(n(\hat x_i+2\delta)\big)\Big).\tag{6.11}$$

Since $Q_n\big(n(\hat x_k+\delta)\big)\to 0$ and $L\left(n(\hat x_i+\delta)\right)n^\alpha(\hat x_i+\delta)^\alpha-L\left(n(\hat x_i+2\delta)\right)n^\alpha(\hat x_i+2\delta)^\alpha\to-\infty$ for each i,

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log(\mathrm{III})_n\\
&\ge \liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\Big(-Q_n\big(n(\hat x_k+\delta)\big)\Big)+\sum_{i=1}^k\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\Big(Q_n\big(n(\hat x_i+\delta)\big)-Q_n\big(n(\hat x_i+2\delta)\big)\Big)\\
&=\sum_{i=1}^k\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\Big(ne^{-L(n(\hat x_i+\delta))n^\alpha(\hat x_i+\delta)^\alpha}\Big(1-e^{L(n(\hat x_i+\delta))n^\alpha(\hat x_i+\delta)^\alpha-L(n(\hat x_i+2\delta))n^\alpha(\hat x_i+2\delta)^\alpha}\Big)\Big)\\
&=\sum_{i=1}^k\liminf_{n\to\infty}\left(\frac{-L\left(n(\hat x_i+\delta)\right)n^\alpha(\hat x_i+\delta)^\alpha}{L(n)n^\alpha}+\frac{\log\left(1-e^{L(n(\hat x_i+\delta))n^\alpha(\hat x_i+\delta)^\alpha-L(n(\hat x_i+2\delta))n^\alpha(\hat x_i+2\delta)^\alpha}\right)}{L(n)n^\alpha}\right)\\
&=-\sum_{i=1}^k(\hat x_i+\delta)^\alpha.
\end{aligned}\tag{6.12}$$

This along with

$$\liminf_{n\to\infty}\frac{1}{n^\alpha L(n)}\log(\mathrm{I})_n=\liminf_{n\to\infty}\frac{1}{n^\alpha L(n)}\log\left(e^{n^\alpha L(n)\left(\Lambda_f(\hat x_1,\dots,\hat x_k)-\epsilon\right)}\right)=\Lambda_f(\hat x_1,\dots,\hat x_k)-\epsilon$$

and

$$\liminf_{n\to\infty}\frac{1}{n^\alpha L(n)}\log(\mathrm{II})_n=\liminf_{n\to\infty}\frac{1}{n^\alpha L(n)}\log\left(e^{L(n)\sum_{i=1}^k(n(\hat x_i+\delta))^\alpha}\right)=\sum_{i=1}^k(\hat x_i+\delta)^\alpha,$$

we arrive atwhich implies

$$\Lambda_f(\hat{x}) - \epsilon \le \liminf_{n\to\infty} \frac{1}{L(n)n^\alpha}\Upsilon_f(n). \tag{6.13}$$

Letting $\epsilon \to 0$, we obtain the lower bound of (6.8).

Turning to the upper bound, consider

$$D_{R,n} \triangleq \{(y_1, y_2, \dots, y_k) : Q_n^{\leftarrow}(y_1)/n \le R\},$$

and decompose $\Upsilon_f(n)$ into two parts:

$$\begin{aligned}\Upsilon_f(n) = &\int_{D_{R,n}} e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(x_1)/n,\dots,Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^k x_i} dx_1\dots dx_k \\ &+ \int_{D_{R,n}^c} e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(x_1)/n,\dots,Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^k x_i} dx_1\dots dx_k.\end{aligned}$$

We first evaluate the integral over $D_{R,n}^c$. Since $|f| \le M$,

$$\begin{aligned}&\int_{D_{R,n}^c} e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(x_1)/n,\dots,Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^k x_i} dx_1\dots dx_k \\ &= \int e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(x_1)/n,\dots,Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^k x_i} \mathbb{1}_{\{Q_n^{\leftarrow}(x_1)/n>R\}} dx_1\dots dx_k \\ &= \int e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(x_1)/n,\dots,Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^k x_i} \mathbb{1}_{\{x_1\le Q_n(nR)\}} dx_1\dots dx_k \\ &\le \int e^{L(n)n^\alpha M} e^{-\sum_{i=1}^k x_i} \mathbb{1}_{\{x_1\le Q_n(nR)\}} dx_1\dots dx_k \le e^{L(n)n^\alpha M}\left(1-e^{-Q_n(nR)}\right) \\ &\le e^{L(n)n^\alpha M} Q_n(nR).\end{aligned} \tag{6.14}$$

Turning to the integral over $D_{R,n}$, fix $\epsilon > 0$ and pick $\{\check{x}^{(1)}, \dots, \check{x}^{(m)}\} \subset \mathbb{R}_+^k$ in such a way that $\left\{\prod_{j=1}^k [\check{x}_j^{(l)} - \epsilon, \check{x}_j^{(l)} + \epsilon]\right\}_{l=1,\dots,m}$ covers A_R. Set

$$H_{R,n,l} \triangleq \left\{(y_1,\dots,y_k) \in \mathbb{R}_+^k : Q_n^{\leftarrow}(y_1)/n \in \left[\check{x}_1^{(l)} - \epsilon, \check{x}_1^{(l)} + \epsilon\right], \dots, Q_n^{\leftarrow}(y_1+\ldots+y_k)/n \in \left[\check{x}_k^{(l)} - \epsilon, \check{x}_k^{(l)} + \epsilon\right]\right\}.$$

Then $D_{R,n} \subseteq \bigcup_{l=1}^m H_{R,n,l}$, and hence,

$$\begin{aligned}&\int_{D_{R,n}} e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(y_1)/n,\dots,Q_n^{\leftarrow}(y_1+\cdots+y_k)/n\right)} e^{-\sum_{i=1}^k y_i} dy_1\cdots dy_k \\ &\le \sum_{l=1}^m \int_{H_{R,n,l}} e^{L(n)n^\alpha f\left(Q_n^{\leftarrow}(y_1)/n,\dots,Q_n^{\leftarrow}(y_1+\ldots+y_k)/n\right)} e^{-\sum_{i=1}^k y_i} dy_1\cdots dy_k \\ &= \sum_{l=1}^m \int_{H_{R,n,l}} e^{L(n)n^\alpha \Lambda_f\left(Q_n^{\leftarrow}(y_1)/n,\dots,Q_n^{\leftarrow}(y_1+\cdots+y_k)/n\right)} e^{L(n)\sum_{i=1}^k Q_n^{\leftarrow}(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_1..dy_k \\ &\le \sum_{l=1}^m \int_{H_{R,n,l}} e^{L(n)n^\alpha \Lambda_f(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_k)} e^{L(n)\sum_{i=1}^k Q_n^{\leftarrow}(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_1\cdots dy_k \\ &= \sum_{l=1}^m \underbrace{e^{L(n)n^\alpha \Lambda_f(\hat{x}_1,\hat{x}_2,\dots\hat{x}_k)} \int_{H_{R,n,l}} e^{L(n)\sum_{i=1}^k Q_n^{\leftarrow}(\sum_{j=1}^i y_j)^\alpha - \sum_{i=1}^k y_i} dy_1\cdots dy_k}_{\triangleq \mathrm{H}(R,n,l)},\end{aligned} \tag{6.15}$$

where the first equality is obtained by adding and subtracting $L(n)\sum_{i=1}^{k} Q_n^{\leftarrow}(\sum_{j=1}^{i} y_j)^\alpha$ to the exponent of the integrand. Since

$$Q_n^{\leftarrow}(\textstyle\sum_{j=1}^{i} y_j)/n \in \big[\check{x}_i^{(l)} - \epsilon,\, \check{x}_i^{(l)} + \epsilon\big] \implies Q_n\big(n(\check{x}_i^{(l)} + \epsilon)\big) \le \textstyle\sum_{j=1}^{i} y_j \le Q_n\big(n(\check{x}_i^{(l)} - \epsilon)\big),$$

we can bound the integral in (6.15) as follows:

$$\begin{aligned}
&\int_{H_{R,n,l}} e^{L(n)\sum_{i=1}^{k} Q_n^{\leftarrow}(\sum_{j=1}^{i} y_j)^\alpha - \sum_{i=1}^{k} y_i}\, dy_1 \cdots, dy_k \\
&\le \int_{H_{R,n,l}} e^{L(n)\sum_{i=1}^{k}\left(n(\check{x}_i^{(l)} - \epsilon)\right)^\alpha - \sum_{i=1}^{k} y_i}\, dy_1 \ldots dy_k \\
&\le \int_{H_{R,n,l}} e^{L(n)\sum_{i=1}^{k}\left(n(\check{x}_i^{(l)} - \epsilon)\right)^\alpha - Q_n\left(n(\check{x}_k^{(l)} + \epsilon)\right)}\, dy_1 \ldots dy_k \\
&= e^{L(n)\sum_{i=1}^{k}\left(n(\check{x}_i^{(l)} - \epsilon)\right)^\alpha - Q_n\left(n(\check{x}_k^{(l)} + \epsilon)\right)} \int_{H_{R,n,l}} dy_1 \ldots dy_k \\
&= e^{L(n)n^\alpha \sum_{i=1}^{k}(\check{x}_i^{(l)} - \epsilon)^\alpha - Q_n\left(n(\check{x}_k^{(l)} + \epsilon)\right)} \prod_{i=1}^{k}\Big(Q_n\big(n(\check{x}_i^{(l)} - \epsilon)\big) - Q_n\big(n(\check{x}_i^{(l)} + \epsilon)\big)\Big). \qquad (6.16)
\end{aligned}$$

With (6.15) and (6.16), a straightforward calculation as in the lower bound leads to

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathrm{H}(R,n,l) \\
&\le \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log\left(e^{L(n)n^\alpha \Lambda_f(\hat{x}_1,\hat{x}_2,\ldots\hat{x}_k)}\right) + \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log\left(e^{L(n)n^\alpha \sum_{i=1}^{k}(\hat{x}_i^{(l)} - \epsilon)^\alpha - Q_n\left(n(\hat{x}_k^{(l)} + \epsilon)\right)}\right) \\
&\qquad + \sum_{i=1}^{k} \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log\Big(Q_n\big(n(\hat{x}_i^{(l)} - \epsilon)\big) - Q_n\big(n(\hat{x}_i^{(l)} + \epsilon)\big)\Big) \\
&= \Lambda_f(\hat{x}_1,\ldots,\hat{x}_k).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \Upsilon_f(n) &= \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log\left(e^{L(n)n^\alpha M} Q_n(nR)\right) \vee \max\left\{\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log H(R,n,l)\right\} \\
&\le (M - R^\alpha) \vee \Lambda_f(\hat{x}_1,\ldots,\hat{x}_k) = (M - R^\alpha) \vee \sup_{x\in\mathbb{R}_+^k}\{f(x) - \check{I}_k(x)\}.
\end{aligned}$$

Since R was arbitrary, we can send $R \to \infty$ to arrive at the desired upper bound of (6.8). □

The following corollary is immediate from Lemma 6.2 and Theorem 4.14 of Ganesh et al. (2004).

