MATHEMATICS OF OPERATIONS RESEARCH
Articles in Advance, pp. 1–24
ISSN 0364-765X (print), ISSN 1526-5471 (online)
https://pubsonline.informs.org/journal/moor

Lyapunov Conditions for Differentiability of Markov Chain Expectations

Chang-Han Rhee[a,*] **Peter W. Glynn**[b]

[a] Industrial Engineering and Management Sciences, Northwestern University, Evanston, Illinois 60208; [b] Management Science and Engineering, Stanford University, Stanford, California 94305
*Corresponding author
Contact: chang-han.rhee@northwestern.edu, https://orcid.org/0000-0002-1651-4677 (C-HR); glynn@stanford.edu, https://orcid.org/0000-0003-1370-6638 (PWG)

Received: March 17, 2021
Revised: April 27, 2022
Accepted: June 1, 2022
Published Online in Articles in Advance: December 30, 2022

MSC2020 Subject Classification: 60J05

https://doi.org/10.1287/moor.2022.1328

Abstract. We consider a family of Markov chains whose transition dynamics are affected by model parameters. Understanding the parametric dependence of (complex) performance measures of such Markov chains is often of significant interest. The derivatives and their continuity of the performance measures w.r.t. the parameters play important roles, for example, in numerical optimization of the performance measures, and quantification of the uncertainties in the performance measures when there are uncertainties in the parameters from the statistical estimation procedures. In this paper, we establish conditions that guarantee the smoothness of various types of intractable performance measures—such as the stationary and random horizon discounted performance measures—of general state space Markov chains and provide probabilistic representations for the derivatives.

Funding: C.-H. Rhee is supported by the National Science Foundation [Grant CMMI-2146530].

Keywords: Markov chain • sensitivity analysis • derivative estimation

1. Introduction

Let $X = (X_n : n \geq 0)$ be a Markov chain taking values in a state space S. For the purpose of this paper, the state space S may be discrete or continuous. In many applications settings, it is natural to consider the behavior of X as a function of a parameter θ that affects the transition dynamics of the process. In particular, suppose that for each θ in some open neighborhood of $\theta_0 \in \mathbb{R}^d$, $P(\theta) = (P(\theta, x, dy) : x, y \in S)$ defines the one-step transition kernel of X associated with parameter choice θ. In such a setting, computing the derivative of some application-specific expectation is often of interest.

Such derivatives play a key role when one is numerically optimizing an objective function, defined as a Markov chain's expected value, over the decision parameter θ. In addition, such derivatives describe the sensitivity of the expected value under consideration to perturbations in θ. Such sensitivities are valuable in statistical applications, and arise when one applies (for example) the delta method in conjunction with estimating equations involving some expectation of the observed Markov chain; see, for example, Lehmann and Casella [14]. More generally, sensitivity analysis is important when one is interested in understanding how robust the model is to uncertainties in the input parameters.

In particular, suppose that θ is a vector of statistical parameters, and that a data set of size n has been collected to estimate the underlying true parameter θ^*. In significant generality, the associated estimator $\hat{\theta}_n$ for θ^* will satisfy a central limit theorem (CLT) of the form

$$n^{1/2}(\hat{\theta}_n - \theta^*) \Rightarrow N(0, C)$$

as $n \to \infty$, where $\Rightarrow$ denotes weak convergence and $N(0, C)$ is a normally distributed random column vector with mean 0 and covariance matrix C; see, for example, Ibragimov and Has'minskii [10]. In many applications, one wishes to understand how the uncertainty in our estimator $\hat{\theta}_n$ of θ^* propagates through the model associated with X to produce uncertainty in output measures of interest. Suppose, for example, that the decision maker focuses on a performance measure of the form $\alpha(\theta) = \mathbf{E}^{\theta} Z$, where Z is some appropriately chosen random variable (rv) and $\mathbf{E}^{\theta}(\cdot)$ is the expectation operator under which X evolves according to $\mathbf{P}(\theta)$. If $\alpha(\cdot)$ is differentiable at θ^*, then

$$n^{1/2}(\alpha(\hat{\theta}_n) - \alpha(\theta^*)) \Rightarrow \nabla\alpha(\theta^*) N(0, C)$$

as $n \to \infty$, where $\nabla\alpha(\theta)$ is the (row) gradient vector evaluated at θ; see Serfling [18]. If, in addition, $\nabla\alpha(\cdot)$ is continuous at θ^* and C can be consistently estimated from the observed data via an estimator C_n, the interval

$$\left[\alpha(\hat{\theta}_n) - z\frac{\sigma_n}{\sqrt{n}}, \alpha(\hat{\theta}_n) + z\frac{\sigma_n}{\sqrt{n}}\right] \tag{1.1}$$

is an asymptotic $100(1-\delta)\%$ confidence interval for $\alpha(\theta^*)$ (provided $\nabla\alpha(\theta^*)C\nabla\alpha(\theta^*)^T > 0$), where z is chosen so that $P(-z \le N(0,1) \le z) = 1 - \delta$ and $\sigma_n = \sqrt{\nabla\alpha(\hat{\theta}_n)C_n\nabla\alpha(\hat{\theta}_n)^T}$. (We note that the continuity of $\nabla\alpha(\cdot)$ at θ^* is needed for the consistency of the estimator $\nabla\alpha(\hat{\theta}_n)$ for $\nabla\alpha(\theta^*)$.) The confidence interval (1.1) provides the modeler with the desired sensitivity and robustness of the model described by X to the statistical uncertainties present in the estimation of θ^*. In summary, the delta method rests on three conditions: (i) the existence of the derivative of $\alpha(\cdot)$ at θ^*, (ii) the continuity of the derivative at θ^*, and (iii) one's ability to compute the derivative. This paper provides close to the best possible conditions for verifying the existence and the continuity of the derivatives in the general state space Markov chain settings and provides probabilistic representations that lead to simulation estimators for the computation of those derivatives.

The problem of determining such differentiability has a long history and has been addressed through various approaches, including weak differentiation (Pflug [16], Vázquez-Abad and Kushner [19]), likelihood ratio (Glynn and L'Ecuyer [3]), measure-valued differentiation (Heidergott and Vázquez-Abad [8], Heidergott et al. [9]), and derivative regeneration (Glasserman [2]). However, most of the previous approaches are limited to special classes of problems. For example, the results in Vázquez-Abad and Kushner [19] and Pflug [16] are limited to bounded performance functionals; Glasserman [2] imposes special structures in the the transition dynamics of the Markov chains and their parametrization; Glynn and L'Ecuyer [3] assume for random horizon expectations that the associated stopping times have finite exponential moments, and for stationary expectations that the Markov chain is geometrically ergodic. Heidergott and Vázquez-Abad [8] provide conditions that do not require the geometric ergodicity for random horizon and stationary performance measures based on a measure-valued differentiation approach. However, their sufficient conditions are difficult to verify in general and still require that the associated stopping times possess at least finite second moment. Also based on measure-valued differentiation, Heidergott et al. [9] study stationary expectations and provide sufficient conditions verifiable based on the model building blocks without such restrictions. However, their sufficient conditions require geometric ergodicity of the Markov chain. In this paper, on the other hand, we provide sufficient conditions (verifiable based on the one-step transition dynamics) for random horizon expectations that do not require any moment conditions for the associated stopping times—hence, allowing even infinite horizon expectations. For stationary expectations, we provide (again, easily verifiable) sufficient conditions that do not require geometric ergodicity.

We illustrate that our differentiability criterion is close to minimal via the example of the G/G/1 queue waiting time sequence with heavy tailed service times. If $(X_n : n \ge 0)$ is the waiting time sequence for the G/G/1 queue, it is well known that the service time distribution needs to possess a finite $(p+1)$ th moment in order for the stationary expectation $\alpha(\theta) = \mathbf{E}^\theta X_\infty^p$ to be finite. When we assume that the service time has a Pareto tail, such a moment condition corresponds to the case where the shape parameter r of the Pareto distribution satisfies $r > p + 1$; hence, this is a necessary condition for the existence of the derivative. We show that with our Lyapunov strategy, one can successfully prove the differentiability of α with this minimal condition.

For both random horizon expectations and stationary expectations, we provide two different sets of sufficient conditions—one based on operator-theoretic arguments and the other one based on Lyapunov conditions. These two approaches complement each other. The operator ideas are simpler to apply, and immediately imply existence of derivatives over the entire space of functions with finite weighted norm. The Lyapunov approach, on the other hand, allows one to craft a Lyapunov function that is specially tuned to the specific functional of interest, and hence allows one to obtain the weakest conditions for the given functional.

We point out that the theory developed in this paper extends easily to nonexplosive Markov jump processes, because the expectations discussed in Sections 2 through 4 correspond to linear systems involving the embedded discrete time Markov chains.

The rest of the paper is organized as follows. Section 2 develops a preliminary theory for both random horizon expectations and stationary expectations based on simple and clean operator-theoretic arguments. Section 3 provides more general criteria for differentiability of random horizon expectations based on stochastic Lyapunov arguments. In Section 4, we apply the Lyapunov approach to studying differentiability for stationary expectations. Section 5 concludes the paper with a brief discussion of the Lyapunov conditions for general random horizon expectations that cannot be written in the form studied in the previous sections.

2. Operator-Theoretic Criteria for Differentiability

We start by studying differentiability in a setting in which one can use operator arguments to establish existance of derivatives. In this operator setting, the proofs and theorem statements are especially straightforward.

It should be noted that the probabilistic representations of the derivatives, whose existence are studied in this section, lead immediately to related simulation algorithms that can be deployed to compute the derivatives numerically. The specific forms of such simulation estimators will be discussed in Section 3 and Section 4, where we provide more general conditions for the differentiability.

Consider a Markov chain $X = (X_n : n \ge 0)$ living on state space S, with one-step transition kernel $P = (P(x, dy) : x, y \in S)$, where

$$P(x, dy) = P(X_{n+1} \in dy | X_n = x)$$

for $x, y \in S$. We focus first on expectations of the form

$$u^*(x) = \mathbf{E}_x \sum_{j=0}^{T-1} \exp\left(\sum_{k=0}^{j-1} g(X_k)\right) f(X_j) + \exp\left(\sum_{k=0}^{T-1} g(X_k)\right) f(X_T), \tag{2.1}$$

where $T = \inf\{n \ge 0 : X_n \in C^c\}$ is the first hitting time of the target set $C^c \subseteq S$, $f : S \to \mathbb{R}_+$, $g : S \to \mathbb{R}$, and $\mathbf{E}_x(\cdot) \triangleq \mathbf{E}(\cdot | X_0 = x)$.

In (2.1), we permit the possibility that $C^c = \emptyset$, in which case $T = \infty$ almost surely (a.s.), and u^* is then to be interpreted as the infinite horizon discounted reward:

$$u^*(x) = \mathbf{E}_x \sum_{j=0}^{\infty} \exp\left(\sum_{k=0}^{j-1} g(X_k)\right) f(X_j).$$

In addition to subsuming infinite horizon discounted rewards, (2.1) also includes expected hitting times ($g \equiv 0$, $f = 1$ on C and $f = 0$ on C^c), exit probabilities ($g \equiv 0$, $f = 0$ on C, and $f(x) = I(x \in B)$ for $x \in C^c$, when one is considering $P(X_T \in B | X_0 = x)$), and many other natural Markov chain expectations.

It is easy to verify that

$$u^* = \sum_{n=0}^{\infty} K^n \tilde{f}, \tag{2.2}$$

where $K = (K(x, dy) : x, y \in C)$ is the nonnegative kernel for which

$$K(x, dy) = \exp(g(x)) P(x, dy) \tag{2.3}$$

for $x, y \in S$, and

$$\tilde{f}(x) = f(x) + \int_{C^c} \exp(g(x)) P(x, dy) f(y)$$

for $x \in C$. Here, we are taking advantage in (2.1) of the (common) notational convention that for a function $h : B \to \mathbb{R}$, a measure η on B, and kernels Q_1 and Q_2 on B, the scalar ηh, the function $Q_1 h$, the measure ηQ_1, and the kernel $Q_1 Q_2$ are respectively defined via

$$\eta h = \int_B h(y) \eta(dy),$$
$$(Q_1 h)(x) = \int_B h(y) Q_1(x, dy),$$
$$(\eta Q_1)(A) = \int_B \eta(dx) Q_1(x, A),$$
$$(Q_1 Q_2)(x, A) = \int_B Q_1(x, dy) Q_2(y, A),$$

whenever the right-hand sides are well-defined. Furthermore, we define the kernels Q^n via $Q^0(x, dy) = \delta_x(dy)$ (where $\delta_x(.)$ is a unit point mass at x), and $Q^n = QQ^{(n-1)}$ for $n \ge 1$.

Our goal is to use operator-theoretic tools to study the differentiability of (2.2). To this end, we start by defining the appropriate linear spaces that underlie this approach. Given a measurable space $(B, \mathcal{B})$, measurable

$w: B \to [1, \infty)$, and $h: B \to \mathbb{R}$, let $\|h\|_w = \sup\{|h(x)|/w(x) : x \in B\}$ and $L_w = \{h \in L : \|h\|_w < \infty\}$, where L is the set of measurable functions. For a linear operator $Q: L_w \to L_w$ and a functional $\eta: L_w \to \mathbb{R}$, set

$$|||Q|||_w = \sup_{h \in L_w : \|h\|_w \neq 0} \frac{\|Qh\|_w}{\|h\|_w}$$

and

$$\|\eta\|_w = \sup\{|\eta h| : h \in L_w, \|h\|_w \leq 1\}.$$

Then, let $\mathcal{L}_w = \{Q \in \mathcal{L} : |||Q|||_w < \infty\}$, and $\mathcal{M}_w = \{\eta \in \mathcal{M} : \|\eta\|_w < \infty\}$, where $\mathcal{L}$ and $\mathcal{M}$ are the sets of kernels, and measures, respectively. Each of the spaces L_w, $\mathcal{L}_w$, and $\mathcal{M}_w$ are Banach spaces under their respective norms and addition/scalar multiplication operations. Furthermore, for $Q_1, Q_2 \in \mathcal{L}_w$, $h \in L_w$, and $\eta \in \mathcal{M}_w$, it is easy to show that

$$|||Q_1 Q_2|||_w \leq |||Q_1|||_w \cdot |||Q_2|||_w \tag{2.4}$$

and

$$\begin{aligned} \|Qh\|_w &\leq |||Q|||_w \cdot \|h\|_w, \\ \|\eta Q\|_w &\leq \|\eta\|_w \cdot |||Q|||_w, \\ |\eta h| &\leq \|\eta\|_w \cdot \|h\|_w; \end{aligned} \tag{2.5}$$

see, for example, Dunford et al. [1] for the special case $w \equiv 1$. In view of (2.4), if $|||Q^m|||_w < 1$ for some $m \geq 1$, then $(I - Q)$ is invertible on $\mathcal{L}_w$ and

$$(I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n.$$

Given a parametrized family of kernels $(Q(\theta) \in \mathcal{L}_w : \theta \in (a, b))$, we say that $Q(\cdot)$ is *continuous* in $\mathcal{L}_w$ at $\theta_0 \in (a, b)$ if $|||Q(\theta_0 + h) - Q(\theta_0)|||_w \to 0$ as $h \to 0$, and *differentiable* in $\mathcal{L}_w$ at $\theta_0 \in (a, b)$ with derivative $Q'(\theta_0)$ if there exists a kernel $Q'(\theta_0) \in \mathcal{L}_w$ for which

$$\left|\left|\left| \frac{Q(\theta_0 + h) - Q(\theta_0)}{h} - Q'(\theta_0) \right|\right|\right|_w \to 0$$

as $h \to 0$. If $Q(\cdot)$ is differentiable in a neighborhood of θ_0 with derivative $Q'(\cdot)$, and $Q'(\cdot)$ is continuous at θ_0 in $\mathcal{L}_w$, then we say that $Q(\cdot)$ is continuously differentiable at θ_0. Similarly, given families $(f(\theta) \in L_w : \theta \in (a, b))$ and $(\eta(\theta) \in \mathcal{M}_w : \theta \in (a, b))$, we say that $f(\cdot)$ is *continuous* in L_w at θ_0 if $\|f(\theta_0 + h) - f(\theta_0)\|_w \to 0$ as $h \to 0$, and *differentiable* in L_w at θ_0 if there exists $f'(\theta_0) \in L_w$ such that

$$\left\| \frac{f(\theta_0 + h) - f(\theta_0)}{h} - f'(\theta_0) \right\|_w \to 0$$

as $h \to 0$; and $\eta(\cdot)$ is *continuous* in $\mathcal{M}_w$ at θ_0 if $\|\eta(\theta_0 + h) - \eta(\theta_0)\|_w \to 0$ as $h \to 0$, and *differentiable* in $\mathcal{M}_w$ at θ_0 if there exists $\eta'(\theta_0) \in \mathcal{M}_w$ such that

$$\left\| \frac{\eta(\theta_0 + h) - \eta(\theta_0)}{h} - \eta'(\theta_0) \right\|_w \to 0$$

as $h \to 0$. As in $\mathcal{L}_w$, if $f(\cdot)$ and $\eta(\cdot)$ are differentiable and their derivatives are continuous at θ_0 in L_w and $\mathcal{M}_w$, respectively, we say that they are *continuously differentiable*.

Assuming that $(Q(\theta) : \theta \in (a, b))$ is n-times differentiable in some neighborhood $\mathcal{N}$ of θ_0, with derivative $(Q^{(n)}(\theta) : \theta \in \mathcal{N})$, we say that $Q(\cdot)$ is $(n+1)$-times differentiable in $\mathcal{L}_w$ at θ_0 if $(Q^{(n)}(\theta) : \theta \in \mathcal{N})$ is differentiable at θ_0, with corresponding derivative $Q^{(n+1)}(\theta_0)$. We can analogously define $f^{(n+1)}(\theta_0)$ and $\eta^{(n+1)}(\theta_0)$ in the spaces L_w and $\mathcal{M}_w$, respectively. (We restrict our discussion in this paper to scalar θ, because the vector case introduces no new mathematical issues.)

We can now state our first result, pertaining to the differentiability of u^*.

Theorem 2.1. *Suppose there exists $w: C \to [1, \infty)$ and $\theta_0 \in (a, b)$ for which*

a. $|||K^m(\theta_0)|||_w < 1$ for some $m \geq 1$;

b. $K(\cdot)$ is (continuously) differentiable in $\mathcal{L}_w$ at θ_0, with derivative $K'(\theta_0)$;

c. $\tilde{f}(\cdot)$ is (continuously) differentiable in L_w at θ_0, with derivative $\tilde{f}'(\theta_0)$.

Then,

i. $(I - K(\theta))$ *is invertible on* L_w *for* θ *in a neighborhood of* θ_0*;*

ii. Setting $G(\theta) = (I - K(\theta))^{-1}$, $G(\cdot)$ *is (continuously) differentiable in* $\mathcal{L}_w$ *at* θ_0*, and*

$$G'(\theta_0) = G(\theta_0)K'(\theta_0)G(\theta_0);$$

iii. $u^*(\theta) = \sum_{n=0}^{\infty} K^n(\theta)\tilde{f}(\theta)$ *is (continuously) differentiable in* L_w *at* θ_0*, with*

$$(u^*)'(\theta_0) = G'(\theta_0)\tilde{f}(\theta_0) + G(\theta_0)\tilde{f}'(\theta_0). \tag{2.6}$$

If, in addition, $K(\cdot)$ *and* $\tilde{f}(\cdot)$ *are n-times (continuously) differentiable in* $\mathcal{L}_w$ *and* L_w*, respectively, at* θ_0*, then* $Q(\cdot)$ *and* $u^*(\cdot)$ *are n-times (continuously) differentiable at* θ_0 *in* $\mathcal{L}_w$ *and* L_w*, respectively, and* $Q^{(n)}(\theta_0)$ *and* $(u^*)^{(n)}(\theta_0)$ *can be recursively computed via*

$$G^{(n)}(\theta_0) = \sum_{j=0}^{n-1} \binom{n}{j} G^{(j)}(\theta_0)K^{(n-j)}(\theta_0)G(\theta_0) \tag{2.7}$$

and

$$(u^*)^{(n)}(\theta_0) = G(\theta_0)\left(\tilde{f}^{(n)}(\theta_0) + \sum_{j=0}^{n-1}\binom{n}{j} K^{(n-j)}(\theta_0)(u^*)^{(j)}(\theta_0)\right), \tag{2.8}$$

where, as usual, $K^{(0)}(\theta) \equiv K(\theta)$ *and* $\tilde{f}^{(0)}(\theta) = \tilde{f}(\theta)$.

Proof. Part (i) is obvious. For part (ii), note that assumptions (a) and (b) imply that there exists a neighborhood $\mathcal{N}$ of θ_0 for which $\sup_{\theta\in\mathcal{N}} |||K^m(\theta)|||_w < 1$ and $\sup_{\theta\in\mathcal{N}} |||K(\theta)|||_w < \infty$, from which it follows that $\sup_{\theta\in\mathcal{N}} |||G(\theta)|||_w < \infty$. Furthermore, because $(I - K(\theta_0 + h))G(\theta_0 + h) = G(\theta_0 + h)(I - K(\theta_0 + h)) = I$, evidently

$$(G(\theta_0 + h) - G(\theta_0))(I - K(\theta_0)) = G(\theta_0 + h)(K(\theta_0 + h) - K(\theta_0)),$$

so that

$$G(\theta_0 + h) - G(\theta_0) = G(\theta_0 + h)(K(\theta_0 + h) - K(\theta_0))G(\theta_0). \tag{2.9}$$

Clearly, this implies that $|||G(\theta_0 + h) - G(\theta_0)|||_w \le |||G(\theta_0 + h)|||_w |||K(\theta_0 + h) - K(\theta_0)|||_w |||G(\theta_0)|||_w \to 0$ as $h \to 0$, so $G(\cdot)$ is continuous in $\mathcal{L}_w$ at θ_0. Consequently, (2.9) implies that $G(\cdot)$ is differentiable in $\mathcal{L}_w$ at θ_0, with $G'(\theta_0) = G(\theta_0)K'(\theta_0)G(\theta_0)$. In case K' is continuous, continuity of G' is also immediate from this expression.

For part (iii), the result follows analogously from the identity

$$u^*(\theta_0 + h) - u^*(\theta_0) = G(\theta_0)(\tilde{f}(\theta_0 + h) - \tilde{f}(\theta_0)) + (G(\theta_0 + h) - G(\theta_0))\tilde{f}(\theta_0 + h).$$

The proof for the n-fold derivatives for $n \ge 2$ is very similar and therefore omitted. □

Remark 2.1. Suppose that $K(\cdot)$ possesses a density $(k(\cdot, x, y) : x, y \in C)$ that is n-times differentiable (with (pointwise) derivative $(k^{(n)}(\cdot, x, y) : x, y \in C)$). For $\epsilon > 0$ and $0 \le j \le n$, let $\tilde{\omega}_\epsilon^{(j)}(x, y) = \sup_{|\theta-\theta_0|<\epsilon} |k^{(j)}(\theta, x, y)|$. Then, the conditions

$$\sup_{x\in C} \int_C K^m(\theta_0, x, dy)\frac{w(y)}{w(x)} < 1 \quad \text{for some } m \ge 1, \tag{2.10}$$

$$\sup_{x\in C} \int_C \omega_\epsilon^{(j)}(x, y)\frac{w(y)}{w(x)} K(\theta_0, x, dy) < \infty, \tag{2.11}$$

and

$$\sup_{x\in C} \int_{C^c} (1 + \tilde{\omega}_\epsilon^{(j)}(x, y))\frac{|f(y)|}{w(x)} K(\theta_0, x, dy) < \infty, \tag{2.12}$$

for $j = 0, \ldots, n$ imply (a), (b), and (c) of Theorem 2.1, implying the validity of (2.7) and (2.8).

There is an analogous differentiability result for measures. For a given initial distribution μ on C, let ν be the measure defined by

$$\nu(dy) = \mathbf{E}_\mu \sum_{j=0}^{T-1} \exp\left(\sum_{k=0}^{j-1} g(X_k)\right) \mathbb{I}(X_j \in dy)$$

for $y \in S$, where $\mathbf{E}_\mu(\cdot) \triangleq \int_C \mu(dx)\mathbf{E}_x(\cdot)$. Then,

$$\nu = \sum_{n=0}^{\infty} \mu K^n,$$

where K is defined as in (2.3). Assume that $\mu(\cdot)$ and $K(\cdot)$ now depend on the parameter θ (so that ν does as well). The following result has a proof identical to that of Theorem 2.1, and is therefore omitted.

Theorem 2.2. *Suppose there exists $w: C \to [1,\infty)$ and $\theta_0 \in (a,b)$ for which*

a. $|||K^m(\theta_0)|||_w < 1$ for some $m \geq 1$;

b. $K(\cdot)$ is (continuously) differentiable in $\mathcal{L}_w$ at θ_0, with derivative $K'(\theta_0)$;

c. $\mu(\cdot)$ is (continuously) differentiable in $\mathcal{M}_w$ at θ_0, with derivative $\mu'(\theta_0)$.