Corollary 6.1. *$(Q_n^{\leftarrow}(\Gamma_1)/n, \ldots, Q_n^{\leftarrow}(\Gamma_k)/n, U_1, \ldots, U_k)$ satisfies a large deviation principle in $\mathbb{R}_+^k \times [0,1]^k$ with speed $L(n)n^\alpha$ and the good rate function*

$$\hat{I}_k(x_1,\ldots,x_k,u_1,\ldots,u_k) \triangleq \begin{cases} \sum_{i=1}^{k} x_i^\alpha & \text{if } x_1 \ge x_2 \ge \cdots \ge x_k \text{ and } u_1,\ldots,u_k \in [0,1], \\ \infty, & \text{otherwise.} \end{cases} \qquad (6.17)$$

Recall that $\hat{J}_n^{\leqslant k} = \frac{1}{n}\sum_{i=1}^{k} Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}$ and the rate function I_k defined in (2.5). We next prove a sample path LDP for $\hat{J}_n^{\leqslant k}$.

Lemma 6.3. *$\hat{J}_n^{\leqslant k}$ satisfies the LDP in $(\mathbb{D}, \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and the rate function I_k.*

Proof. First, we note that I_k is indeed a rate function since the sublevel sets of I_k equal the intersection between the sublevel sets of I and a closed set $\mathbb{D}_{\leqslant k}$, and I is a rate function (Lemma 6.1).

Next, we prove the LDP in $\mathbb{D}_{\leqslant k}$ w.r.t. the relative topology induced by $\mathcal{T}_{J_1}$. (Note that I_k is a rate function in $\mathbb{D}_{\leqslant k}$ as well.) Set $T_k(x,u) \triangleq \sum_{i=1}^k x_i \mathbb{1}_{[u_i,1]}$. Since

$$\inf_{(x,u)\in T_k^{-1}(\xi)} \hat{I}_k(x,u) = I_k(\xi)$$

for $\xi \in \mathbb{D}_{\leqslant k}$, the LDP in $\mathbb{D}_{\leqslant k}$ is established once we show that for any closed set $F \subseteq \mathbb{D}_{\leqslant k}$,

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\hat{J}_n^{\leqslant k} \in F\right) \leq - \inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u), \tag{6.18}$$

and for any open set $G \subseteq \mathbb{D}_{\leqslant k}$,

$$- \inf_{(x,u)\in T_k^{-1}(G)} \hat{I}_k(x,u) \leq \liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\hat{J}_n^{\leqslant k} \in G\right). \tag{6.19}$$

We start with the upper bound. Note that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\hat{J}_n^{\leqslant k} \in F\right) \\
&= \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\Big(\big(Q_n^{\leftarrow}(\Gamma_1), \dots, Q_n^{\leftarrow}(\Gamma_k), U_1, \dots, U_k\big) \in T_k^{-1}(F)\Big) \\
&\leq \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\Big(\big(Q_n^{\leftarrow}(\Gamma_1), \dots, Q_n^{\leftarrow}(\Gamma_k), U_1, \dots, U_k\big) \in T_k^{-1}(F)^-\Big) \\
&\leq - \inf_{(x_1,\dots,x_k,u_1,\dots,u_k)\in T_k^{-1}(F)^-} \hat{I}_k(x_1,\dots,x_k,u_1,\dots,u_k).
\end{aligned}$$

In view of (6.18), it is therefore enough for the upper bound to show that

$$\inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u) \leq \inf_{(x,u)\in T_k^{-1}(F)^-} \hat{I}_k(x,u).$$

To prove this, we proceed with proof by contradiction. Suppose that

$$c \triangleq \inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u) > \inf_{(x,u)\in T_k^{-1}(F)^-} \hat{I}_k(x,u). \tag{6.20}$$

Pick an $\epsilon > 0$ in such a way that $\inf_{(x,u)\in T_k^{-1}(F)^-} \hat{I}_k(x,u) < c - 2\epsilon$. Then there exists $(x^*,u^*) \in T_k^{-1}(F)^-$ such that $\hat{I}_k(x^*,u^*) < c-2\epsilon$. Let $\bar{I}_k(x_1,\dots,x_k,u_1,\dots,u_k) \triangleq \sum_{i=1}^k x_i^\alpha$. Since $\bar{I}_k$ is continuous, one can find $(x',u') = (x_1',\dots,x_k',u_1',\dots,u_k') \in T_k^{-1}(F)$ sufficiently close to (x^*,u^*) so that $\bar{I}_k(x',u') < c-\epsilon$. Note that for any permutation $p : \{1,\dots,k\} \to \{1,\dots,k\}$, $(x'',u'') \triangleq (x_{p(1)}',\dots,x_{p(k)}',u_{p(1)}',\dots,u_{p(k)}')$ also belongs to $T_k^{-1}(F)$ and $\bar{I}_k(x'',u'') = \bar{I}_k(x',u')$ due to the symmetric structure of T_k and $\bar{I}_k$. If we pick p so that $x_{p(1)}' \geq \cdots \geq x_{p(k)}'$, then $\hat{I}_k(x'',u'') = \bar{I}_k(x',u') < c - \epsilon \leq \inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u)$, which contradicts to $(x'',u'') \in T_k^{-1}(F)$. Therefore, (6.20) cannot be the case, which proves the upper bound.

Turning to the lower bound, consider an open set $G \subseteq \mathbb{D}_{\leqslant k}$.

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\hat{J}_n^{\leqslant k} \in G\right) \\
&= \liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\Big(\big(Q_n^{\leftarrow}(\Gamma_1),\dots,Q_n^{\leftarrow}(\Gamma_k),U_1,\dots,U_k\big) \in T_k^{-1}(G)\Big) \\
&\geq \liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\big(Q_n^{\leftarrow}(\Gamma_1),\dots,Q_n^{\leftarrow}(\Gamma_k),U_1,\dots,U_k\big) \in T_k^{-1}(G)^\circ\right) \\
&\geq -\inf_{(x_1,\dots,x_k,u_1,\dots,u_k)\in T_k^{-1}(G)^\circ} \hat{I}_k(x_1,\dots,x_k,u_1,\dots,u_k).
\end{aligned}$$

In view of (6.19), we are done if we prove that

$$\inf_{(x,u)\in T_k^{-1}(G)^\circ} \hat{I}_k(x,u) \leq \inf_{(x,u)\in T_k^{-1}(G)} \hat{I}_k(x,u). \tag{6.21}$$

Let (x,u) be an arbitrary point in $T_k^{-1}(G)$ so that $T_k(x,u)\in G$. We will show that there exists $(x^*,u^*)\in T_k^{-1}(G)^\circ$ such that $I_k(x^*,u^*)\leq I_k(x,u)$. Note first that if $u_i\in\{0,1\}$ for some i, then x_i has to be 0 since $G\subseteq\mathbb{D}_{\leqslant k}$. This means that we can replace u_i with an arbitrary number in $(0,1)$ without changing the value of I_k and T_k. Therefore, we assume w.l.o.g. that $u_i>0$ for each $i=1,\dots,k$. Now, suppose that $u_i=u_j$ for some $i\neq j$. Then one can find (x',u') such that $T_k(x',u')=T_k(x,u)$ by setting

$$(x',u') \triangleq (x_1,\dots,\underbrace{x_i+x_j}_{i^{\text{th}}\text{ coordinate}},\dots,\underbrace{0}_{j^{\text{th}}\text{ coordinate}},\dots,x_k,u_1,\dots,\underbrace{u_i}_{k+i^{\text{th}}\text{ coordinate}},\dots,\underbrace{u'_j}_{k+j^{\text{th}}\text{coordinate}},\dots,u_k),$$

where u'_j is an arbitrary number in $(0,1)$; in particular, we can choose u'_j so that $u'_j\neq u_l$ for $l=1,\dots,k$. It is easy to see that $\bar{I}_k(x',u')\leq\hat{I}_k(x,u)$. Now one can permute the coordinates of (x',u') as in the upper bound to find (x'',u'') such that $T_k(x'',u'')=T_k(x,u)$ and $\hat{I}_k(x'',u'')\leq\hat{I}_k(x,u)$. Iterating this procedure until there is no $i\neq j$ for which $u_i=u_j$, we can find (x^*,u^*) such that $T_k(x^*,u^*)=T_k(x,u)$, u_i^*'s are all distinct in $(0,1)$, and $I_k(x^*,u^*)\leq I_k(x,u)$. Note that since T_k is continuous at (x^*,u^*), $T_k(x^*,u^*)\in G$, and G is open, we conclude that $(x^*,u^*)\in T_k^{-1}(G)^\circ$. Therefore,

$$\inf_{(x,u)\in T_k^{-1}(G)^\circ} I_k(x,u) \leq I_k(x,u).$$

Since (x,u) was arbitrarily chosen in $T_k^{-1}(G)$, (6.21) is proved. Along with the upper bound, this proves the LDP in $\mathbb{D}_{\leqslant k}$. Finally, since $\mathbb{D}_{\leqslant k}$ is a closed subset of $\mathbb{D}$, $\mathbf{P}(\hat{J}_n^{\leqslant k}\notin\mathbb{D}_{\leqslant k})=0$, and $I_k=\infty$ on $\mathbb{D}\setminus\mathbb{D}_{\leqslant k}$, Lemma 4.1.5 of Dembo and Zeitouni (2010) applies proving the desired LDP in $\mathbb{D}$. □

Now we are ready to prove Lemma 2.1.

Proof of Lemma 2.1. Recall that

$$\bar{J}_n^k \stackrel{\mathcal{D}}{=} \underbrace{\frac{1}{n}\sum_{i=1}^k Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}}_{=\hat{J}_n^{\leqslant k}} - \underbrace{\frac{1}{n}\mathbb{1}\{\tilde{N}_n<k\}\sum_{i=\tilde{N}_n+1}^{k} Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}}_{=\check{J}_n^{\leqslant k}}.$$

Let F be a closed set and note that

$$\begin{aligned}
\mathbf{P}(\bar{J}_n^k\in F) &= \mathbf{P}(\hat{J}_n^{\leqslant k}-\check{J}_n^{\leqslant k}\in F) \leq \mathbf{P}\big(\hat{J}_n^{\leqslant k}-\check{J}_n^{\leqslant k}\in F, \mathbb{1}\{N(n)<k\}=0\big)+\mathbf{P}\big(\mathbb{1}\{N(n)<k\}\neq 0\big) \\
&\leq \mathbf{P}(\hat{J}_n^{\leqslant k}\in F)+\mathbf{P}(N(n)<k).
\end{aligned}$$

From Lemma 6.3,

$$\limsup_{n\to\infty}\frac{\log\mathbf{P}(\bar{J}_n^k\in F)}{L(n)n^\alpha}\le\limsup_{n\to\infty}\frac{\log\mathbf{P}(\hat{J}_n^{\leqslant k}\in F)}{L(n)n^\alpha}\vee\limsup_{n\to\infty}\frac{\log\mathbf{P}(N(n)<k)}{L(n)n^\alpha}$$
$$\le-\inf_{\xi\in F}I_k(\xi),$$

since $\limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}(N(n)<k)=-\infty$.

Turning to the lower bound, let G be an open set. Since the lower bound is trivial in case $\inf_{x\in G}I_k(x)=\infty$, we focus on the case $\inf_{x\in G}I_k(x)<\infty$. In this case,

$$\begin{aligned}
\liminf_{n\to\infty}\frac{\log\mathbf{P}(\bar{J}_n^k\in G)}{L(n)n^\alpha}&\ge\liminf_{n\to\infty}\frac{\log\mathbf{P}(\bar{J}_n^k\in G,\,N(n)\ge k)}{L(n)n^\alpha}=\liminf_{n\to\infty}\frac{\log\mathbf{P}(\hat{J}_n^{\leqslant k}\in G,\,N(n)\ge k)}{L(n)n^\alpha}\\
&\ge\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\Big(\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)-\mathbf{P}(N(n)<k)\Big)\\
&=\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\left(\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)\left(1-\frac{\mathbf{P}(N(n)<k)}{\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)}\right)\right)\\
&=\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\left\{\log\Big(\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)\Big)+\log\left(1-\frac{\mathbf{P}(N(n)<k)}{\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)}\right)\right\}\\
&=\liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)\ge-\inf_{\xi\in G}I_k(\xi).
\end{aligned}$$

The last equality holds since

$$\lim_{n\to\infty}\frac{\mathbf{P}(N(n)<k)}{\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)}=\lim_{n\to\infty}\left\{\exp\left(\frac{\log\mathbf{P}(N(n)<k)}{L(n)n^\alpha}-\frac{\log\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)}{L(n)n^\alpha}\right)\right\}^{L(n)n^\alpha}=0 \tag{6.22}$$

which in turn follows from

$$\limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}(N(n)<k)=-\infty\qquad\text{and}\qquad\limsup_{n\to\infty}\frac{-1}{L(n)n^\alpha}\log\mathbf{P}(\hat{J}_n^{\leqslant k}\in G)\le\inf_{x\in G}I_k(x)<\infty.$$

□

6.4 Proof of Lemma 2.2

Proof of Lemma 2.2. Since the inequality is obvious if $\inf_{\xi\in\mathbb{D}}I(\xi)=\infty$, we assume that $\inf_{\xi\in\mathbb{D}}I(\xi)<\infty$. Then, there exists a $\xi_0\in G$ such that $I(\xi_0)\le I(G)+\delta$. Since G is open, we can pick $\epsilon>0$ such that $B_{J_1}(\xi_0;2\epsilon)\subseteq A$, and hence, $B_{J_1}(\xi_0;\epsilon)\subseteq G^{-\epsilon}$. Note that since $I(\xi_0)<\infty$, ξ_0 has the representation $\xi_0=\sum_{i=1}^\infty x_i\mathbb{1}_{[u_i,1]}$ where $x_i\ge0$ for all $i=1,2,\ldots$, and u_i's all distinct in $(0,1)$. Note also that since $I(\xi_0)=\sum_{i=1}^\infty x_i^\alpha<\infty$ with $\alpha<1$, $\sum_{i=1}^\infty x_i$ has to be finite as well. There exists K such that $k\ge K$ implies $\sum_{i=k+1}^\infty x_i<\epsilon$. If we fix $k\ge K$ and let $\xi_1\triangleq\sum_{i=1}^k x_i\mathbb{1}_{[u_i,1]}$, then $I_k(\xi_1)\le I(\xi_0)$ while $d_{J_1}(\xi_0,\xi_1)\le\|\xi_0-\xi_1\|_\infty<\sum_{i=k+1}^\infty x_i<\epsilon$. That is, $\xi_1\in B_{J_1}(\xi_0;\epsilon)\subseteq A^{-\epsilon}$. Therefore, $\inf_{\xi\in A^{-\epsilon}}I_k(\xi)\le I(\xi_1)\le I(\xi_0)\le\inf_{\xi\in A}I(\xi)+\delta$. □

6.5 Proof of Lemma 2.3

In our proof of Lemma 2.3, the following lemmas—Lemma 6.4 and Lemma 6.5 play key roles.

Lemma 6.4. *Let $\beta > \alpha$. For each $\epsilon > \delta > 0$,*

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\max_{1\le j\le 2n} \sum_{i=1}^{j} \left(Z_i \mathbb{1}_{\{Z_i \le n\delta\}} - \mathbf{E}Z \right) > n\epsilon \right) \le -(\epsilon/3)^\alpha (\epsilon/\delta)^{1-\beta}. \tag{6.23}$$

Proof. We refine an argument developed in Jelenković and Momčilović (2003). Note that for any $s > 0$ such that $1/s \le n\delta$,

$$\mathbf{E}e^{sZ\mathbb{1}_{\{Z\le n\delta\}}} = \mathbf{E}e^{sZ\mathbb{1}_{\{Z\le n\delta\}}}\mathbb{1}_{\{Z\ge\frac{1}{s}\}} + \mathbf{E}e^{sZ\mathbb{1}_{\{Z\le n\delta\}}}\mathbb{1}_{\{Z<\frac{1}{s}\}} = (I) + (II), \tag{6.24}$$

and

$$\begin{aligned}
(I) &= \int_{[1/s,n\delta]} e^{sy}\mathrm{d}\mathbf{P}(Z\le y) + \int_{(n\delta,\infty)} \mathrm{d}\mathbf{P}(Z\le y) \\
&= \Big[e^{sy}\mathbf{P}(Z\le y)\Big]_{(1/s)-}^{(n\delta)+} - s\int_{[1/s,n\delta]} e^{sy}\mathbf{P}(Z\le y)\mathrm{d}y + \mathbf{P}(Z>n\delta) \\
&= e^{sn\delta}\mathbf{P}(Z\le n\delta) - e\mathbf{P}(Z<1/s) - s\int_{[1/s,n\delta]} e^{sy}\mathrm{d}y + s\int_{[1/s,n\delta]} e^{sy}\mathbf{P}(Z>y)\mathrm{d}y + \mathbf{P}(Z>n\delta) \\
&= e^{sn\delta}\mathbf{P}(Z\le n\delta) - e\mathbf{P}(Z<1/s) - e^{sn\delta} + e + s\int_{[1/s,n\delta]} e^{sy}\mathbf{P}(Z>y)\mathrm{d}y + \mathbf{P}(Z>n\delta) \\
&= -e^{sn\delta}\mathbf{P}(Z>n\delta) + e\mathbf{P}(Z\ge 1/s) + s\int_{[1/s,n\delta]} e^{sy}\mathbf{P}(Z>y)\mathrm{d}y + \mathbf{P}(Z>n\delta) \\
&\le s\int_{[1/s,n\delta]} e^{sy}\mathbf{P}(Z>y)\mathrm{d}y + e\mathbf{P}(Z\ge 1/s) + \mathbf{P}(Z>n\delta) \\
&\le s\int_{[1/s,n\delta]} e^{sy}\mathbf{P}(Z>y)\mathrm{d}y + s^2(e+1)\mathbf{E}Z^2
\end{aligned} \tag{6.25}$$

where the last inequality is from $\mathbf{P}(Z\ge n\delta) \le \mathbf{P}(Z\ge 1/s) \le s^2\,\mathbf{E}Z^2$; while

$$(II) \le \int_0^{1/s} e^{sy}\mathrm{d}\mathbf{P}(Z\le y) \le \int_0^{1/s} (1+sy+(sy)^2)\mathrm{d}\mathbf{P}(Z\le y) \le 1 + s\,\mathbf{E}Z + s^2\,\mathbf{E}Z^2. \tag{6.26}$$

Therefore, from (6.24), (6.25) and (6.26), if $1/s \le n\delta$,

$$\begin{aligned}
\mathbf{E}e^{sZ\mathbb{1}_{\{Z\le n\delta\}}} &\le s\int_{\frac{1}{s}}^{n\delta} e^{sy}\mathbf{P}\left(Z>y\right)dy + 1 + s\mathbf{E}Z + s^2(e+2)\mathbf{E}Z^2 \\
&= s\int_{\frac{1}{s}}^{n\delta} e^{sy-q(y)}dy + 1 + s\mathbf{E}Z + s^2(e+2)\mathbf{E}Z^2 \\
&\le sn\delta\left(e^{sn\delta-q(n\delta)} + e^{1-q(\frac{1}{s})}\right) + 1 + s\mathbf{E}Z + s^2(e+2)\mathbf{E}Z^2
\end{aligned} \tag{6.27}$$

where $q(x) \triangleq -\log\mathbf{P}(X>x) = L(x)x^\alpha$ and the last inequality is since $e^{sy-q(y)}$ is increasing due to the assumption that $L(y)y^{\alpha-1}$ is non-increasing. Now, from the Markov inequality,