Then, $\nu(\theta) = \sum_{n=0}^{\infty} \mu(\theta)K^m(\theta)$ is (continuously) differentiable in $\mathcal{M}_w$ in θ_0, with

$$\nu'(\theta_0) = \mu'(\theta_0)G(\theta_0) + \nu(\theta_0)G'(\theta_0).$$

If, in addition, $K(\cdot)$ and $\mu(\cdot)$ are n-times (continuously) differentiable in $\mathcal{L}_w$ and $\mathcal{M}_w$, respectively, at θ_0, then $\nu(\cdot)$ is n-times (continuously) differentiable in $\mathcal{M}_w$, and $\nu^{(n)}(\theta_0)$ can be recursively computed via

$$\nu^{(n)}(\theta_0) = \left(\mu^{(n)}(\theta_0) + \sum_{j=0}^{n-1} \binom{n}{j} \nu^{(j)} K^{(n-j)}(\theta_0)\right) G(\theta_0).$$

We finish this section with a short operator-theoretic argument establishing existence of a derivative for the stationary distribution under the assumption of geometric ergodicity (see condition (a) in the following, which is the key Lyapunov condition that implies geometric ergodicity in chapter 15 of Meyn and Tweedie [15]).

Theorem 2.3. *Suppose that there exists a subset $A \subseteq S$, $\epsilon, c > 0$, $\lambda, r \in (0,1)$, an integer $m \geq 1$, a probability measure φ on S, and $w: S \to [1,\infty)$ such that*

a. $(P(\theta_0)w)(x) \leq rw(x) + cI(x \in A)$ for $x \in S$;

b. $P^m(\theta, x, dy) \geq \lambda\varphi(dy)$ for $x \in A$, $y \in S$, and $|\theta - \theta_0| < \epsilon$;

c. $P(\cdot)$ is (continuously) differentiable in $\mathcal{L}_w$ at θ_0.

Then, X is positive Harris recurrent for θ in a neighborhood of θ_0, and the stationary distributions $\pi(\theta) \in \mathcal{M}_w$ for θ in a neighborhood of θ_0 are (continuously) differentiable in $\mathcal{M}_w$ at θ_0. Furthermore, if $\Pi(\theta_0)$ is the kernel defined by $\Pi(\theta_0, x, dy) = \pi(\theta_0, dy)$ for $x, y \in S$, $(I - P(\theta_0) + \Pi(\theta_0))$ has an inverse on $\mathcal{L}_w$ and

$$\pi'(\theta_0) = \pi(\theta_0)P'(\theta_0)(I - P(\theta_0) + \Pi(\theta_0))^{-1}. \tag{2.13}$$

If, in addition, $P(\cdot)$ is n-times (continuously) differentiable in $\mathcal{L}_w$ at θ_0, then $\pi(\cdot)$ is n-times (continuously) differentiable in $\mathcal{M}_w$ at θ_0, and $\pi^{(n)}(\theta_0)$ can be recursively computed via

$$\pi^{(n)}(\theta_0) = \sum_{j=0}^{n-1} \binom{n}{j} \pi^{(j)}(\theta_0)P^{(n-j)}(\theta_0)(I - P(\theta_0) + \Pi(\theta_0))^{-1}.$$

Remark 2.2. Note that theorem 4 of Glynn and L'Ecuyer [3] is closely related to the Theorem 2.3. See also remark 11 and the Kendall set assumption in Glynn and L'Ecuyer [3]. Heidergott et al. [9] and Heidergott and Hordijk [6] also impose similar assumptions to establish the measure-valued derivative of the stationary distribution.

Proof. In view of (a) and (c), there exists $r' < 1$ such that

$$(P(\theta_0 + h)w)(x) \leq r'w(x) + cI(x \in A) \tag{2.14}$$

for $x \in S$ and $|h|$ sufficiently small. Assumptions (a) and (b), and the fact that $w \geq 1$ implies that X is positive Harris recurrent for θ in a neighborhood of θ_0. We can now appeal to theorem 2.3 of Glynn and Meyn [4] to establish that $(I - P(\theta_0) + \Pi(\theta_0))$ is invertible on $\mathcal{L}_w$, with $(I - P(\theta_0) + \Pi(\theta_0))^{-1} \in \mathcal{L}_w$.

Furthermore, according to Glynn and Zeevi [5], (2.14) implies that $\pi(\theta_0 + h)w \leq c/(1 - r')$, and hence $\|\pi(\theta_0 + h)\|_w \leq c/(1 - r')$. Also,

$$\begin{aligned}(\pi(\theta_0 + h) - \pi(\theta_0))(I - P(\theta_0)) &= \pi(\theta_0 + h)(I - P(\theta_0))\\ &= \pi(\theta_0 + h)(P(\theta_0 + h) - P(\theta_0)).\end{aligned}$$

In addition, $\nu\Pi(\theta_0) = \pi(\theta_0)$ for any probability ν on S. So $(\pi(\theta_0 + h) - \pi(\theta_0))\Pi(\theta_0) = 0$. Consequently,

$$(\pi(\theta_0 + h) - \pi(\theta_0))(I - P(\theta_0) + \Pi(\theta_0)) = \pi(\theta_0 + h)(P(\theta_0 + h) - P(\theta_0)),$$

from which it follows that

$$\pi(\theta_0+h)-\pi(\theta_0)=\pi(\theta_0+h)(P(\theta_0+h)-P(\theta_0))(I-P(\theta_0)+\Pi(\theta_0))^{-1}. \tag{2.15}$$

Thus,

$$\|\pi(\theta_0+h)-\pi(\theta_0)\|_w \le \frac{c}{1-r'}|||P(\theta_0+h)-P(\theta_0)|||_w\cdot|||(I-P(\theta_0)+\Pi(\theta_0))^{-1}|||_w. \tag{2.16}$$

Because $P(\cdot)$ is differentiable in $\mathcal{L}_w$, $|||P(\theta_0+h)-P(\theta_0)|||_w\to 0$ as $h\to 0$, so $\pi(\theta_0+h)\to\pi(\theta_0)$ in $\mathcal{M}_w$ as $h\to 0$. Letting $h\to 0$ in (2.15) then yields (2.13).

For the continuity of the derivative in case $P(\cdot)$ is continuously differentiable, note first that (a) and (b) imply that $|||(P(\theta_0)-\Pi(\theta_0))^m|||_w<1$ for some $m\ge 1$; this along with the continuity of $P(\cdot)$ and $\pi(\cdot)$, in turn, implies that $\sup_{|h|\le h_0}|||(P(\theta_0+h)-\Pi(\theta_0+h))^m|||_w<1$ for a small enough h_0. Therefore, we conclude that $|||(I-P(\theta_0+h)+\Pi(\theta_0+h))^{-1}|||_w$ is bounded (uniformly with respect to (w.r.t.) h). From this, it is easy to see that the same argument as for (2.13) works with $\theta=\theta_0+h$ instead of θ_0 and proves that

$$\pi'(\theta_0+h)=\pi(\theta_0+h)P'(\theta_0+h)(I-P(\theta_0+h)+\Pi(\theta_0+h))^{-1}. \tag{2.17}$$

Now,

$$\pi'(\theta_0+h)-\pi'(\theta_0)=\left(\pi'(\theta_0+h)-\frac{\pi(\theta_0)-\pi(\theta_0+h)}{-h}\right)-\left(\pi'(\theta_0)-\frac{\pi(\theta_0+h)-\pi(\theta_0)}{h}\right)=\text{(I)}-\text{(II)},$$

where we have already seen that (II) converges to 0. To show that (I) also vanishes, note that

$$\begin{aligned}(\pi(\theta_0)-\pi(\theta_0+h))(I-P(\theta_0+h)+\Pi(\theta_0+h))&=(\pi(\theta_0)-\pi(\theta_0+h))(I-P(\theta_0+h))\\&=\pi(\theta_0)(I-P(\theta_0+h))=\pi(\theta_0)(P(\theta_0)-P(\theta_0+h)),\end{aligned}$$

and hence,

$$\frac{\pi(\theta_0)-\pi(\theta_0+h)}{-h}=\pi(\theta_0)\frac{P(\theta_0+h)-P(\theta_0)}{h}(I-P(\theta_0+h)+\Pi(\theta_0+h))^{-1}. \tag{2.18}$$

From (2.17), (2.18), the continuity of $\pi(\cdot)$, the continuous differentiability of $P(\cdot)$, and the uniform boundedness of the norm of $(I-P(\theta_0+h)+\Pi(\theta_0+h))^{-1}$, we conclude that (I) vanishes. Therefore, $\pi'(\cdot)$ is continuous at θ_0.

Finally, as in Theorem 2.1, the proof for the n-fold derivatives for $n\ge 2$ follows similar lines, and is therefore omitted. □

We conclude this section with a brief discussion of the role of w. Note that the growth rate of w decides the extent of the performance measures to which the theorems in this section apply. For example, the $\mathcal{M}_w$-differentiability of $\pi(\theta)$ in Theorem 2.3 establishes the differentiability of the stationary expectations of f for all f's that are majorized by w. On the other hand, the sufficient conditions are also stated in terms of w, and it tends to be harder to establish such sufficient conditions when w's grow faster. Therefore, in the context of optimization or sensitivity analysis, the choice of w should be made in such a way that it covers sufficiently wide range of performance measures and objective functions for the purpose of the tasks at hand, while the sufficient conditions are satisfied at the same time.

The condition (a) of Theorem 2.1, 2.2, and 2.3 are the key inequalities that ensure the differentiability of the expectations of our interest. Note that the condition (a) of Theorem 2.1 and 2.2 are equivalent to $\int_C K^m(\theta_0)w(x)\le rw(x)$ for some $r\in(0,1)$, and hence, we see that w plays the role of a Lyapunov function in all three main theorems. It should be noted that the operator-theoretic formulation in this section allows simple statements at the cost of stronger conditions. For example, Theorem 2.3 establishes, in the presence of a single Lyapunov function w, the n-fold differentiability of the stationary distribution $\pi(\cdot)$ in $\mathcal{M}_w$, but the existence of such w requires geometric ergodicity; compare this to Theorem 4.1, which involves two Lyapunov conditions but does not require geometry ergodicity. Establishing sufficient conditions that are closer to necessary (at the cost of slightly more involved sufficient conditions) is the overarching subject of the rest of this paper.

3. Lyapunov Criteria for Differentiability of Random Horizon Expectations

Let $\Lambda=(a,b)$ be an open interval containing θ_0. For each $\theta\in\Lambda$, let $\mathbf{E}_x^\theta(\cdot)\triangleq\mathbf{E}^\theta(\cdot|X_0=x)$ be the expectation operator associated with X, when X is driven by the one-step transition kernel $P(\theta)$. As in Section 2, we consider

$$u^*(\theta,x)=\mathbf{E}_x^\theta\sum_{j=0}^{T-1}\exp\left(\sum_{k=0}^{j-1}g(X_k)\right)f(X_j)+\exp\left(\sum_{k=0}^{T-1}g(X_k)\right)f(X_T) \tag{3.1}$$

for each $x \in C$ given $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$, $\emptyset \neq C \subseteq S$, and $T = \inf\{n \geq 0 : X_n \in C^c\}$. Our goal, in this section, is to provide Lyapunov conditions under which $u^*(\theta) = (u^*(\theta, x) : x \in C)$ is differentiable at θ_0, and to provide an expression for the derivative $u^{*\prime}(\theta)$.