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{j}\left(Z_i\mathbb{1}_{\{Z_i\le n\delta\}}-\mathbf{E}Z\right)>n\epsilon\right)\\
&\le \mathbf{P}\left(\exp\left(s\sum_{i=1}^{j}Z_i\mathbb{1}_{\{Z_i\le n\delta\}}\right)>\exp\left(s(n\epsilon+j\mathbf{E}Z)\right)\right)\\
&\le \exp\left\{-s\big(n\epsilon+j\mathbf{E}Z\big)+j\log\left(\mathbf{E}e^{sZ\mathbb{1}_{\{Z\le n\delta\}}}\right)\right\}\\
&\le \exp\left\{-s(n\epsilon+j\mathbf{E}Z)+j\left(sn\delta\left(e^{sn\delta-q(n\delta)}+e^{1-q(\frac{1}{s})}\right)+s\mathbf{E}Z+s^2(e+2)\mathbf{E}Z^2\right)\right\}\\
&= \exp\left\{-sn\epsilon+jsn\delta\left(e^{sn\delta-q(n\delta)}+e^{1-q(\frac{1}{s})}\right)+js^2(e+2)\mathbf{E}Z^2\right\}\\
&\le \exp\left\{-sn\epsilon+2n^2s\delta\left(e^{sn\delta-q(n\delta)}+e^{1-q(\frac{1}{s})}\right)+2ns^2(e+2)\mathbf{E}Z^2\right\} \qquad (6.28)
\end{aligned}
$$

for $j\le 2n$, where the third inequality is from (6.27) and the generic inequality $\log(x+1)\le x$. Fix $\gamma\in(0,(\epsilon/\delta)^{1-\beta})$ and and set $s=\frac{\gamma q(n\epsilon)}{n\epsilon}$. From now on, we only consider sufficiently large n's such that $1/s<n\delta$. To establish an upper bound for (6.28), we next examine $e^{sn\delta-q(n\delta)}$ and $e^{1-q(\frac{1}{s})}$. Note that $q(n\epsilon)\le q(n\delta)(\epsilon/\delta)^{\beta}$ for sufficiently large n's. Therefore,

$$sn\delta-q(n\delta)=\frac{\gamma q(n\epsilon)}{n\epsilon}n\delta-q(n\delta)\le\gamma q(n\delta)(\epsilon/\delta)^{\beta-1}-q(n\delta)=-q(n\delta)\left(1-\gamma(\delta/\epsilon)^{1-\beta}\right)<0,$$

and hence,

$$e^{sn\delta-q(n\delta)}\le e^{-q(n\delta)\left(1-\gamma(\delta/\epsilon)^{1-\beta}\right)}. \qquad (6.29)$$

For $e^{1-q(\frac{1}{s})}$, note that $1-q(1/s)\le 1-\gamma^{-\beta}q(n\epsilon)^{1-\beta}$ since $q(n\epsilon)\le Q\left(1/s\right)\gamma^{\beta}q(n\epsilon)^{\beta}$. Therefore,

$$e^{1-q(\frac{1}{s})}\le e^{1-\gamma^{-\beta}q(n\epsilon)^{1-\beta}}. \qquad (6.30)$$

Plugging s into (6.28) along with (6.29) and (6.30),

$$
\begin{aligned}
&\max_{0\le j\le 2n}\mathbf{P}\left(\sum_{i=1}^{j}\left(Z_i\mathbb{1}_{\{Z_i\le n\delta\}}-\mathbf{E}Z\right)>n\epsilon\right)\\
&\le \exp\left\{-\gamma q(n\epsilon)+\frac{2\gamma\delta nq(n\epsilon)}{\epsilon}\left(e^{-q(n\delta)\left(1-\gamma(\delta/\epsilon)^{1-\beta}\right)}+e^{1-\gamma^{-\beta}q(n\epsilon)^{1-\beta}}\right)+\frac{2\gamma^2(e+2)\mathbf{E}Z^2}{\epsilon^2}\frac{q(n\epsilon)^2}{n}\right\}.
\end{aligned}
$$

Since

$$\limsup_{n\to\infty}\frac{1}{L(n)n^{\alpha}}\frac{2\gamma\delta nq(n\epsilon)}{\epsilon}\left(e^{-q(n\delta)\left(1-\gamma(\delta/\epsilon)^{1-\beta}\right)}+e^{1-\gamma^{-\beta}q(n\epsilon)^{1-\beta}}\right)=0,$$

and

$$\limsup_{n\to\infty}\frac{1}{L(n)n^{\alpha}}\frac{2\gamma^2(e+2)\mathbf{E}Z^2}{\epsilon^2}\frac{q(n\epsilon)^2}{n}=0,$$

we conclude that

$$\limsup_{n\to\infty}\frac{1}{L(n)n^{\alpha}}\log\max_{0\le j\le 2n}\mathbf{P}\left(\sum_{i=1}^{j}\left(Z_i\mathbb{1}_{\{Z_i\le n\delta\}}-\mathbf{E}Z\right)>n\epsilon\right)=\limsup_{n\to\infty}\frac{-\gamma q(n\epsilon)}{L(n)n^{\alpha}}=-\epsilon^{\alpha}\gamma.$$

From Etemadi's inequality,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\max_{0\le j\le 2n} \sum_{i=1}^{j} \left(Z_i \mathbb{1}_{\{Z_i\le n\delta\}} - \mathbf{E}Z\right) > 3n\epsilon\right)\\
&\le \limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log\left\{3 \max_{0\le j\le 2n} \mathbf{P}\left(\sum_{i=1}^{j} \left(Z_i \mathbb{1}_{\{Z_i\le n\delta\}} - \mathbf{E}Z\right) > n\epsilon\right)\right\} = -\epsilon^\alpha \gamma.
\end{aligned}$$

Since this is true for arbitrary γ's such that $\gamma \in (0, (\epsilon/\delta)^{1-\beta})$, we arrive at the conclusion of the lemma. □

Lemma 6.5. *For every ϵ, $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\sup_{1\le j\le 2n} \sum_{i=1}^{j} \left(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\right) > n\epsilon\right) = -\infty.$$

Proof. Note first that there is n_0 such that $\mathbf{E}\left(Z_i \mathbb{1}_{\{Z_i>n\delta\}}\right) \le \frac{\epsilon}{3}$ for $n \ge n_0$. For $n \ge n_0$ and $j \le 2n$,

$$\begin{aligned}
\mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\right) > n\epsilon\right) &= \mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z\mathbb{1}_{\{Z\le n\delta\}} - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\right) > n\epsilon - j\mathbf{E}Z\mathbb{1}_{\{Z>n\delta\}}\right)\\
&\le \mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z\mathbb{1}_{\{Z\le n\delta\}} - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\right) > n\epsilon - j\epsilon/3\right)\\
&\le \mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z\mathbb{1}_{\{Z\le n\delta\}} - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\right) > \frac{n\epsilon}{3}\right).
\end{aligned}$$

Let $Y_i^{(n)} \triangleq \mathbf{E}(Z_i \mathbb{1}_{\{Z_i\le n\delta\}}) - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}$, then $\mathbf{E}Y_i^{(n)} = 0$, $\mathbf{var}\, Y_i^{(n)} \le \mathbf{E}Z^2$, and $Y_i^{(n)} \le \mathbf{E}Z$ almost surely. Note that from Bennet's inequality,

$$\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z\mathbb{1}_{\{Z_i\le n\delta\}} - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\right) > \frac{n\epsilon}{3}\right)\\
&\le \exp\left[-\frac{j\mathbf{var}\, Y^{(n)}}{(\mathbf{E}Z)^2}\left\{\left(1 + \frac{n\epsilon\,\mathbf{E}Z}{3j\,\mathbf{var}\, Y^{(n)}}\right)\log\left(1 + \frac{n\epsilon\,\mathbf{E}Z}{3j\,\mathbf{var}\, Y^{(n)}}\right) - \left(\frac{n\epsilon\,\mathbf{E}Z}{3j\,\mathbf{var}\, Y^{(n)}}\right)\right\}\right]\\
&\le \exp\left[-\frac{j\mathbf{var}\, Y^{(n)}}{(\mathbf{E}Z)^2}\left\{\left(\frac{n\epsilon\,\mathbf{E}Z}{3j\,\mathbf{var}\, Y^{(n)}}\right)\log\left(1 + \frac{n\epsilon\,\mathbf{E}Z}{3j\,\mathbf{var}\, Y^{(n)}}\right) - \left(\frac{n\epsilon\,\mathbf{E}Z}{3j\,\mathbf{var}\, Y^{(n)}}\right)\right\}\right]\\
&\le \exp\left[-\left\{\left(\frac{n\epsilon}{3\,\mathbf{E}Z}\right)\log\left(1 + \frac{n\epsilon\,\mathbf{E}Z}{3j\,\mathbf{var}\, Y^{(n)}}\right) - \left(\frac{n\epsilon}{3\,\mathbf{E}Z}\right)\right\}\right]\\
&\le \exp\left[-n\left\{\left(\frac{\epsilon}{3\,\mathbf{E}Z}\right)\log\left(1 + \frac{\epsilon\,\mathbf{E}Z}{6\,\mathbf{E}Z^2}\right) - \left(\frac{\epsilon}{3\,\mathbf{E}Z}\right)\right\}\right]
\end{aligned}$$

for $j \le 2n$. Therefore, for $n \ge n_0$ and $j \le 2n$,

$$\mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\right) > n\epsilon\right) \le \exp\left[-n\left\{\left(\frac{\epsilon}{3\,\mathbf{E}Z}\right)\log\left(1 + \frac{\epsilon\,\mathbf{E}Z}{6\,\mathbf{E}Z^2}\right) - \left(\frac{\epsilon}{3\,\mathbf{E}Z}\right)\right\}\right].$$

Now, from Etemadi's inequality,

$$\limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}\left(\sup_{1\le j\le 2n}\sum_{i=1}^{j}\left(\mathbf{E}Z - Z_i\mathbb{1}_{\{Z_i\le n\delta\}}\right) > 3n\epsilon\right)$$
$$\le \limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\left\{3\max_{1\le j\le 2n}\mathbf{P}\left(\sum_{i=1}^{j}\left(\mathbf{E}Z - Z_i\mathbb{1}_{\{Z_i\le n\delta\}}\right) > n\epsilon\right)\right\}$$
$$\le \limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\left\{3\exp\left[-n\left\{\left(\frac{\epsilon}{3\,\mathbf{E}Z}\right)\log\left(1+\frac{\epsilon\,\mathbf{E}Z}{6\,\mathbf{E}Z^2}\right)-\left(\frac{\epsilon}{3\,\mathbf{E}Z}\right)\right\}\right]\right\} = -\infty.$$

Replacing ϵ with $\epsilon/3$, we arrive at the conclusion of the lemma. □

Now we are ready to prove Lemma 2.3.