Note that if f is nonnegative, then $u^*(\theta)$ is always well-defined. Furthermore, by conditioning on X_1, it is easily seen that

$$u^*(\theta, x) = f(x) + \int_{C^c} \exp(g(x))P(\theta, x, dy)f(y) + \int_C \exp(g(x))P(\theta, x, dy)u^*(\theta, y)$$

for $x \in C$, and hence

$$u^*(\theta) = \tilde{f}(\theta) + K(\theta)u^*(\theta), \tag{3.2}$$

where as in Section 2,

$$\tilde{f}(\theta, x) = f(x) + \int_{C^c} \exp(g(x))P(\theta, x, dy)f(y)$$

for $x \in C$, and $K(\theta) = (K(\theta, x, dy) : x, y \in C)$ is the nonnegative kernel on C for which

$$K(\theta, x, dy) = \exp(g(x))P(\theta, x, dy).$$

Given (3.2), formal differentiation of both sides of the equation yields

$$u^{*\prime}(\theta_0) = \tilde{f}'(\theta_0) + K'(\theta_0)u^*(\theta_0) + K(\theta_0)u^{*\prime}(\theta_0), \tag{3.3}$$

so that $u^{*\prime}(\theta)$ should satisfy the linear system

$$(I - K(\theta_0))u^{*\prime}(\theta_0) = \tilde{f}'(\theta_0) + K'(\theta_0)u^*(\theta_0). \tag{3.4}$$

When $|C|$ is finite and g is negative, it will frequently be the case that the matrix $K(\theta_0)$ has spectral radius less than 1, in which case $I - K(\theta_0)$ is invertible and

$$(I - K(\theta_0))^{-1} = \sum_{n=0}^{\infty} K^n(\theta_0). \tag{3.5}$$

In this case,

$$u^{*\prime}(\theta_0) = \sum_{n=0}^{\infty} K^n(\theta_0)\left(\tilde{f}'(\theta_0) + K'(\theta_0)u^*(\theta_0)\right).$$

But (3.2) and (3.5) further imply that

$$u^*(\theta_0) = \sum_{n=0}^{\infty} K^n(\theta_0)\tilde{f}(\theta_0), \tag{3.6}$$

and hence we arrive at the formula

$$u^{*\prime}(\theta_0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} K^m(\theta_0)K'(\theta_0)K^n(\theta_0)\tilde{f}(\theta_0) + \sum_{m=0}^{\infty} K^m(\theta_0)\tilde{f}'(\theta_0). \tag{3.7}$$

The remainder of this section is largely concerned with rigorously extending the formula (3.7) to the general state space setting, under Lyapunov criteria that are close to minimal (and easily checkable from the model building blocks). We start by observing that when f is nonnegative, Fubini's theorem implies that

$$\begin{aligned} u^*(\theta, x) &= \sum_{j=0}^{\infty} \mathbf{E}_x^{\theta} \exp\left(\sum_{k=0}^{j-1} g(X_k)\right) f(X_j) I(T > j) \\ &\quad + \sum_{j=0}^{\infty} \mathbf{E}_x^{\theta} \exp\left(\sum_{k=0}^{j-1} g(X_k)\right) I(T \geq j) f(X_j) I(X_j \in C^c) \\ &= \sum_{j=0}^{\infty} (K^j(\theta)\tilde{f}(\theta))(x), \end{aligned} \tag{3.8}$$

thereby rigorously verifying (3.6). To simplify the notation in the remainder of this paper, we set $K = K(\theta_0)$ and put

$$G = \sum_{n=0}^{\infty} K^n. \tag{3.9}$$

Our path to providing rigorous conditions under which (3.7) holds involves the following key *absolute continuity* assumption:

A1. The kernels $(K(\theta): \theta \in \Lambda)$ are absolutely continuous with respect to K, in the sense that there exists a (measurable) density $(k(\theta, x, y): x \in C, y \in S)$ such that

$$K(\theta, x, dy) = k(\theta, x, y)K(x, dy)$$

for $\theta \in \Lambda$, $x \in C, y \in S$.

Our absolute continuity condition is often a mild hypothesis. For example, when X has a transition density with respect to a reference measure η, A1 is in force when the support of the density is independent of θ.

We also need to assume that $K(\theta)$ is suitably differentiable at θ_0.

A2. There exists $\epsilon > 0$ such that for each $x \in C$ and $y \in S$, $k(\cdot, x, y)$ is continuously differentiable, with derivative $k'(\cdot, x, y)$, in $[\theta_0 - \epsilon, \theta_0 + \epsilon]$.

Set $\omega_\epsilon(x, y) = \sup\{|k'(\theta, x, y)| : |\theta - \theta_0| < \epsilon\}$, $k'(x, y) = k'(\theta_0, x, y)$, and $K'(x, dy) = k'(x, y)K(x, dy)$. (Note that K' is a signed kernel, and not nonnegative.)

Our hypotheses are stated in terms of $K(\theta)$, not $P(\theta)$, in order to offer the extra generality needed to cover settings in which derivatives involving parameters in the discount factor $\exp(g(\cdot))$ are of interest. Such derivatives are commonly considered in the finance literature when attempting to hedge uncertainty in the so-called "short rate." (The resulting derivative is called *rho* in the finance context.)

Finally, we also need to assume $\tilde{f}(\theta)$ is suitably differentiable at θ_0. To permit derivatives in parameters that involve the discount factor, we write $\tilde{f}(\theta)$ in the form

$$\tilde{f}(\theta, x) = f(x) + \int_{C^c} K(\theta, x, dy) f(y). \tag{3.10}$$

A3. Assume that

$$\tilde{r}_\epsilon(x) \triangleq \int_{C^c} \omega_\epsilon(x, y)|f(y)|K(\theta_0, x, dy) < \infty$$

for $x \in C$.

In many applications, $f \equiv 0$ on C^c, and hence, $\tilde{f}(\theta)$ is independent of θ and A3 need not be verified (e.g. expected hitting times).

Throught the rest of this section, we will slightly abuse notation and let $K(\theta)h(x)$ denote $\int_C h(y)K(x, dy)$. We are now ready to state the main theorem of this section.

Theorem 3.1. *Assume A1, A2, and A3. Suppose there exists $\epsilon > 0$ and two finite-valued nonnegative functions v_0 and v_1 defined on C for which*

$$(K(\theta)v_0)(x) \le v_0(x) - |\tilde{f}(\theta, x)| \tag{3.11}$$

for $x \in C$ and $|\theta - \theta_0| < \epsilon$, and

$$(Kv_1)(x) \le v_1(x) - \int_C \omega_\epsilon(x, y)v_0(y)K(x, dy) - \tilde{r}_\epsilon(x) \tag{3.12}$$

for $x \in C$. Then, $u^*(\cdot, x)$ is differentiable at θ_0 and

$$u^{*\prime}(\theta_0) = \int_C\int_C\int_C G(x, dy)K'(y, dz)G(z, dw)\tilde{f}(w) + \int_C\int_{C^c} G(x, dy)K'(y, dz)f(z). \tag{3.13}$$

If, in addition,

$$\int_C \omega_\epsilon(x,y)v_1(y)K(x,dy) < \infty \tag{3.14}$$

and (3.12) *holds in a neighborhood of* θ_0, *that is,*

$$(K(\theta)v_1)(x) \le v_1(x) - \int_C \omega_\epsilon(x,y)v_0(y)K(x,dy) - \tilde{r}_\epsilon(x), \qquad \forall\theta \in [\theta_0-\epsilon, \theta_0+\epsilon], \tag{3.12$'$}$$

then $u^{*\prime}(\cdot, x)$ *is continuous on* $[\theta_0-\epsilon, \theta_0+\epsilon]$.

Recalling the definition of G, we see that (3.13) is indeed the general state space analog of (3.7). The functions v_0 and v_1 appearing in Theorem 3.1 are often called (stochastic) Lyapunov functions. A standard means of guessing good choices for v_0 and v_1 is to recognize that $u^*(\theta)$ satisfies (3.11) with equality if $\tilde{f}$ is nonnegative, whereas

$$\int_C \left[\int_C K(y,dz)\omega_\epsilon(y,z)v_0(z) + \tilde{r}_\epsilon(y)\right] G(x,dy)$$

satisfies (3.12) with equality. When $C \subseteq \mathbb{R}^m$ is unbounded, one can often approximate the large x behavior of these functions, and use these approximations as choices for v_1 and v_2, respectively.

The proof of Theorem 3.1 rests on the following easy bound.

Proposition 3.1. *Suppose that* $Q = (Q(x,dy) : x,y \in C)$ *is a nonnegative kernel and that* $f : C \to \mathbb{R}_+$. *If* $v : C \to \mathbb{R}_+$ *is a finite-valued function for which*

$$Qv \le v - f, \tag{3.15}$$

then,

$$\sum_{n=0}^{\infty} Q^n f \le v. \tag{3.16}$$

Proof. Note that (3.15) implies that $Qv \le v$, and hence $Q^n v \le v$ for $n \ge 0$. It follows that $Q^n\, v$ is finite-valued for $n \ge 0$. Inequality (3.15) can be rewritten as

$$f \le v - Qv. \tag{3.17}$$

Applying Q^j to both sides of (3.17), we get

$$Q^j f \le Q^j v - Q^{j+1} v \tag{3.18}$$

Summing both sides of (3.18) over $j = 0, 1, \ldots, n$, we find that

$$\sum_{j=0}^{n} Q^j f \le v - Q^{n+1} v \le v.$$

Sending $n \to \infty$ yields (3.16). □

Proof of Theorem 3.1. For the purposes of this proof, ϵ is taken as the smallest of the ϵ's appearing in A2, A3, and the statement of the theorem. We start by observing that Proposition 3.1, applied to the Lyapunov bound (3.11), guarantees that

$$\sum_{n=0}^{\infty} K^n(\theta)|\tilde{f}(\theta)| \le v_0,$$

and hence Fubini's theorem implies that $u^*(\theta)$ is finite-valued, $u^*(\theta) = \sum_{n=0}^{\infty} K^n(\theta)\tilde{f}(\theta)$, and $|u^*(\theta)| \le v_0$. Because $u^*(\theta)$ is finite-valued (as is $K(\theta)u^*(\theta)$), we can write

$$u^*(\theta_0+h) - u^*(\theta_0) = K(\theta_0+h)u^*(\theta_0+h) - K(\theta_0)u^*(\theta_0) + \tilde{f}(\theta_0+h) - \tilde{f}(\theta_0),$$

and hence,

$$(I-K)(u^*(\theta_0+h) - u^*(\theta_0)) = (K(\theta_0+h) - K(\theta_0))u^*(\theta_0+h) + (\tilde{f}(\theta_0+h) - \tilde{f}(\theta_0)). \tag{3.19}$$

For $|h| < \epsilon$,

$$\begin{aligned}&\left|\int_C (K(\theta_0+h,x,dy) - K(\theta_0,x,dy))u^*(\theta_0+h,y)\right|\\ &\leq \int_C |k(\theta_0+h,x,y) - k(\theta_0,x,y)|K(x,dy)v_0(y)\\ &\leq |h|\int_C \sup_{|\theta-\theta_0|<\epsilon} |k'(\theta,x,y)|K(x,dy)v_0(y)\\ &= |h|\int_C \omega_\epsilon(x,y)K(x,dy)v_0(y).\end{aligned}$$

Similarly, for $|h| < \epsilon$,

$$\begin{aligned}&|\tilde{f}(\theta_0+h,x) - \tilde{f}(\theta_0,x)|\\ &\leq |h|\int_{C^c} \omega_\epsilon(x,y)K(x,dy)|f(y)|\\ &\leq |h|\tilde{r}_\epsilon(x).\end{aligned}$$

Recall that we assumed that $\tilde{r}_\epsilon(x) < \infty$ in Assumption A3. Consequently, Proposition 3.1, together with the Lyapunov bound (3.12), ensures that

$$\int_C G(x,dy)\left(\left|\int_C (K(\theta_0+h,y,dz) - K(\theta_0,y,dz))u^*(\theta_0+h,z)\right| + \left|\tilde{f}(\theta_0+h,y) - \tilde{f}(\theta_0,y)\right|\right) \leq |h|v_1(x).$$

It follows from (3.19) that $u^*(\theta,x)$ is continuous at θ_0 and

$$\begin{aligned}\frac{u^*(\theta_0+h,x) - u^*(\theta_0,x)}{h} &= \int_C G(x,dy)\bigg[\int_C \frac{k(\theta_0+h,y,z) - k(\theta_0,y,z)}{h}u^*(\theta_0+h,z)K(y,dz)\\ &\qquad + \int_{C^c}\frac{k(\theta_0+h,y,z) - k(\theta_0,y,z)}{h}f(z)K(y,dz)\bigg].\end{aligned}$$

But,

$$\frac{k(\theta_0+h,y,z) - k(\theta_0,y,z)}{h} \to k'(y,z) \tag{3.20}$$

and

$$u^*(\theta_0+h,z) \to u^*(\theta_0,z) \tag{3.21}$$

as $h \to 0$. Also,

$$\left|\frac{k(\theta_0+h,y,z) - k(\theta_0,y,z)}{h}u^*(\theta_0+h,z)\right| \leq \omega_\epsilon(y,z)v_0(z) \tag{3.22}$$

for $y,z \in C$, and

$$\frac{|k(\theta_0+h,y,z) - k(\theta_0,y,z)|}{h} \leq \omega_\epsilon(y,z) \tag{3.23}$$

for $y \in C$, $z \in C^c$. The Lyapunov bound (3.12), together with Proposition 3.1, guarantees that

$$\int_C G(x,dy)\left(\int_C \omega_\epsilon(y,z)v_0(z)K(y,dz) + \int_{C^c}\omega_\epsilon(y,z)|f(z)|K(y,dz)\right) < \infty. \tag{3.24}$$

In view of (3.20) through (3.24), the dominated convergence theorem therefore establishes that $u^*(\theta,x)$ is differentiable at θ_0, and

$$u^{*\prime}(\theta_0,x) = \int_C G(x,dy)\int_C k'(y,z)u^*(\theta_0,z)K(y,dz) + \int_C G(x,dy)\int_{C^c} k'(y,z)f(z)K(y,dz), \tag{3.25}$$

which is equivalent to (3.13).

Turning to the continuity of $u^{*\prime}(\cdot, x)$, note one can easily check that

$$u^{*\prime}(\theta) = \tilde{f}'(\theta) + K'(\theta)u^*(\theta) + K(\theta)u^{*\prime}(\theta)$$

For $\theta \in [\theta_0 - \epsilon, \theta_0 + \epsilon]$, where $K'(\theta)u^*(\theta, x) = \int_C k'(\theta, x, y)u^*(\theta, y)K(x, dy)$, and hence,

$$\begin{aligned} u^{*\prime}(\theta + h) - u^{*\prime}(\theta) = {} & G(\theta)(\tilde{f}'(\theta + h) - \tilde{f}'(\theta)) + G(\theta)((K'(\theta + h) - K'(\theta))u^*(\theta + h)) \\ & + G(\theta)(K'(\theta)(u^*(\theta + h) - u^*(\theta))) + G(\theta)((K(\theta + h) - K(\theta))u^{*\prime}(\theta + h)). \end{aligned}$$

Now, a similar argument (via dominated convergence and the Lyapunov conditions) as the one that leads to (3.25)—along with (3.14), and (3.12′)—shows that $u^{*\prime}(\theta + h) - u^{*\prime}(\theta) \to 0$ for $\theta \in [\theta_0 - \epsilon, \theta_0 + \epsilon]$. □

Our proof also yields the following (computable) bound on $u^{*\prime}(\theta_0)$, namely,

$$|u^{*\prime}(\theta_0, x)| \le v_1(x), \tag{3.26}$$

for $x \in C$. Moreover, (3.13) implies that

$$U' \triangleq \sum_{m+n<T} \exp\left(\sum_{i=0}^{m+n} g(X_i)\right) k'(\theta_0, X_m, X_{m+1}) f(X_{m+n+1}) \tag{3.27}$$

is an unbiased estimator for $u^{*\prime}(\theta_0)$.

In many applications, the parameter θ enters the dynamics in a very specific way, which allows further simplification of the result. In particular, whenever S is a separable metric space, we can always express X as the solution to a stochastic recursion; see, for example, Kifer [12]. Namely, we can find a mapping $r: S \times S' \to S$ and a sequence $(Z_n : n \ge 1)$ of independent and identically distributed (iid) S'-valued random elements such that

$$X_{n+1} = r(X_n, Z_{n+1}) \tag{3.28}$$

for $n \ge 0$. Suppose that θ affects the dynamics of X only through the distribution of the Z_n's. Assume that for $z \in S'$,

$$P^{\theta}(Z_1 \in dz) = p(\theta, z)P^{\theta_0}(Z_1 \in dz), \tag{3.29}$$

where $p(\cdot, z)$ is continuously differentiable for $z \in S'$. If $u^*(\theta, x)$ is defined as in (3.1), then $u^*(\cdot, x)$ is differentiable at θ_0 and $u^{*\prime}(\theta_0, x)$ is given by (3.13) (where $K'(x, dy) = \mathbf{E}^{\theta_0}\mathbb{I}(r(x, Z_1) \in dy)p'(\theta_0, Z_1)$), provided that there exists $\epsilon > 0$ and finite-valued nonnegative functions v_0 and v_1 defined on $C \subseteq S$ for which

$$\mathbf{E}^{\theta_0} v_0(r(x, Z_1))p(\theta, Z_1) \le v_0(x) - |\tilde{f}(\theta, x)| \tag{3.30}$$

for $x \in C$ and $|\theta - \theta_0| < \epsilon$, and

$$\begin{aligned} \mathbf{E}^{\theta_0} v_1(r(x, Z_1)) \le v_1(x) - {} & \mathbf{E}^{\theta_0} v_0(r(x, Z_1)) \sup_{|\theta - \theta_0| < \epsilon} |p'(\theta, Z_1)| \mathbb{I}(r(x, Z_1) \in C) \\ & - \mathbf{E}^{\theta_0} |f(r(x, Z_1))| \sup_{|\theta - \theta_0| < \epsilon} |p'(\theta, Z_1)| \mathbb{I}(r(x, Z_1) \in C^c), \end{aligned}$$

for $x \in C$; the proof is essentially identical to that of Theorem 3.1 and is omitted.

According to Theorem 3.1, for functions f satisfying the Lyapunov bound,

$$u^{*\prime}(\theta_0, x) = \int_S \nu'(x, dy) f(y),$$

where

$$\nu'(w, dz) = \begin{cases} \int_C G(w, dx) \int_C K'(x, dy) \int_C G(y, dz), & w, z \in C \\ \int_C G(w, dx) \int_{C^c} K'(x, dz), & w \in C, z \in C^c. \end{cases}$$

Hence, our derivative can be represented in terms of a signed measure. (In general, $\nu'(x, S)$ is nonzero in this setting.)

Our approach also extends, in a straightforward way, to higher-order derivatives. Formal differentiation of (3.2) n times yields the identity

$$u^{*(n)}(\theta) = \tilde{f}^{(n)}(\theta) + \sum_{j=0}^{n} \binom{n}{j} K^{(n-j)}(\theta) u^{*(j)}(\theta),$$

which suggests that the n^{th} order derivative $u^{*(n)}(\theta)$ can then be recursively computed from $u^{*(0)}(\theta), \ldots, u^{*(n-1)}(\theta)$ by solving the linear (integral) equation

$$(I - K(\theta))u^{*(n)}(\theta) = \tilde{f}^{(n)}(\theta) + \sum_{j=0}^{n-1} \binom{n}{j} K^{(n-j)}(\theta)u^{*(j)}(\theta). \tag{3.31}$$

In particular, it should follow that

$$u^{*(n)}(\theta) = G\left(\tilde{f}^{(n)}(\theta) + \sum_{j=0}^{n-1} \binom{n}{j} K^{(n-j)}(\theta)u^{*(j)}(\theta)\right). \tag{3.32}$$

Rigorous verification of (3.32) can be implemented with a family $v_0, v_1, \ldots, v_n$ of Lyapunov functions. Specifically, assume that the densities $k(\cdot, x, y)$ (for $x \in S, y \in S$) are n-times continuously differentiable in some neighborhood $[\theta_0 - \epsilon, \theta_0 + \epsilon]$ of θ_0, and set

$$\omega_\epsilon^{(j)}(x,y) = \sup_{|\theta-\theta_0|<\epsilon} |k^{(j)}(\theta, x, y)|$$

for $x \in C$, $y \in S$.

Theorem 3.2. *Suppose that there exists $\epsilon > 0$ and a family of finite-valued nonnegative functions $v_0, v_1, \ldots, v_n$ defined on C for which*

$$(K(\theta)v_0)(x) \le v_0(x) - |\tilde{f}(\theta, x)|$$

for $x \in C$ and $|\theta - \theta_0| < \epsilon$;

$$(K(\theta)v_l)(x) \le v_l(x) - \sum_{j=0}^{l-1} \binom{l}{j} \int_C \omega_\epsilon^{(l-j)}(x,y)v_j(y)K(\theta, x, dy) - \int_{C^c} \omega_\epsilon^{(l)}(x,y)|f(y)|K(\theta, x, dy)$$

for $x \in C$, $|\theta - \theta_0| < \epsilon$, and $1 \le l \le n$; and

$$\int_C \omega_\epsilon^{(n)}(x,y)v_n(y)K(x,dy) < \infty$$

for $x \in C$. Then, $u^*(\cdot, x)$ is n-times continuously differentiable at θ_0, and the derivative can be recursively computed from the equations

$$u^{*(l)}(\theta_0, x) = \int_C G(x,dy) \int_C \sum_{j=0}^{l-1} \binom{l}{j} k^{(l-j)}(\theta_0, x, y)u^{*(j)}(y)K(x,dy) + \int_C G(x,dy) \int_{C^c} k^{(l)}(y,z)f(z)K(y,dz).$$

The proof of Theorem 3.2 mirrors that of Theorem 3.1, and is therefore omitted. As in the proof of Theorem 3.1, the argument establishes the bound $|u^{*(n)}(\theta_0, x)| \le v_n(x)$ for $x \in C$ on the n^{th} order derivative.

The previous results that come closest to our results in this section are Glynn and L'Ecuyer [3] and Heidergott and Vázquez-Abad [8]. Glynn and L'Ecuyer [3] study more general functionals over the random horizon than those studied in this section, however, they require geometric moments of the associated stopping times, which is often a much stronger condition than necessary. The sufficient conditions in Heidergott and Vázquez-Abad [8] do not assume geometric moments of the stopping times. However, they study a quite restricted class of random horizon expectations—that is, the ones of the form $\mathbf{E}_x^\theta \sum_{j=0}^T f(X_j)$—and they require that the stopping time possess at least finite second moment. Moreover, their sufficient conditions for differentiability are not given in terms of conditions that can be checked directly from the transition function of the Markov chain (unlike the Lyapunov criteria used in our paper). Instead, they provide conditions that require the expectation of a certain functional of the parametrized maximum of the Markov chains over the random horizon to be finite:

$$\mathbf{E}\left[\sup_{\theta\in\Lambda_\epsilon} \tau_\theta(\alpha) \sum_{i=1}^{\sup_{\theta\in\Lambda_\epsilon}\tau_\theta(\alpha)} \sup_{\theta\in\Lambda_\epsilon} |f(X_m^\theta)|\right] < \infty,$$

where $\{X^\theta\}_{\theta\in\Lambda_\epsilon}$ is the coupled family of Markov chains with the corresponding transition distributions $\{P(\theta,x,dy)\}_{\theta\in\Lambda_\epsilon}$ and $\{\tau_\theta(\alpha)\}$ is the associated stopping times. As a result, verifying such conditions typically requires cleverly bounding the above functionals with some other random variable that does not depend on θ's and then bounding the expectation of the random variable.

4. Lyapunov Criteria for Differentiability of Stationary Expectations

Perhaps the most commonly occurring expectations that arise in applications are those associated with steady-state behavior. Our Lyapunov approach is also well-suited to establishing differentiability in this context. As in Section 3, it is informative to first study the problem nonrigorously.

A stationary distribution $\pi(\theta)=(\pi(\theta,dx):x\in S)$ of the Markov chain X associated with one-step transition kernel $P(\theta)$ will satisfy

$$\pi(\theta)=\pi(\theta)P(\theta). \tag{4.1}$$

Differentiating both sides of (4.1) with respect to θ, we obtain

$$\pi'(\theta)=\pi'(\theta)P(\theta)+\pi(\theta)P'(\theta),$$

which leads to the equation

$$\pi'(\theta)(I-P(\theta))=\pi(\theta)P'(\theta).$$

This equation is similar to (3.4). However, unlike (3.4), the operator $I-P(\theta)$ appearing here will never be invertible, even when $|S|<\infty$. In addition, $I-P(\theta)$ is acting on a measure rather than a function in this setting. Thus, a different approach is needed here.

For a given function $f:S\to\mathbb{R}$, set $\alpha(\theta)=\pi(\theta)f$. Thus,

$$\begin{aligned}\alpha(\theta_0+h)-\alpha(\theta_0)&=\pi(\theta_0+h)f-\pi(\theta_0)f\\&=\pi(\theta_0+h)f_c,\end{aligned} \tag{4.2}$$

where $f_c(x)=f(x)-\pi(\theta_0)f$. Whereas $I-P(\theta_0)$ is singular, the *Poisson's equation*

$$(I-P(\theta_0))g=f_c \tag{4.3}$$

is, under suitable technical conditions, generally solvable for g (because of the special structure of the right-hand side, namely $\pi(\theta_0)f_c=0$). Substituting (4.3) into (4.2), we get

$$\begin{aligned}\alpha(\theta_0+h)-\alpha(\theta_0)&=\pi(\theta_0+h)(I-P(\theta_0))g\\&=\pi(\theta_0+h)(P(\theta_0+h)-P(\theta_0))g.\end{aligned} \tag{4.4}$$

This suggests that

$$\alpha'(\theta_0)=\pi(\theta_0)P'(\theta_0)g. \tag{4.5}$$

We now turn to making this argument rigorous.

We start by assuming that $(P(\theta):\theta\in\Lambda)$ itself satisfies the absolute continuity condition:

A4. The family of one-step transition kernels $(P(\theta):\theta\in\Lambda)$ is absolutely continuous with respect to $P(\theta_0)$, in the sense that there exists a density $(p(\theta,x,y):\theta\in\Lambda,x,y\in S)$ for which

$$P(\theta,x,dy)=p(\theta,x,y)P(\theta_0,x,dy)$$

for $x,y\in S$, and $\theta\in\Lambda$. Furthermore, there exists $\epsilon>0$ for which $p(\cdot,x,y)$ is continuously differentiable on $[\theta_0-\epsilon,\theta_0+\epsilon]$ for each $x,y\in S$.