Proof of Lemma 2.3.

$$\begin{aligned}
&\mathbf{P}\left(\|\bar K_n^k\|_\infty > \epsilon\right)\\
&\le \mathbf{P}\left(\|\bar K_n^k\|_\infty > \epsilon, N(nt)\ge k\right) + \mathbf{P}\left(\|\bar K_n^k\|_\infty > \epsilon, N(nt) < k\right)\\
&\le \mathbf{P}\left(\|\bar K_n^k\|_\infty > \epsilon,\ N(nt)\ge k,\ Z_{R_n^{-1}(k)}\le n\delta\right) + \mathbf{P}\left(\|\bar K_n^k\|_\infty > \epsilon,\ N(nt)\ge k,\ Z_{R_n^{-1}(k)} > n\delta\right) + \mathbf{P}\left(N(nt) < k\right)\\
&\le \mathbf{P}\left(\|\bar K_n^k\|_\infty > \epsilon,\ N(nt)\ge k,\ Z_{R_n^{-1}(k)}\le n\delta\right) + \mathbf{P}\left(N(nt)\ge k,\ Z_{R_n^{-1}(k)} > n\delta\right) + \mathbf{P}\left(N(nt) < k\right). \qquad (6.31)
\end{aligned}$$

An explicit upper bound for the second term can be obtained:

$$\begin{aligned}
\mathbf{P}\left(N(nt)\ge k,\ Z_{R_n^{-1}(k)} > n\delta\right) &\le \mathbf{P}\left(Q_n^{\leftarrow}(\Gamma_k) > n\delta\right) = \mathbf{P}\left(\Gamma_k \le Q_n(n\delta)\right) = \int_0^{Q_n(n\delta)}\frac{1}{k!}t^{k-1}e^{-t}dt\\
&= \int_0^{n\nu[n\delta,\infty)}\frac{1}{k!}t^{k-1}e^{-t}dt \le \int_0^{n\nu(n\delta,\infty)} t^{k-1}dt = \frac{1}{k}n^k e^{-kL(n\delta)n^\alpha\delta^\alpha}.
\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}\left(Q_n^{\leftarrow}(\Gamma_k) > n\delta\right) \le -k\delta^\alpha. \qquad (6.32)$$

Turning to the first term of (6.31), we consider the following decomposition:

$$\begin{aligned}
&\mathbf{P}\left(\|\bar K_n^k\|_\infty > \epsilon,\ N(nt)\ge k,\ Z_{R_n^{-1}(k)}\le n\delta\right)\\
&= \underbrace{\mathbf{P}\left(N(nt)\ge k,\ Z_{R_n^{-1}(k)}\le n\delta,\ \sup_{t\in[0,1]}\bar K_n^k(t) > \epsilon\right)}_{\triangleq\text{(i)}} + \underbrace{\mathbf{P}\left(N(nt)\ge k,\ Z_{R_n^{-1}(k)}\le n\delta,\ \sup_{t\in[0,1]} -\bar K_n^k(t) > \epsilon\right)}_{\triangleq\text{(ii)}}.
\end{aligned}$$

Since $Z_{R_n^{-1}(k)} \le n\delta$ implies $\mathbb{1}_{\{R_n(i)>k\}} \le \mathbb{1}_{\{Z_i \le n\delta\}}$,

$$\begin{aligned}
\text{(i)} &\le \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \big(Z_i \mathbb{1}_{\{R_n(i)>k\}} - \mathbf{E}Z\big) > n\epsilon,\ N(nt) \ge k,\ Z_{R_n^{-1}(k)} \le n\delta\right) \\
&\le \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \big(Z_i \mathbb{1}_{\{Z_i\le n\delta\}} - \mathbf{E}Z\big) > n\epsilon\right) = \mathbf{P}\left(\sup_{0\le j\le N(n)} \sum_{i=1}^{j} \big(Z_i \mathbb{1}_{\{Z_i\le n\delta\}} - \mathbf{E}Z\big) > n\epsilon\right) \\
&\le \mathbf{P}\left(\sup_{0\le j\le 2n} \sum_{i=1}^{j} \big(Z_i \mathbb{1}_{\{Z_i\le n\delta\}} - \mathbf{E}Z\big) > n\epsilon, N(n) < 2n\right) + \mathbf{P}\big(N(n)\ge 2n\big) \\
&\le \mathbf{P}\left(\sup_{0\le j\le 2n} \sum_{i=1}^{j} \big(Z_i \mathbb{1}_{\{Z_i\le n\delta\}} - \mathbf{E}Z\big) > n\epsilon\right) + \mathbf{P}\big(N(n)\ge 2n\big).
\end{aligned}$$

From Lemma 6.4 and the fact that the second term decays at an exponential rate,

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \mathbf{P}\left(Z_{R_n^{-1}(k)} \le n\delta,\ \sup_{t\in[0,1]} \bar{K}_n^k(t) > \epsilon\right) \le -(\epsilon/3)^\alpha(\epsilon/\delta)^{1-\beta}. \tag{6.33}$$

Turning to (ii),

$$\begin{aligned}
\text{(ii)} &\le \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \big(\mathbf{E}Z - Z_i \mathbb{1}_{\{R_n(i)>k\}}\big) > n\epsilon\right) \\
&= \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \Big(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}} + Z_i\left(\mathbb{1}_{\{Z_i\le n\delta\}} - \mathbb{1}_{\{R_n(i)>k\}}\right)\Big) > n\epsilon\right) \\
&\le \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \Big(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}} + Z_i \mathbb{1}_{\{Z_i\le n\delta\}\cap\{R_n(i)\le k\}}\Big) > n\epsilon\right) \\
&\le \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \Big(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\Big) + kn\delta > n\epsilon\right) = \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \Big(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\Big) > n(\epsilon - k\delta)\right) \\
&\le \mathbf{P}\left(\sup_{0\le j\le 2n} \sum_{i=1}^{j} \Big(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\Big) > n(\epsilon-k\delta), N(nt) < 2n\right) + \mathbf{P}(N(nt)\ge 2n) \\
&\le \mathbf{P}\left(\sup_{0\le j\le 2n} \sum_{i=1}^{j} \Big(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i\le n\delta\}}\Big) > n(\epsilon-k\delta)\right) + \mathbf{P}(N(nt)\ge 2n).
\end{aligned}$$

Applying Lemma 6.5 to the first term and noticing that the second term vanishes at an exponential rate, we conclude that for δ and k such that $k\delta < \epsilon$

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(Z_{R_n^{-1}(k)} \le n\delta,\ \sup_{t\in[0,1]} -\bar{K}_n^k(t) > \epsilon\right) = -\infty. \tag{6.34}$$

From (6.33) and (6.34),

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \mathbf{P}\left(Z_{R_n^{-1}(k)} \le n\delta, \|\bar{K}_n^k\|_\infty > \epsilon\right) \le -(\epsilon/3)^\alpha(\epsilon/\delta)^{1-\beta}. \tag{6.35}$$

This together with (6.31) and (6.32),

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \mathbf{P}\left(\|\bar{K}_n^k\|_\infty > \epsilon\right) \le \max\{-(\epsilon/3)^\alpha(\epsilon/\delta)^{1-\beta},\ -k\delta^\alpha\}$$

for any δ and k such that $k\delta < \epsilon$. Choosing, for example, $\delta = \frac{\epsilon}{2k}$ and letting $k \to \infty$, we arrive at the conclusion of the lemma. □

6.6 Proof of Theorem 2.3

We follow a similar program as in Section 2.1 and the earlier subsections of this Section. Let $\bar{Q}_n^{(i)}(j) \triangleq Q_n^{\leftarrow}(\Gamma_j^{(i)})/n$ where $Q_n^{\leftarrow}(t) = \inf\{s > 0 : n\nu[s,\infty) < t\}$ and $\Gamma_l^{(i)} = E_1^{(i)} + \cdots + E_l^{(i)}$ where $E_j^{(i)}$'s are independent standard exponential random variables. Let $U_j^{(i)}$ be independent uniform random variables in [0,1] and $Z_n^{(i)} \triangleq \left(\bar{Q}_n^{(i)}(1), \dots, \bar{Q}_n^{(i)}(k), U_1^{(i)}, \dots, U_k^{(i)}\right)$. The following corollary is an immediate consequence of Corollary 6.1 and Theorem 4.14 of Ganesh et al. (2004).

Corollary 6.2. *$\left(Z_n^{(1)}, \dots, Z_n^{(d)}\right)$ satisfies the LDP in $\prod_{i=1}^d \left(\mathbb{R}_+^k \times [0,1]^k\right)$ with the rate function $\hat{I}_k^d(z^{(1)}, \dots, z^{(d)}) \triangleq \sum_{j=1}^d \hat{I}_k(z^{(j)})$ where $z^{(j)} = (x_1^{(j)}, \dots, x_k^{(j)}, u_1^{(j)}, \dots, u_k^{(j)})$ for each $j \in \{1, \dots, d\}$.*

Let $\hat{J}_n^{\leqslant k\,(i)} \triangleq \sum_{j=1}^k \bar{Q}_n^{(i)}(j) \mathbb{1}_{[U_j^{(i)},1]}$.

Lemma 6.6. *$\left(\hat{J}_n^{\leqslant k\,(1)}, \dots, \hat{J}_n^{\leqslant k\,(d)}\right)$ satisfies the LDP in $\prod_{i=1}^d \mathbb{D}([0,1],\mathbb{R})$ with speed $L(n)n^\alpha$ and the rate function*

$$I_k^d(\xi_1, \dots \xi_d) \triangleq \sum_{i=1}^d I_k(\xi_i) = \begin{cases} \sum_{i=1}^d \sum_{t:\xi_i(t)\neq\xi_i(t-)} (\xi_i(t) - \xi_i(t-))^\alpha & \text{if } \xi_i \in \mathbb{D}_{\leqslant k} \text{ for } i = 1, \dots, d, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Since I_{k_i} is lower semi-continuous in $\prod_{i=1}^d \mathbb{D}([0,1],\mathbb{R})$ for each i, $I_{k_1,\dots,k_d}$ is a sum of lower semi-continuous functions, and hence, is lower semi-continuous itself. The rest of the proof for the LDP upper bound and the lower bounds mirrors that of one dimensional case (Lemma 6.3) closely, and hence, omitted. □

Proof of Lemma 2.4. Again, we consider the same distributional relation for each coordinate as in the 1-dimensional case:

$$\bar{J}_n^{k\,(i)} \stackrel{\mathcal{D}}{=} \underbrace{\frac{1}{n}\sum_{j=1}^k Q_n^{(i)}(j)\mathbb{1}_{[U_j,1]}}_{=\hat{J}_n^{\leqslant k\,(i)}} - \underbrace{\frac{1}{n}\mathbb{1}\{\tilde{N}_n^{(i)} < k\} \sum_{j=\tilde{N}_n^{(i)}+1}^{k} Q_n^{(i)}(j)\mathbb{1}_{[U_j,1]}}_{=\check{J}_n^{\leqslant k\,(i)}}.$$

Let F be a closed set and write

$$\begin{aligned}\mathbf{P}((\bar{J}_n^{k\,(1)}, \dots, \bar{J}_n^{k\,(d)}) \in F) &\le \mathbf{P}\Big((\hat{J}_n^{\leqslant k\,(1)}, \dots, \hat{J}_n^{\leqslant k\,(d)}) \in F,\ \textstyle\sum_{i=1}^d \mathbb{1}\{\tilde{N}_n^{(i)} < k\} = 0\Big) + \sum_{i=1}^d \mathbf{P}\big(\mathbb{1}\{N_n^{(i)} < k\} \neq 0\big) \\ &\le \mathbf{P}\Big((\hat{J}_n^{\leqslant k\,(1)}, \dots, \hat{J}_n^{\leqslant k\,(d)}) \in F\Big) + \sum_{i=1}^d \mathbf{P}\big(\mathbb{1}\{N_n^{(i)} < k\} \neq 0\big).\end{aligned}$$

From Lemma 6.6 and the principle of the largest term,

$$\limsup_{n\to\infty}\frac{\log \mathbf{P}\big((\bar J_n^{k\,(1)},\dots,\bar J_n^{k\,(d)})\in F\big)}{L(n)n^\alpha} \le \limsup_{n\to\infty}\frac{\log \mathbf{P}\big((\hat J_n^{\leqslant k\,(1)},\dots,\hat J_n^{\leqslant k\,(d)})\in F\big)}{L(n)n^\alpha} \vee \max_{i=1,\dots,d}\limsup_{n\to\infty}\frac{\log \mathbf{P}\big(\tilde N_n^{(i)}<k\big)}{L(n)n^\alpha}$$
$$\le -\inf_{(\xi_1,\dots,\xi_d)\in F} I_k^d(\xi_1,\dots,\xi_d).$$

Turning to the lower bound, let G be an open set. Since the lower bound is trivial in case $\inf_{x\in G} I_k(x)=\infty$, we focus on the case $\inf_{x\in G} I_k(x)<\infty$. In this case, from the similar reasoning as for (6.22),

$$\begin{aligned}
\liminf_{n\to\infty}\frac{\log \mathbf{P}((\bar J_n^{k\,(1)},\dots,\bar J_n^{k\,(d)})\in G)}{L(n)n^\alpha} &\ge \liminf_{n\to\infty}\frac{\log \mathbf{P}\big((\bar J_n^{k\,(1)},\dots,\bar J_n^{k\,(d)})\in G,\ \sum_{i=1}^d \mathbb{1}\{\tilde N_n^{(i)}\ge k\}=0\big)}{L(n)n^\alpha}\\
&= \liminf_{n\to\infty}\frac{\log \mathbf{P}\big((\hat J_n^{\leqslant k\,(1)},\dots,\hat J_n^{\leqslant k\,(d)})\in G,\ \sum_{i=1}^d \mathbb{1}\{\tilde N_n^{(i)}\ge k\}=0\big)}{L(n)n^\alpha}\\
&\ge \liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\Big(\mathbf{P}\big((\hat J_n^{\leqslant k\,(1)},\dots,\hat J_n^{\leqslant k\,(d)})\in G\big)-d\mathbf{P}(\tilde N_n^{(1)}<k)\Big)\\
&= \liminf_{n\to\infty}\frac{1}{L(n)n^\alpha}\log \mathbf{P}\big((\hat J_n^{\leqslant k\,(1)},\dots,\hat J_n^{\leqslant k\,(d)})\in G\big)\\
&\ge -\inf_{(\xi_1,\dots,\xi_d)\in G} I_k^d(\xi_1,\dots,\xi_d).
\end{aligned}$$

□

The proof of Lemma 2.5 is completely analogous to the one-dimensional case, and therefore omitted.

A M_1' topology and goodness of the rate function

Let $\tilde{\mathbb{D}}[0,1]$ be the space of functions from $[0,1]$ to $\mathbb{R}$ such that the left limit exists at each $t\in(0,1]$, the right limit exists at each $t\in[0,1)$, and

$$\xi(t)\in[\xi(t-)\wedge\xi(t+),\,\xi(t-)\vee\xi(t+)] \tag{A.1}$$

for each $t\in[0,1]$ where we interpret $\xi(0-)$ as 0 and $\xi(1+)$ as $\xi(1)$.

Definition 1. *For $\xi\in\tilde{\mathbb{D}}$, define the extended completed graph $\Gamma'(\xi)$ of ξ as*

$$\Gamma'(\xi)\triangleq\{(u,t)\in\mathbb{R}\times[0,1]: u\in[\xi(t-)\wedge\xi(t+),\ \xi(t-)\vee\xi(t+)]\}$$

where $\xi(0-)\triangleq 0$ and $\xi(1+)\triangleq\xi(1)$. Define an order on the graph $\Gamma'(\xi)$ by setting $(u_1,t_1)<(u_2,t_2)$ if either

- *$t_1<t_2$; or*
- *$t_1=t_2$ and $|\xi(t_1-)-u_1|<|\xi(t_2-)-u_2|$.*

We call a continuous nondecreasing function $(u,t)=\big((u(s),t(s)),\ s\in[0,1]\big)$ from $[0,1]$ to $\mathbb{R}\times[0,1]$ a parametrization of $\Gamma'(\xi)$—or a parametrization of ξ—if $\Gamma'(\xi)=\{(u(s),t(s)):s\in[0,1]\}$.

Definition 2. *Define the M_1' metric on $\mathbb{D}$ as follows*

$$d_{M_1'}(\xi,\zeta)\triangleq\inf_{\substack{(u,t)\in\Gamma'(\xi)\\(v,r)\in\Gamma'(\zeta)}}\{\|u-v\|_\infty+\|t-r\|_\infty\}.$$

Let $\mathbb{D}^{\uparrow} \triangleq \{\xi \in \mathbb{D} : \xi \text{ is nondecreasing and } \xi(0) \geq 0\}$.

Proposition A.1. *Suppose that $\hat{\xi}_0 \in \tilde{\mathbb{D}}$ with $\hat{\xi}_0(0) \geq 0$ and $\xi_n \in \mathbb{D}^{\uparrow}$ for each $n \geq 1$. If $T \triangleq \{t \in [0,1] : \xi_n(t) \to \hat{\xi}_0(t)\}$ is dense on $[0,1]$ and $1 \in T$, then $\xi_n \overset{M_1'}{\to} \xi_0 \in \mathbb{D}^{\uparrow}$ where $\xi_0(t) \triangleq \lim_{s \downarrow t} \hat{\xi}_0(s)$ for $t \in [0,1)$ and $\xi_0(1) \triangleq \hat{\xi}_0(1)$.*

Proof. It is easy to check that $\hat{\xi}_0$ has to be non-negative and non-decreasing, and for such $\hat{\xi}_0$, ξ_0 should be in $\mathbb{D}^{\uparrow}$. Let (x,t) be a parametrization of $\Gamma'(\hat{\xi}_0)$, and let $\epsilon > 0$ be given. Note that $\Gamma'(\xi_0)$ and $\Gamma'(\hat{\xi}_0)$ coincide. Therefore, the proposition is proved if we show that there exists an integer N_0 such that for each $n \geq N_0$, $\Gamma'(\xi_n)$ can be parametrized by some (y,r) such that

$$\|x-y\|_\infty + \|t-r\|_\infty \leq \epsilon. \tag{A.2}$$

We start with making an observation that one can always construct a finite number of points $S = \{s_i\}_{i=0,1,\dots,m} \subseteq [0,1]$ such that

(S1) $0 = s_0 < s_1 < \cdots < s_m = 1$

(S2) $t(s_i) - t(s_{i-1}) < \epsilon/4$ for $i = 1, \dots, m$

(S3) $x(s_i) - x(s_{i-1}) < \epsilon/8$ for $i = 1, \dots, m$

(S4) if $t(s_{k-1}) < t(s_k) < t(s_{k+1})$ then $t(s_k) \in T$

(S5) if $t(s_{k-1}) < t(s_k) = t(s_{k+1})$, then $t(s_{k-1}) \in T$; if, in addition, $k-1 > 0$, then $t(s_{k-2}) < t(s_{k-1})$

(S6) if $t(s_{k-1}) = t(s_k) < t(s_{k+1})$, then $t(s_{k+1}) \in T$; if, in addition, $k+1 < m$, then $t(s_{k+1}) < t(s_{k+2})$