Set $\omega_\epsilon(x,y)=\sup_{|\theta-\theta_0|<\epsilon}|p'(\theta,x,y)|$. Our next assumption involves a (uniform) minorization condition over the set A, which is standard in the theory of Harris recurrent Markov chains; see, for example, Meyn and Tweedie [15]:

A5. There exists $\epsilon>0$, a subset $A\subseteq S$, an integer $n\geq 1$, $\lambda>0$, and a probability φ for which

$$P^n(\theta,x,dy)\geq\lambda\varphi(dy)$$

for $x\in A$, $y\in S$, and $|\theta-\theta_0|<\epsilon$.

For $a,b\in\mathbb{R}$, let $a\vee b\triangleq\max(a,b)$. We can now state our main theorem on differentiability of stationary expectations.

Theorem 4.1. *Assume that A4 and A5 hold. Let $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ be a function for which $\kappa(x) \geq x$ and $\kappa(x)/x \to \infty$ as $x \to \infty$. Suppose that there exist positive constants ϵ, c_0, and c_1, and nonnegative finite-valued functions q, v_0, and v_1 for which*

$$(P(\theta)v_0)(x) \leq v_0(x) - (q(x) \vee 1) + c_0\mathbb{I}(x \in A), \tag{4.6}$$

$$(P(\theta)v_1)(x) \leq v_1(x) - \kappa\left(\int_S (1 \vee \omega_\epsilon(x,y))(v_0(y)+1)P(\theta,x,dy)\right) + c_1\mathbb{I}(x \in A), \tag{4.7}$$

for $x \in S$, $|\theta - \theta_0| < \epsilon$, and

$$\sup_{x \in A} v_0(x) < \infty \tag{4.8}$$

Then,

i. There exists an open interval $\mathcal{N}$ containing θ_0 for which X is a positive recurrent Harris chain under $P(\theta)$ for each $\theta \in \mathcal{N}$;

ii. There exists a unique stationary distribution $\pi(\theta)$ satisfying $\pi(\theta) = \pi(\theta)P(\theta)$ for each $\theta \in \mathcal{N}$ and $\pi(\theta)q \leq c_0$ for $\theta \in \mathcal{N}$;

iii. For each f such that $|f(x)| \leq q(x) \vee 1$ for $x \in S$, there exists a solution g (denoted $g = \Gamma f$) of Poisson's equation satisfying

$$((I - P(\theta_0))g)(x) = f(x) - \pi(\theta_0)f$$

for $x \in S$, and $|g(x)| = |(\Gamma f)(x)| \leq a(v_0(x)+1)$ for $x \in S$, where a is a finite constant;

iv. For each f such that $|f(x)| \leq q(x) \vee 1$, $\alpha(\theta) = \pi(\theta)f$ is continuously differentiable at θ_0, and

$$\alpha'(\theta_0) = \int_S \pi(\theta_0,dx)\int_S p'(\theta_0,x,y)(\Gamma f)(y)P(\theta_0,x,dy). \tag{4.9}$$

Proof. It is a standard fact that A5, (4.6), and (4.8) imply that X is a positive recurrent Harris chain under $P(\theta)$ for $\theta \in \mathcal{N}$ (where $\mathcal{N}$ is selected so that A5, (4.6), and (4.8) are all in force); see, for example, (Meyn and Tweedie [15]). As a consequence, there exists a unique stationary distribution $\pi(\theta)$ for each $\theta \in \mathcal{N}$. Furthermore, (4.6) implies that the bound $\pi(\theta)q \leq c_0$ holds for $\theta \in \mathcal{N}$; see, for example, corollary 4 of Glynn and Zeevi [5]. Because X is Harris recurrent (and (4.6) holds), one can now invoke theorem 2.3 of Glynn and Meyn [4] to obtain (iii).

Turning to (iv), note that (4.7) guarantees that $\pi(\theta)v_0 < \infty$ for $\theta \in \mathcal{N}$, so that $\pi(\theta)|\Gamma f| < \infty$.

With (i), (ii), and (iii) having been verified, we can now appeal to (4.4) to write

$$\begin{aligned}\pi(\theta_0+h)f - \pi(\theta_0)f &= \pi(\theta_0+h)(P(\theta_0+h) - P(\theta_0))\Gamma f \\ &= \int_S \pi(\theta_0+h,dx)\int_S (P(\theta_0+h,x,dy) - P(\theta_0,x,dy))(\Gamma f)(y).\end{aligned} \tag{4.10}$$

Set $s(x) = \int_S \omega_\epsilon(x,y)(v_0(y)+1)P(\theta_0,x,dy)$ and put $\mathbb{I}_m(x) = \mathbb{I}(s(x) \geq m)$, $\mathbb{I}_m^c(x) = \mathbb{I}(s(x) < m)$.

Observe that because $|p(\theta_0+h,x,y) - p(\theta_0,x,y)|/h \leq \omega_\epsilon(x,y)$, and $|(\Gamma f)(y)| \leq a(v_0(y)+1)$,

$$\begin{aligned}&\int_S \pi(\theta_0+h,dx)\mathbb{I}_m(x)\left|\left(\frac{P(\theta_0+h)-P(\theta_0)}{h}(\Gamma f)\right)(x)\right| \\ &\leq \int_S \pi(\theta_0+h,dx)\mathbb{I}_m(x)\int_S \omega_\epsilon(x,y)\,a(v_0(y)+1)P(\theta_0,x,dy) \\ &\leq a\int_S \pi(\theta_0+h,dx)\mathbb{I}_m(x)s(x) \\ &\leq \frac{a}{\inf\left\{\dfrac{\kappa(s(y))}{s(y)} : s(y) \geq m\right\}}\int_S \pi(\theta_0+h,dx)\frac{\kappa(s(x))}{s(x)}s(x) \\ &\leq \frac{a}{\inf\left\{\dfrac{\kappa(s(y))}{s(y)} : s(y) \geq m\right\}}\int_S \pi(\theta_0+h,dx)\kappa\left(\int_S (1 \vee \omega_\epsilon(x,y))(v_0(y)+1)P(\theta_0,x,dy)\right) \\ &\leq \frac{a}{\inf\left\{\dfrac{\kappa(s(y))}{s(y)} : s(y) \geq m\right\}}c_1,\end{aligned} \tag{4.11}$$

where the last inequality follows from (4.7) and corollary 4 of Glynn and Zeevi [5].

On the other hand,

$$\int_S \pi(\theta_0+h,dx)\mathbb{I}_m^c(x)\frac{P(\theta_0+h)-P(\theta_0)}{h}(\Gamma f)(x) \triangleq \int_S \pi(\theta_0+h,dx)s_h^m(x) = \pi(\theta_0+h)s_h^m,$$

where

$$|s_h^m(x)| \le a\int_S \omega_\epsilon(x,y)(v_0(y)+1)P(\theta_0,x,dy)\mathbb{I}(s(x)<m) \le am, \tag{4.12}$$

so s_h^m is bounded. It follows that

$$\pi(\theta_0+h)s_h^m - \pi(\theta_0)s_h^m = \pi(\theta_0+h)(P(\theta_0+h)-P(\theta_0))(\Gamma s_h^m).$$

Note that $\left|\frac{\Gamma s_h^m}{am}\right| \le a(v_0+1)$ because $\left|\frac{s_h^m}{am}\right| \le q \vee 1$ from (4.12), and hence,

$$\begin{aligned}|\pi(\theta_0+h)s_h^m - \pi(\theta_0)s_h^m| &\le a^2 m|h|\int_S \pi(\theta_0+h,dx)\int_S \omega_\epsilon(x,y)P(\theta_0,x,dy)(v_0(y)+1)\\ &\le a^2 m|h|\int_S \pi(\theta_0+h,dx)s(x)\\ &\le a^2 m|h|c_1 \to 0 \end{aligned} \tag{4.13}$$

as $h\to 0$. Finally,

$$\int_S \pi(\theta_0,dx)s_h^m(x) = \int_S \pi(\theta_0,dx)\mathbb{I}_m^c(x)\int_S \frac{p(\theta_0+h,x,y)-p(\theta_0,x,y)}{h}P(\theta_0,x,dy)(\Gamma f)(y),$$

and

$$\frac{p(\theta_0+h,x,y)-p(\theta_0,x,y)}{h} \to p'(\theta_0,x,y)$$

as $h \searrow 0$. Furthermore, $|p(\theta_0+h,x,y)-p(\theta_0,x,y)|/h \le \omega_\epsilon(x,y)$, $(\Gamma f)(y) \le a(v_0(y)+1)$, and

$$\int_S \pi(\theta_0,dx)\int_S \omega_\epsilon(x,y)P(\theta_0,x,dy)(v_0(y)+1) \le \int_S \pi(\theta_0,dx)s(x) \le c_1,$$

so the dominated convergence theorem implies that

$$\int_S \pi(\theta_0,dx)s_h^m(x) \to \int_S \pi(\theta_0,dx)\mathbb{I}_m^c(x)\cdot \int_S p'(\theta_0,x,y)P(\theta_0,x,dy)(\Gamma f)(y) \tag{4.14}$$

as $h\searrow 0$.

If we first let $h\to 0$ and then let $m\to\infty$, (4.10) through (4.14) imply part (iv) of our theorem.

Finally, turning to the continuity of the derivative, note that the exactly same argument as earlier gives $\alpha'(\theta_0+h) = \int_S \pi(\theta_0+h,dx)p'(\theta_0+h,x,y)\Gamma_{\theta_0+h}f(x)P(\theta_0,x,dy)$ where $\Gamma_{\theta_0+h}f$ is the solution g of the Poisson equation $g - P(\theta_0+h)g = f - \pi(\theta_0+h)f$. Because

$$\alpha'(\theta_0+h)-\alpha'(\theta_0) = \left(\alpha'(\theta_0+h) - \frac{\alpha((\theta_0+h)+(-h))-\alpha(\theta_0+h)}{-h}\right) - \left(\alpha'(\theta_0) - \frac{\alpha(\theta_0+h)-\alpha(\theta_0)}{h}\right),$$

and we have seen that the second term vanishes as $h\to 0$, we are done if we show that the first term also vanishes. Similarly as in (4.4), $\alpha(\theta_0)-\alpha(\theta_0+h) = \pi(\theta_0)(P(\theta_0)-P(\theta_0+h))\Gamma_{\theta_0+h}f$. Therefore,

$$\begin{aligned}\alpha'(\theta_0+h) - \frac{\alpha((\theta_0+h)+(-h))-\alpha(\theta_0+h)}{-h} &= \alpha'(\theta_0+h) + \frac{\alpha(\theta_0)-\alpha(\theta_0+h)}{h}\\ &= \int_S \pi(\theta_0+h,dx)p'(\theta_0+h,x,y)\Gamma_{\theta_0+h}f(x)P(\theta_0,x,dy)\\ &\quad + \int_S \pi(\theta_0,dx)\frac{p(\theta_0,x,y)-p(\theta_0+h,x,y)}{h}\Gamma_{\theta_0+h}f(x)P(\theta_0,x,dy)\\ &= \int_S (\pi(\theta_0+h,dx)-\pi(\theta_0,dx))p'(\theta_0+h,x,y)\Gamma_{\theta_0+h}f(x)P(\theta_0,x,dy) \qquad (4.15)\\ &\quad + \int_S \pi(\theta_0,dx)\left(p'(\theta_0+h,x,y) + \frac{p(\theta_0,x,y)-p(\theta_0+h,x,y)}{h}\right)\Gamma_{\theta_0+h}f(x)P(\theta_0,x,dy). \qquad (4.16)\end{aligned}$$

Upon a perusal of the proof of theorem 2.3 of Glynn and Meyn [4], one can see that the uniform minorization condition A5 and the uniform Lyapunov inequality (4.6) imply $|\Gamma_{\theta_0+h} f(x)| \le a(v_0(x)+1)$ with the same constant a as in (iii). One can prove that (4.15) vanishes as $h \to 0$ by the same argument as (4.11) and (4.13). On the other hand, (4.16) vanishes by the continuous differentiability condition A4 of p and the dominated convergence along with (4.7). □

As for Theorem 3.1, the proof also establishes a computable bound on $|\alpha'(\theta_0)|$, namely $|\alpha'(\theta_0)| \le ac_1$ where a is the constant in (iii). Also, the representation (4.9) of the derivative leads to the simulation estimator as well. Assuming without loss of generality that X possesses an atom A, (4.9) can be written as an infinite sum of expectations:

$$\alpha'(\theta_0) = \sum_{i=1}^{\infty} \mathbf{E}^{\theta_0}_{\pi(\theta_0)} p'(\theta_0, X_0, X_1)(f(X_i) - \pi(\theta_0)f)\mathbb{I}(\tau_1(A) \ge i),$$

where $\tau_1(A) = \inf\{n \ge 1 : X_n \in A\}$. Such a quantity can be estimated via cross-spectral density estimation methods; see, for example, Rosenblatt [17]. Again, as in Section 3, we can further simplify the conditions when X is the solution to the stochastic recursion (3.28), in which the parameter θ affects only the distribution Z_1. When $p(\cdot, z)$ is continuously differentiable, (4.7) may be simplified as

$$(P(\theta)v_1)(x) \le v_1(x) - \kappa\left(\mathbf{E}^{\theta_0}\left(1 \vee \sup_{|\theta-\theta_0|<\epsilon} |p'(\theta, Z_1)|\right)(v_0(r(x, Z_1)) + 1)p(\theta, Z_1)\right) + c_1\mathbb{I}(x \in A). \tag{4.17}$$