One way to construct such a set is to start with S such that (S1), (S2), and (S3) are satisfied. This is always possible because x and t are continuous and non-decreasing. Suppose that (S4) is violated for some three consecutive points in S, say s_{k-1}, s_k, s_{k+1}. We argue that it is always possible to eliminate this violation by either adding an additional point $\hat{s}_k$ or moving s_k slightly. More specifically, if there exists $\hat{s}_k \in (s_{k-1}, s_{k+1}) \setminus \{s_k\}$ such that $t(\hat{s}_k) = t(s_k)$, add $\hat{s}_k$ to S. If there is no such $\hat{s}_k$, $t(\cdot)$ has to be strictly increasing at s_k, and hence, from the continuity of x and t along with the fact that T is dense, we can deduce that there has to be $\tilde{s}_k \in (s_{k-1}, s_{k+1})$ such that $t(\tilde{s}_k) \in T$ and $|t(\tilde{s}_k) - t(s_k)|$ and $|x(\tilde{s}_k) - x(s_k)|$ are small enough so that (S2) and (S3) are still satisfied when we replace s_k with $\tilde{s}_k$ in S. Iterating this procedure, we can construct S so that (S1)-(S4) are satisfied. Now turning to (S5), suppose that it is violated for three consecutive points s_{k-1}, s_k, s_{k+1} in S. Since T is dense and t is continuous, one can find $\hat{s}_k$ between s_{k-1} and s_k such that $t(s_{k-1}) < t(\hat{s}_k) < t(s_k)$ and $t(\hat{s}_k) \in T$. Note that after adding $\hat{s}_k$ to S, (S2), (S3), and (S4) should still hold while the number of triplets that violate (S5) is reduced by one. Repeating this procedure for each triplet that violates (S5), one can construct a new S which satisfies (S1)-(S5). One can also check that the same procedure for the triplets that violate (S6) can reduce the number of triplets that violate (S6) while not introducing any new violation for (S2), (S3), (S4), and (S5). Therefore, S can be augmented so that the resulting finite set satisfies (S6) as well. Set $\hat{S} \triangleq \{s_i \in S : t(s_i) \in T,\ t(s_{i-1}) < t(s_i) \text{ in case } i > 0,\ t(s_i) < t(s_{i+1}) \text{ in case } i < m\}$ and let N_0 be such that $n \geq N_0$ implies $|\xi_n(t(s_i)) - \hat{\xi}_0(t(s_i))| < \epsilon/8$ for all $s_i \in \hat{S}$. Now we will fix $n \geq N_0$ and proceed to showing that we can re-parametrize an arbitrary parametrization (y', r') of $\Gamma(\xi_n)$ to obtain a new parametrization (y, r) such that (A.2) is satisfied. Let (y', r') be an arbitrary parametrization of $\Gamma(\xi_n)$. For each i such that $s_i \in \hat{S}$, let $s_i' \triangleq \max\{s \geq 0 : r'(s) = t(s_i)\}$ so that $r'(s_i') = t(s_i)$ and $\xi_n(r'(s_i')) = y'(s_i')$. For i's such that $s_i \in S \setminus \hat{S}$, note that there are three possible cases: $t(s_i) \in (0,1)$, $t(s_i) = 0$, and $t(s_i) = 1$. Since the other cases can be handled in similar (but simpler) manners, we focus on the case $t(s_i) \in (0,1)$. In this case, one can check that there exist k and j such that $k \le i \le k+j$, $t(s_{k-1}) < t(s_k) = t(s_{k+j}) < t(s_{k+j+1})$, and $s_{k-1}, s_{k+j+1} \in \hat{S}$. Here we assume that $k > 1$; the case $k = 1$ is essentially identical but simpler—hence omitted. Note that from the monotonicity of $\hat{\xi}_0$ and (A.1),

$$x(s_{k-2}) \le \hat{\xi}_0(t(s_{k-2})+) \le \hat{\xi}_0(t(s_{k-1})-) \le \hat{\xi}_0(t(s_{k-1})) \le \hat{\xi}_0(t(s_{k-1})+) \le \hat{\xi}_0(t(s_k)-) \le x(s_k),$$

i.e., $\hat{\xi}_0(t(s_{k-1})) \in [x(s_{k-2}), x(s_k)]$, which along with (S3) implies $|\hat{\xi}_0(t(s_{k-1})) - x(s_{k-1})| < \epsilon/8$. From this, (S5), and the constructions of s'_{k-1} and N_0,

$$\begin{aligned}|y'(s'_{k-1}) - x(s_{k-1})| &= |\xi_n(r'(s'_{k-1})) - x(s_{k-1})| \\ &= |\xi_n(r'(s'_{k-1})) - \hat{\xi}_0(t(s_{k-1}))| + |\hat{\xi}_0(t(s_{k-1})) - x(s_{k-1})| \\ &= |\xi_n(t(s_{k-1})) - \hat{\xi}_0(t(s_{k-1}))| + |\hat{\xi}_0(t(s_{k-1})) - x(s_{k-1})| < \epsilon/4.\end{aligned}$$

Following the same line of reasoning, we can show that $|y'(s'_{k+j+1}) - x(s_{k+j+1})| < \epsilon/4$. Noting that both x and y' are nondecreasing, there have to exist $s'_k, s'_{k+1}, \ldots, s'_{k+j}$ such that $s'_{k-1} < s'_k < \cdots < s'_{k+j} < s'_{k+j+1}$ and $|y'(s'_l) - x(s_l)| < \epsilon/4$ for $l = k, k+1, \ldots, k+j$. Note also that from (S2),

$$t(s_l) - \epsilon/4 = t(s_k) - \epsilon/4 < t(s_{k-1}) = r'(s'_{k-1}) \le r'(s'_l) \le r'(s'_{k+j+1}) = t(s_{k+j+1}) < t(s_{k+j}) + \epsilon/4 = t(s_l) + \epsilon/4,$$

and hence, $|r'(s'_l) - t(s_l)| < \epsilon/4$ for $l = k, \ldots, k+j$ as well. Repeating this procedure for the i's for which s'_i is not designated until there is no such i's are left, we can construct $s'_1, \ldots, s'_m$ in such a way that

$$|y'(s'_i) - x(s_i)| < \epsilon/4 \qquad \text{and} \qquad |r'(s'_i) - t(s_i)| < \epsilon/4$$

for all i's. Now, define a (piecewise linear) map $\lambda : [0,1] \to [0,1]$ by setting $\lambda(s_i) = s'_i$ at each s_i's and interpolating (s_i, s'_i)'s in between. Then, $y \triangleq y' \circ \lambda$ and $r \triangleq r' \circ \lambda$ consist a parametrization (y, r) of $\Gamma(\xi_n)$ such that $|x(s_i) - y(s_i)| < \epsilon/4$ and $|t(s_i) - r(s_i)| < \epsilon/4$ for each $i = 1, \ldots, m$. Due to the monotonicity of x, y, t, and r along with (S2) and (S3), we conclude that $\|y - x\|_\infty < \epsilon/2$ and $\|t - r\|_\infty < \epsilon/2$, proving (A.2). □

Proposition A.2. *Let K be a subset of $\mathbb{D}^\uparrow$. If $M \triangleq \sup_{\xi \in K} \|\xi\|_\infty < \infty$ then K is relatively compact w.r.t. the M'_1 topology.*

Proof. Let $\{\xi_n\}_{n=1,2,\ldots}$ be a sequence in K. We will prove that there exists a subsequence $\{\xi_{n_k}\}_{k=1,2,\ldots}$ and $\xi_0 \in \mathbb{D}$ such that $\xi_{n_k} \overset{M'_1}{\to} \xi_0$ as $k \to \infty$. Let $T \triangleq \{t_n\}_{n=1,2,\ldots}$ be a dense subset of $[0,1]$ such that $1 \in T$. By the assumption, $\sup_{n=1,2,\ldots} |\xi_n(t_1)| < M$, and hence there is a subsequence $\{n_k^{(1)}\}_{k=1,2,\ldots}$ of $\{1,2,\ldots\}$ such that $\xi_{n_k^{(1)}}(t_1)$ converges to a real number $x_1 \in [-M, M]$. For each $i \ge 1$, given $\{n_k^{(i)}\}$, one can find a further subsequence $\{n_k^{(i+1)}\}_{k=1,2,\ldots}$ of $\{n_k^{(i)}\}_{k=1,2,\ldots}$ in such a way that $\xi_{n_k^{(i+1)}}(t_{i+1})$ converges to a real number x_{i+1}. Let $n_k \triangleq n_k^{(k)}$ for each $k = 1, 2, \ldots$. Then, $\xi_{n_k}(t_i) \to x_i$ as $k \to \infty$ for each $i = 1, 2, \ldots$. Define a function $\hat{\xi}_0 : T \to \mathbb{R}$ on T so that $\hat{\xi}_0(t_i) = x_i$. We claim that $\hat{\xi}_0$ has left limit everywhere; more precisely, we claim that for each $s \in (0,1]$, if a sequence $\{s_n\} \subseteq T \cap [0, s)$ is such that $s_n \to s$ as $n \to \infty$, then $\hat{\xi}_0(s_n)$ converges as $n \to \infty$. (With a similar argument, one can show that $\hat{\xi}_0$ has right limit everywhere—i.e., for each $s \in [0,1)$, if a sequence $\{s_n\} \subseteq T \cap (s, 1]$ is such that $s_n \to s$ as $n \to \infty$, then $\hat{\xi}_0(s_n)$ converges as $n \to \infty$.) To prove this claim, we proceed with proof by contradiction; suppose that the conclusion of the claim is not true—i.e., $\hat{\xi}_0(s_n)$ is not convergent. Then, there exist a $\epsilon > 0$ and a subsequence r_n of s_n such that

$$|\hat{\xi}_0(r_{n+1}) - \hat{\xi}_0(r_n)| > \epsilon. \tag{A.3}$$

Note that since $\hat{\xi}_0$ is a pointwise limit of nondecreasing functions $\{\xi_{n_k}\}$ (restricted on T),

- $\hat{\xi}_0$ is also nondecreasing on T, (monotonicity)
- $\sup_{t\in T}|\hat{\xi}_0(t)| < M$. (boundedness)

However, these two are contradictory to each other since the monotonicity together with (A.3) implies $\hat{\xi}_0(r_{N_0+j}) > \hat{\xi}_0(r_{N_0}) + j\epsilon$, which leads to the contradiction to the boundedness for a large enough j. This proves the claim.

Note that the above claim means that $\hat{\xi}_0$ has both left and right limit at each point of $T\cap(0,1)$, and due to the monotonicity, the function value has to be between the left limit and the right limit. Since T is dense in $[0,1]$, we can extend $\hat{\xi}$ from T to $[0,1]$ by setting $\hat{\xi}_0(t) \triangleq \lim_{\substack{t_i\to t\\ t_i>t}} \hat{\xi}_0(t_i)$ for $t\in[0,1]\setminus T$. Note that such $\hat{\xi}_0$ is an element of $\tilde{\mathbb{D}}$ and satisfies the conditions of Proposition A.1. We therefore conclude that $\xi_{n_k}\to\xi_0\in\mathbb{D}^\uparrow$ in M_1' as $k\to\infty$, where $\xi_0(t)\triangleq\lim_{s\downarrow t}\hat{\xi}_0(s)$ for $t\in[0,1)$ and $\xi_0(1)\triangleq\hat{\xi}_0(1)$. This proves that K is indeed relatively compact. □

Recall that our rate function for one-sided compound poisson processes is as follows:

$$I_{M_1'}(\xi) = \begin{cases}\sum_{t\in[0,1]}\big(\xi(t)-\xi(t-)\big)^\alpha & \text{if } \xi \text{ is a non-decreasing pure jump function with } \xi(0)\geq 0,\\ \infty & \text{otherwise.}\end{cases}$$

Proposition A.3. *$I_{M_1'}$ is a good rate function w.r.t. the M_1' topology.*

Proof. In view of Proposition A.2, it is enough to show that the sublevel sets of $I_{M_1'}$ are closed. Let a be an arbitrary finite constant, and consider the sublevel set $\Psi_{I_{M_1'}}(a)\triangleq\{\xi\in\mathbb{D}: I_{M_1'}(\xi)\leq a\}$. Let $\xi^c\in\mathbb{D}$ be any given path that does not belong to $\Psi_{I_{M_1'}}(a)$. We will show that there exists $\epsilon>0$ such that $d_{M_1'}\big(\xi^c,\Psi_{I_{M_1'}}(a)\big)\geq\epsilon$. Note that $\Psi_{I_{M_1'}}(a)^{\mathsf{c}} = A\cup B\cup C\cup D$ where

$A=\{\xi\in\mathbb{D}:\xi(0)<0\}$,
$B=\{\xi\in\mathbb{D}:\xi$ is not a non-decreasing function$\}$,
$C=\{\xi\in\mathbb{D}:\xi$ is non-decreasing but not a pure jump function$\}$,
$D=\{\xi\in\mathbb{D}:\xi$ is a non-decreasing pure jump function with $\xi(0)\geq 0$ and $\sum_{t\in[0,1]}\big(\xi(t)-\xi(t-)\big)^\alpha > a\}$.