With A5, (4.6), and (4.8) also in force, this ensures the differentiability of $\alpha(\cdot)$ at θ_0, with $\alpha'(\theta_0)$ given by

$$\alpha'(\theta_0) = \int_S \pi(\theta_0, dx)\mathbf{E}^{\theta_0}(\Gamma f)(r(x, Z_1))p'(\theta_0, Z_1). \tag{4.18}$$

A useful example on which to illustrate our theory (and an important model in its own right) is that of the waiting time sequence $W = (W_n : n \ge 0)$ for the single-server G/G/1 queue, with first-come, first-serve queue discipline. Let V_n be the service time for the n^{th} customer, and let χ_{n+1} be the interarrival time that elapses between the arrival of the n^{th} and $(n+1)^{\text{st}}$ customer. If W_n is the waiting time (exclusive of service) for customer n, the W_n's satisfy the stochastic recursion

$$W_{n+1} = [W_n + V_n - \chi_{n+1}]^+ \tag{4.19}$$

for $n \ge 0$, where $[x]^+ \triangleq \max(x, 0)$. Assume that the V_n's are iid, independent of the χ_n's (which are also assumed iid). Then, W is a Markov chain taking values in $S = [0, \infty)$. It is well known that W is a positive recurrent Harris chain if $\mathbf{E}V_0 < \mathbf{E}\chi_1$, and that $\mathbf{E}V_0^{p+1} < \infty$ is then a necessary and sufficient condition for guaranteeing the finiteness of πf_p, where $f_p(x) = x^p$ (with $p > 0$); see, for example, Kiefer and Wolfowitz [11]. This suggests that it then typically will be the case that the pth moment should be differentiable when $\mathbf{E}V_0^{p+1} < \infty$. This can be immediately seen in the setting of the M/M/1 queue, where this follows directly from the Pollaczek-Khintchine formula.

We consider this problem in the special case in which the service times are finite mean Pareto random variables (rv's), and θ influences the scale parameter of the Pareto distribution. In other words, we consider the setting in which

$$P^{\theta}(V_0 > v) = (1 + \theta v)^{-\alpha}$$

for $\alpha > 1$. In this case, the density of V_0 under P^{θ} is given by $\theta h_V(\theta v)$, where $h_V(v) = \alpha(1+v)^{-\alpha-1}$, so that

$$p(\theta, v) = \left(\frac{\theta}{\theta_0}\right)\left(\frac{1 + \theta v}{1 + \theta_0 v}\right)^{-\alpha-1},$$

and

$$p'(\theta, v) = p(\theta, v)\left(\frac{1}{\theta} - (\alpha + 1)\frac{v}{(1 + \theta v)}\right).$$

Note that both the density p and its derivative (with respect to θ) are bounded functions. Furthermore, the rv $p'(\theta_0, V_i)$ has mean zero under P^{θ_0}. For any $c > 0$, the set $A = [0, c]$ is easily seen to satisfy condition A5, and A4 is trivially verified (with $\omega_\epsilon(\cdot)$ bounded). Then, if $v_0(x) = a_1 x^{p+1}$, $v_1(x) = a_2 x^{r+2}$, and $\kappa(x) = x^{\frac{1+r}{1+p}}$ (with $r > p$ and a_1, a_2 chosen suitably), we see that (4.6), (4.7), and (4.8) all hold, guaranteeing the differentiability of $\pi(\theta) f_p$ (according to Theorem 4.1).

For example, to verify (4.6), we note that

$$x^{-p}[(P(\theta)v_0)(x) - v_0(x)] = a_1 x \mathbf{E}^{\theta_0}\left(\left[1+\frac{1}{x}\left(\frac{\theta_0}{\theta}V_0 - \chi_1\right)\right]^+\right)^{p+1} - a_1 x.$$

Observe that as $x \to \infty$,

$$\begin{aligned} x f_{p+1}\left(\left[1+\frac{1}{x}\left(\frac{\theta_0}{\theta}V_0-\chi_1\right)\right]^+\right) - x &= x\left(f_{p+1}(1) + f'_{p+1}(1)\left(\frac{1}{x}\right)\left(\frac{\theta_0}{\theta}V_0-\chi_1\right)\right) - x + o(1) \quad a.s.\\ &= (p+1)\left(\frac{\theta_0}{\theta}V_0-\chi_1\right) + o(1) \quad a.s.\end{aligned}$$

where $o(1)$ represents a function $k(x)$ such that $k(x) \to 0$ as $x \to \infty$ uniformly in a neighborhood of θ_0. In addition, note that for $p > 0$ and $x > 0$, the mean value theorem implies that $f_{p+1}(1+x) = f_{p+1}(1) + f'_{p+1}(1+\xi)x$ for some $\xi \in [0, x]$, so that $f_{p+1}(1+x) = f_{p+1}(1) + (p+1)(1+\xi)^p x \le 1 + (p+1)(1+x)^p x$. Consequently,

$$\begin{aligned} x\left(\left[1+\frac{1}{x}\left(\frac{\theta_0}{\theta}V_0-\chi_1\right)\right]^+\right)^{p+1} - x &\le x\left(1+\frac{1}{x}\frac{\theta_0}{\theta}V_0\right)^{p+1} - x\\ &\le x\left(1+(p+1)\left(1+\frac{1}{x}\frac{\theta}{\theta_0}V_0\right)^p \frac{1}{x}\frac{\theta}{\theta_0}V_0\right) - x\\ &\le (p+1)\left(1+\frac{1}{x}\frac{\theta}{\theta_0}V_0\right)^p \frac{\theta}{\theta_0}V_0.\end{aligned}$$

Because $\mathbf{E}V_0^{p+1} < \infty$, Fatou's lemma applies to ensure that

$$\begin{aligned} \limsup_{x\to\infty} \sup_\theta \mathbf{E}^{\theta_0}\left(x f_{p+1}\left(\left[1+\frac{1}{x}\left(\frac{\theta_0}{\theta}V_0-\chi_1\right)\right]^+\right) - x\right) &\le (p+1)\mathbf{E}^{\theta_0}\sup_\theta\left(\frac{\theta_0}{\theta}V_0-\chi_1\right)\\ &= (p+1)\sup_\theta\left(\frac{1}{\theta(\alpha-1)} - \mathbf{E}\chi_1\right),\end{aligned}$$

as $x \to \infty$ (with convergence that is uniform in a neighborhood of θ_0). If we choose a_1 so that $a_1(p+1)\sup_\theta\left(\frac{1}{\theta(\alpha-1)} - \mathbf{E}\chi_1\right) \le -2$ and c so that

$$a_1 \sup_\theta \mathbf{E}^{\theta_0}\left(x f_{p+1}\left(\left[1+\frac{1}{x}\left(\frac{\theta_0}{\theta}V_0-\chi_1\right)\right]^+\right) - x\right) \le -1$$

for $x \ge c$, then (4.6) is validated. A similar argument applies to (4.7), in view of the boundedness of $\omega_\epsilon(\cdot)$. Our argument therefore establishes that πf_p is differentiable if $\mathbf{E}V_0^q < \infty$ for some $q > p+2$. This is not quite the "correct" result (in that we previously argued that $\mathbf{E}V_0^{p+1} < \infty$ should be sufficient.)

The reason that our argument fails to provide optimal condition here has to do with special random walk structure that is present in the process W that is difficult for general machinery to exploit. The challenge arises in (4.4). Note that the argument just provided for W involves using $v_0 = a_1 f_{p+1}$ as a bound on the solution g to Poisson's equation for f_p. (As we shall see in a moment, g is indeed exactly of order x^{p+1}). The problem is that neither $P(\theta_0+h)f_{p+1}$ nor $P(\theta_0)f_{p+1}$ in (4.4) are integrable with respect to $\pi(\theta_0+h)$ unless $\mathbf{E}V_0^{p+2} < \infty$. This is what leads to the extra moment appearing in our earlier argument for W. Thus, any argument that yields differentiability under the hypothesis $\mathbf{E}V_0^{p+1} < \infty$ must take advantage of the fact that the random walk structure of W yields the integrability of $(P(\theta_0+h) - P(\theta_0))g$ under $\mathbf{E}V_0^{p+1} < \infty$ without demanding the integrability of $P(\theta_0)g$ and $P(\theta_0+h)g$ separately.

It is shown in Glynn and Meyn [4] that, in view of the fact that W regenerates at hitting times of $\{0\}$, the solution g to Poisson's equation for f_p can be expressed as

$$g(x) = \mathbf{E}_x^{\theta_0}\sum_{j=0}^{\tau(0)-1}(f_p(W_j) - \pi(\theta_0)f_p), \tag{4.20}$$

where $\tau(0) = \inf\{n \geq 1 : W_n = 0\}$ is the hitting time of $\{0\}$. Let $Z_j = V_{j-1} - \chi_j$, $S_j = Z_1 + \cdots + Z_j$, (for $j \geq 1$), $\tau_x(0) = \inf\{j \geq 1 : x + S_j \leq 0\}$, $\mu = \mathbf{E}Z_1$, and note that (4.20) implies that

$$(P(\theta_0+h)g)(x) - (P(\theta_0)g)(x) = \mathbf{E}^{\theta_0} g(W_1)[p(\theta_0+h, V_0) - 1] \\ = \mathbf{E}^{\theta_0} \sum_{j=1}^{\tau_x(0)-1} [(x+S_j)^p - \pi(\theta_0) f_p](p(\theta_0+h, V_0) - 1)\mathbb{I}(x + Z_1 > 0). \tag{4.21}$$

But,

$$\sum_{j=1}^{\tau_x(0)-1} (x+S_j)^p (p(\theta_0+h, V_0) - 1) = x^p \sum_{j=1}^{\tau_x(0)-1} \left[\left(1 + \frac{S_j - V_0}{x}\right)^p + p\xi_j(x)^{p-1} \frac{V_0}{x}\right](p(\theta_0+h, V_0) - 1),$$

where $\xi_j(x)$ lies between $1 + S_j/x - V_0/x$ and $1 + S_j/x$. Note that

$$\sum_{j=1}^{\tau_x(0)-1} (1 + S_j/x)^{p-1} \frac{1}{x} = \int_0^{\tau_x(0)-1} (1 + S_{\lceil u \rceil}/x)^{p-1} \frac{1}{x} du = \int_0^{\frac{\tau_x(0)-1}{x}} (1 + \overline{S}_x(u))^{p-1} du,$$

where $\overline{S}_x(u) = S_{\lceil xu \rceil}/x$. Similarly, $\sum_{j=1}^{\tau_x(0)-1} (1 + S_j/x - V_0/x)^{p-1} \frac{1}{x} = \int_0^{\frac{\tau_x(0)-1}{x}} (1 + \overline{S}_x(u) - V_0/x)^{p-1} du$. Because $\overline{S}_x(\cdot)$ converges to a straight line with slope μ, and $\frac{\tau_x(0)-1}{x}$ converges to $1/|\mu|$, while V_0/x vanishes almost surely as $x \to \infty$,

$$\sum_{j=1}^{\tau_x(0)-1} \xi_j(x)^{p-1} \frac{1}{x} \to \int_0^{1/|\mu|} (1 + \mu s)^{p-1} ds \quad a.s. \\ = \frac{1}{|\mu|} \cdot \frac{1}{p}$$

as $x \to \infty$. Furthermore, $p(\theta_0 + h, V_0) - 1$ is a mean zero rv that is independent of $(1 + (S_j - V_0)/x)^p$ for $j \geq 1$ and $\mathbf{E}\tau_x(0) \sim x/|\mu|$ as $x \to \infty$ (where $a_1(x) \sim a_2(x)$ as $x \to \infty$ means that $a_1(x)/a_2(x) \to 1$ as $x \to \infty$). In view of (4.21), this suggests that

$$(P(\theta_0+h)g)(x) - (P(\theta_0)g)(x) \sim \frac{x^p}{|\mu|} \mathbf{E} V_0 (p(\theta_0+h, V_0) - 1)$$

as $x \to \infty$ (i.e., one power lower than the growth of g itself). Thus, this style of argument can successfully deal with the integrability issue discussed earlier, and leads to a validation of the derivative formula (4.9) for W under the assumption $\mathbf{E}V_0^{p+1} < \infty$. A rigorous statement and the remaining details of the proof can be found in Appendix A.