In each case, we will show that ξ^c is bounded away from $\Psi_{I_{M_1'}}(a)$. In case $\xi^c\in A$, note that for any parametrization (x,t) of ξ^c, there has to be $s^*\in[0,1]$ such that $x(s^*)=\xi^c(0)<0$. On the other hand, $y(s)\geq 0$ for all $s\in[0,1]$ for any parametrization (y,r) of ζ such that $\zeta\in\Psi_I(a)$, and hence, $\|x-y\|_\infty\geq\xi^c(0)$. Therefore,

$$d_{M_1'}(\xi^c,\zeta)\geq\inf_{\substack{(x,t)\in\Gamma(\xi^c)\\(y,r)\in\Gamma(\zeta)}}\|x-y\|_\infty\geq|\xi^c(0)|.$$

Since ζ was an arbitrary element of $\Psi_{I_{M_1'}}(a)$, we conclude that $d_{M_1'}(\xi^c,\Psi_I(a))\geq\epsilon$ with $\epsilon=|\xi^c(0)|$.

It is straightforward with similar argument to show that any $\xi^c\in B$ is bounded away from $\Psi_I(a)^c$.

If $\xi^c\in C$, there has to be T_s and T_t such that $0\leq T_s<T_t\leq 1$, ξ^c is continuous on $[T_s,T_t]$, and $c\triangleq\xi^c(T_t)-\xi^c(T_s)>0$. Pick a small enough $\epsilon\in(0,1)$ so that

$$(4\epsilon)^{\alpha-1}(c-5\epsilon)>a. \tag{A.4}$$

Note that since ξ^c is uniformly continuous on $[T_s,T_t]$, there exists $\delta>0$ such that $|\xi^c(t)-\xi^c(s)|<\epsilon$ if $|t-s|\leq\delta$. In particular, we pick δ so that $\delta<\epsilon$ and $T_s+\delta<T_t-\delta$. We claim that

$$d_{M_1'}(\Psi_{I_{M_1'}}(a),\xi^c)\geq\delta.$$

Suppose not. That is, there exists $\zeta \in \Psi_{I_{M_1'}}(a)$ such that $d_{M_1'}(\zeta, \xi^c) < \delta$. Let $(x,t) \in \Gamma(\xi^c)$ and $(y,r) \in \Gamma(\zeta)$ be the parametrizations of ξ^c and ζ, respectively, such that $\|x-y\|_\infty + \|t-r\|_\infty < \delta$. Since $I_{M_1'}(\zeta) \le a < \infty$, one can find a finite set $K \subseteq \{t \in [0,1] : \zeta(t) - \zeta(t-) > 0\}$ of jump times of ζ in such a way that $\sum_{t \notin K} \big(\zeta(t) - \zeta(t-)\big)^\alpha < \epsilon$. Note that since $\epsilon \in (0,1)$, this implies that $\sum_{t \notin K} \big(\zeta(t) - \zeta(t-)\big) < \epsilon$. Let $T_1, \dots, T_k$ denote (the totality of) the jump times of ζ in $K \cap (T_s + \delta, T_t - \delta]$, and let $T_0 \triangleq T_s + \delta$ and $T_{k+1} \triangleq T_t - \delta$. That is, $\{T_1, \dots, T_k\} = K \cap (T_s + \delta, T_t - \delta] = K \cap (T_0, T_{k+1}]$. Note that

- There exist s_0 and s_{k+1} in $[0,1]$ such that

$$y(s_0) = \zeta(T_0), \qquad r(s_0) = T_0, \qquad y(s_{k+1}) = \zeta(T_{k+1}), \qquad r(s_{k+1}) = T_{k+1}.$$

- For each $i = 1, \dots, k$, there exists s_i^+ and s_i^- such that

$$r(s_i^+) = r(s_i^-) = T_i, \qquad y(s_i^+) = \zeta(T_i), \qquad y(s_i^-) = \zeta(T_i-).$$

Since $t(s_{k+1}) \in [r(s_{k+1}) - \delta, r(s_{k+1}) + \delta] \subseteq [T_s, T_t]$, and ξ^c is continuous on $[T_s, T_t]$ and non-decreasing,

$$y(s_{k+1}) \ge x(s_{k+1}) - \delta = \xi^c(t(s_{k+1})) - \delta \ge \xi^c(r(s_{k+1}) - \delta) - \delta = \xi^c(T_{k+1} - \delta) - \delta \ge \xi^c(T_{k+1}) - \epsilon - \delta \ge \xi^c(T_{k+1}) - 2\epsilon.$$

Similarly,

$$y(s_0) \le x(s_0) + \delta = \xi^c(t(s_0)) + \delta \le \xi^c(r(s_0) + \delta) + \delta = \xi^c(T_0 + \delta) + \delta \le \xi^c(T_0) + \epsilon + \delta \le \xi^c(T_0) + 2\epsilon.$$

Subtracting the two equations,

$$y(s_{k+1}) - y(s_0) \ge \xi^c(T_{k+1}) - \xi^c(T_0) - 4\epsilon = c - 4\epsilon.$$

Note that

$$\begin{aligned}\sum_{i=1}^k \big(\zeta(T_i) - \zeta(T_i-)\big) &= \zeta(T_{k+1}) - \zeta(T_0) - \sum_{t \in (T_0, T_{k+1}] \cap K^c} (\zeta(t) - \zeta(t-)) \ge \zeta(T_{k+1}) - \zeta(T_0) - \epsilon \\ &= y(s_{k+1}) - y(s_0) - \epsilon \ge c - 5\epsilon. \end{aligned} \tag{A.5}$$

On the other hand,

$$\begin{aligned} y(s_i^+) - y(s_i^-) &\le (x(s_i^+) + \delta) - (x(s_i^-) - \delta) = x(s_i^+) - x(s_i^-) + 2\delta \le \xi^c(t(s_i^+)) - \xi^c(t(s_i^-)) + 2\delta \\ &\le \xi^c(r(s_i^+) + \delta) - \xi^c(r(s_i^-) - \delta) + 2\delta \le \xi^c(T_i + \delta) - \xi^c(T_i - \delta) + 2\delta \le 2\epsilon + 2\delta \le 4\epsilon. \end{aligned}$$

That is, $(\zeta(T_i) - \zeta(T_i-))^{\alpha-1} = (y(s_i^+) - y(s_i^-))^{\alpha-1} \ge (4\epsilon)^{\alpha-1}$. Combining this with (A.5),

$$I_{M_1'}(\zeta) \ge \sum_{i=1}^k \big(\zeta(T_i) - \zeta(T_i-)\big)^\alpha = \sum_{i=1}^k \big(\zeta(T_i) - \zeta(T_i-)\big)\big(\zeta(T_i) - \zeta(T_i-)\big)^{\alpha-1} \ge (c - 5\epsilon)(4\epsilon)^{\alpha-1} > a,$$

which is contradictory to the assumption that $\zeta \in \Psi_{I_{M_1'}}(a)$. Therefore, the claim that ξ^c is bounded away from $\Psi_{I_{M_1'}}(a)$ by δ is proved.

Finally, suppose that $\xi^c \in D$. That is, there exists $\{(z_i, u_i) \in \mathbb{R}_+ \times [0,1]\}_{i=1,\dots}$ such that $\xi^c = \sum_{i=1}^\infty z_i \mathbb{1}_{[u_i,1]}$ where u_i's are all distinct and $\sum_{i=1}^\infty z_i^\alpha > a$. Pick sufficiently large k and sufficiently small $\delta > 0$ such that $\sum_{i=1}^k (z_i - 2\delta)^\alpha > a$ and $u_{i+1} - u_i > 2\delta$ for $i = 1, \dots, k-1$. We claim that $d_{M_1'}(\zeta, \xi^c) \ge \delta$ for any $\zeta \in \Psi_{I_{M_1'}}(a)$. Suppose not and there is $\zeta \in \Psi_{I_{M_1'}}(a)$ such that $\|x - y\|_\infty + \|t - r\|_\infty < \delta$ for some parametrizations $(x,t) \in \Gamma(\xi^c)$ and $(y,r) \in \Gamma(\zeta)$. Note first that there are s_i^+'s and s_i^-'s for each $i = 1,\dots,k$ such that $t(s_i^-) = t(s_i^+) = u_i$, $x(s_i^-) = \xi^c(u_i-)$, and $x(s_i^+) = \xi^c(u_i)$. Since $y(s_i^+) \ge x(s_i^+) - \delta = \xi^c(u_i) - \delta$ and $y(s_i^-) \le x(s_i^-) + \delta = \xi^c(u_i-) + \delta$,

$$\zeta(r(s_i^+)) - \zeta(r(s_i^-)) \ge y(s_i^+) - y(s_i^-) \ge \xi^c(u_i) - \xi^c(u_i^-) - 2\delta = z_i - 2\delta.$$

Note that by construction, $r(s_i^+) < t(s_i^+) + \delta = u_i + \delta < u_{i+1} - \delta = t(s_{i+1}^-) - \delta < r(s_{i+1}^-)$ for each $i = 1,\dots,k-1$, and hence, along with the subadditivity of $x \mapsto x^\alpha$,

$$I_{M_1'}(\zeta) = \sum_{t\in[0,1]} \big(\zeta(t) - \zeta(t-)\big)^\alpha \ge \sum_{i=1}^k [\zeta(r(s_i^+)) - \zeta(r(s_i^-))]^\alpha \ge \sum_{i=1}^k (z_i - 2\delta)^\alpha > a,$$

which is contradictory to the assumption $\zeta \in \Psi_{I_{M_1'}}(a)$. □

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