This differentiation result for W can also be found in Heidergott and Hordijk [7], with a different (and longer) proof, and with some steps that appear to be incomplete. (In particular, the paper asserts that $\mathbf{E}_x^{\theta_0} \sum_{j=0}^{\tau(0)-1} f_p(W_j)$ is bounded for any fixed θ and p, which implies that our function g grows at most linearly regardless of p; see Lemma 5.5 of the paper). Another related result can be found in Leahu et al. [13], which also proves the differentiability of the G/G/1 queue. However, it requires that the stationary waiting time has $p + 1$ moments, which in turn requires that the service time has $p + 2$ moments; see condition (iv) in theorem 4. As discussed earlier, this is a stronger assumption than necessary.

For general Markov chain stationary expectations, the previous results that come closest to Theorem 4.1 are Glynn and L'Ecuyer [3] and Heidergott et al. [9]. However, the sufficient conditions provided in these papers require geometric ergodicity of the Markov chain, which is a much stronger condition than our Lyapunov conditions. For example, for the G/G/1 queue in the previous example to be geometrically ergodic, the service time distribution needs to possess an exponential moment. It should be noted that in case the Markov chain possesses an atom α, the sufficient condition for the random horizon result in Heidergott and Vázquez-Abad [8] can also be used for checking the differentiability of the stationary expectation of Markov chains taking advantage of the fact that the stationary expectation can be written as $\pi(\theta)g = \mathbf{E}_\alpha^\theta \sum_{n=0}^{\tau(\alpha)-1} g(X_n)/\mathbf{E}_\alpha^\theta \tau(\alpha)$. However, as pointed out in Section 3, the sufficient conditions provided there are often not straightforward to verify.

5. Lyapunov Criteria for Differentiability of General Random Horizon Expectations

In this section, we discuss the differentiability of the Markov chain expectations that cannot be described as solutions of linear systems as in the previous sections. More specifically, let $T \triangleq \inf\{n \geq 0 : X_n \in C^c\}$ and consider for each positive integer k a functional $f_k : C^k \times C^c \to \mathbb{R}$. We are interested in the differentiability of

$$u(\theta, x) = \mathbf{E}_x^\theta f_T(X_0, X_1, \ldots, X_T)$$

for $f_T(X_0, X_1, \dots, X_T)$'s that can be bounded by the functionals of the form studied in Section 2. For example, $f_T(X_0, X_1, \dots, X_T)$ can be a function of the maximum of X_i's between time 0 and T. Set $v_\epsilon(x,y) = \sup\{|p(\theta,x,y)| : |\theta-\theta_0| < \epsilon\}$, $\omega_\epsilon(x,y) = \sup\{|p'(\theta,x,y)/p(\theta,x,y)| : |\theta-\theta_0| < \epsilon\}$, and for each $p > 1$,

$$\hat{v}_\epsilon^p(x) \triangleq \int_{C^c} v_\epsilon^p(x,dy) P(\theta_0,x,dy),$$

$$\hat{\omega}_\epsilon^p(x) \triangleq \int_C \omega_\epsilon^p(x,y) P(\theta_0,x,dy) + \int_C\int_{C^c} \omega_\epsilon^p(y,z) P(\theta_0,y,dz) P(\theta_0,x,dy),$$

and

$$\hat{f}^p(\theta,x) \triangleq f^p(x) + \int_{C^c} \exp(p\cdot g(x)) f^p(y) P(\theta,x,dy).$$

Theorem 5.1. *Assume that A4 in Section 4 holds. Suppose that $f_T(X_0, X_1, \dots, X_T) \le \sum_{j=0}^T \exp\left(\sum_{k=0}^{j-1} g(X_k)\right) f(X_j)$ almost surely for some $g: S \to \mathbb{R}$ and $f: S \to \mathbb{R}_+$, and there exist constants $\epsilon, \gamma \in (0,1)$, $q, r, s > 1$, and nonnegative finite-valued functions v_0, v_1, and v_2 such that $1/q + 1/r + 1/s < 1$ and*

$$\int_C v_\epsilon^q(x,y) v_0(y) P(\theta_0,x,dy) \le v_0(x) - |\hat{v}_\epsilon^q(x)|, \qquad \forall x \in C, \tag{5.1}$$

$$\int_C v_1(y) P(\theta_0,x,dy) \le v_1(x) - |\hat{\omega}_\epsilon^r(x)|, \qquad \forall x \in C, \tag{5.2}$$

$$\int_C \exp(sg(x)) v_2(y) P(\theta_0,x,dy) \le v_2(x) - |\hat{f}^s(\theta_0,x)|, \qquad \forall x \in C, \tag{5.3}$$

$$\int_C v_3(y) P(\theta_0,x,dy) \le \gamma v_3(x) - \mathbf{P}_x^{\theta_0}(X_1 \in C^c), \qquad \forall x \in C. \tag{5.4}$$

Then, $u(\theta,x)$ is differentiable at θ_0 and

$$u'(\theta,x) = \mathbf{E}_x^{\theta_0} f_T(X_0, X_1, \dots, X_T) L_T'(\theta), \tag{5.5}$$

where

$$L_T'(\theta) \triangleq L_T(\theta) \sum_{i=1}^T \frac{p'(\theta, X_{i-1}, X_i)}{p(\theta, X_{i-1}, X_i)} \quad \text{and} \quad L_T(\theta) \triangleq \prod_{i=1}^T p(\theta, X_{i-1}, X_i).$$

If (5.3) and (5.4) hold in a neighborhood of θ_0, then $u'(\cdot,x)$ is continuous at θ_0.

Proof. It is well known (see, for example, Glynn and L'Ecuyer [3]) that

$$u(\theta_0+h) = \mathbf{E}_x^{\theta_0} L(\theta_0+h) f_T(X_0, X_1, \dots, X_T),$$

and hence,

$$\frac{1}{h}\left(\mathbf{E}_x^{\theta_0+h} f_T(X_0, X_1, \dots, X_T) - \mathbf{E}_x^{\theta_0} f_T(X_0, X_1, \dots, X_T)\right) = \mathbf{E}_x^{\theta_0} \frac{L(\theta_0+h) - L(\theta_0)}{h} f_T(X_0, X_1, \dots, X_T)$$

for each $h \in (\theta_0 - \epsilon, \theta_0 + \epsilon)$. Because $(L_T(\theta_0+h) - L_T(\theta_0))/h \to L_T'(\theta_0)$ almost surely as $h \to 0$, (5.5) follows if the integrand on the right-hand side is uniformly integrable. To establish such a uniform integrability, pick $\delta > 1$ sufficiently close to 1 so that $a = (1/q + \delta/r + \delta/s)^{-1} > 1$. We will show that

$$\sup_{|h|<\epsilon} \mathbf{E}_x^{\theta_0} \left| \frac{L_T(\theta_0+h) - L_T(\theta_0)}{h} f_T(X_0, X_1, \dots, X_T)\right|^a < \infty.$$

Note that due to the continuous differentiability of $p(\cdot,x,y)$, $(L_T(\theta_0+h) - L_T(\theta_0))/h = L_T'(\theta^*)$ where $|\theta^* - \theta_0| < h$. From Hölder inequality,

$$\begin{aligned}
&\mathbf{E}_x^{\theta_0} \left| \frac{L_T(\theta_0+h) - L_T(\theta_0)}{h} f_T(X_0, X_1, \dots, X_T)\right|^a \\
&= \mathbf{E}_x^{\theta_0} |f_T(X_0, \dots, X_T) L_T'(\theta^*)|^a = \mathbf{E}_x^{\theta_0} \left| f_T(X_0, \dots, X_T) \prod_{i=1}^T p(\theta^*, X_{i-1}, X_i) \sum_{i=1}^T \frac{p'(\theta^*, X_{i-1}, X_i)}{p(\theta^*, X_{i-1}, X_i)} \right|^a \\
&\le \underbrace{\left(\mathbf{E}_x^{\theta_0} \prod_{i=1}^T v_\epsilon^q(X_{i-1}, X_i)\right)^{a/q}}_{\triangleq\, (\mathrm{I})} \underbrace{\left(\mathbf{E}_x^{\theta_0} \left(\sum_{i=1}^T \omega_\epsilon(X_{i-1}, X_i)\right)^{r/\delta}\right)^{a\delta/r}}_{\triangleq\, (\mathrm{II})} \underbrace{(\mathbf{E}_x^{\theta_0} |f_T(X_0, \dots, X_T)|^{s/\delta})^{a\delta/s}}_{\triangleq\, (\mathrm{III})}.
\end{aligned} \tag{5.6}$$

We proceed to proving (I), (II), and (III) are all finite. Starting with (I), note that from (5.1) following a similar argument as in the proof of Theorem 3.1 along with Proposition 3.1, one can check that this implies that (I) is finite, in particular, bounded by $v_0(x)$. We now move onto (III) and omit the proof for (II) because the argument for (II) is similar to (III) but only slightly easier. We start with observing that (5.4) implies that T possesses a finite exponential moment: $\mathbf{E}_x^{\theta_0}\gamma^{-T}<\infty$. Now we turn to proving $\mathbf{E}_x^{\theta_0}|f_T(X_0,X_1,\ldots,X_T)|^{s/\delta}<\infty$, which, in turn, is implied by the finiteness of $\mathbf{E}_x^{\theta_0}|\sum_{j=0}^T\exp(\sum_{k=0}^{j-1}g(X_k))f(X_j)|^{s/\delta}$ due to the assumption of the current theorem. Note that from (5.3) and again following the same reasoning as in the proof of Theorem 3.1,

$$\mathbf{E}_x^{\theta_0}\sum_{j=0}^T\left(\exp\left(\sum_{k=0}^{j-1}g(X_k)\right)f(X_j)\right)^s<\infty. \tag{5.7}$$

Therefore,

$$\begin{aligned}\mathbf{E}_x^{\theta_0}\left(\sum_{j=0}^T\exp\left(\sum_{k=0}^{j-1}g(X_k)\right)f(X_j)\right)^{s/\delta} &\le \mathbf{E}_x^{\theta_0}\left(T\max_{0\le j\le T}\exp\left(\sum_{k=0}^{j-1}g(X_k)\right)f(X_j)\right)^{s/\delta}\\ &\le \left(\mathbf{E}_x^{\theta_0}T^{\frac{s}{\delta-1}}\right)^{\frac{\delta-1}{\delta}}\left(\mathbf{E}_x^{\theta_0}\max_{0\le j\le T}\left(\exp\left(\sum_{k=0}^{j-1}g(X_k)\right)f(X_j)\right)^s\right)^{\frac{1}{\delta}}\\ &\le \left(\mathbf{E}_x^{\theta_0}T^{\frac{s}{\delta-1}}\right)^{\frac{\delta-1}{\delta}}\left(\mathbf{E}_x^{\theta_0}\sum_{j=0}^T\left(\exp\left(\sum_{k=0}^{j-1}g(X_k)\right)f(X_j)\right)^s\right)^{\frac{1}{\delta}}\\ &<\infty.\end{aligned} \tag{5.8}$$

Note that in case (5.3) and (5.4) hold in the neighborhood of θ_0, the previous argument can be carried out for θ's that are sufficiently close to θ_0 as well to show that $u'(\theta)=\mathbf{E}_x^{\theta_0}f_T(X_0,\ldots,X_T)L_T'(\theta)$, and note that the previous argument in fact proves the uniform integrability of $|f_T(X_0,\ldots,X_T)L_T'(\theta)|$'s in a neighborhood of θ_0. Therefore, the continuity of the derivative follows. □

In the case that X is the solution to the stochastic recursion (3.28), the conditions of Theorem 5.1 simplify again. That is, if (5.4) holds, and for some $q,r,s>1$ such that $1/q+1/r+1/s<1$,

$$\mathbf{E}^{\theta_0}v_2(r(x,Z_1))\exp(s\cdot g(x))p(\theta_0,Z_1)\le v_2(x)-|\hat{f}^s(\theta_0,x)|, \tag{5.9}$$

$$\mathbf{E}\left[\sup_{|\theta-\theta_0|<\epsilon}|p(\theta,Z_1)|^q\right]<\infty,\qquad\text{and}\qquad \mathbf{E}\left[\sup_{|\theta-\theta_0|<\epsilon}|p'(\theta,Z_1)/p(\theta,Z_1)|^r\right]<\infty, \tag{5.10}$$

then $u'(\cdot,x)$ is differentiable at θ_0, and if in addition (5.4) and (5.9) hold for θ's in the neighborhood of θ_0, then the derivative is continuous. The proof is similar, with the only difference arising in bounding what corresponds to (5.6). That is, instead of (5.6), here we need to bound

$$\begin{aligned}&\mathbf{E}_x^{\theta_0}|f_T(X_0,\ldots,X_T)L_T'(\theta^*)|^a\\ &=\mathbf{E}_x^{\theta_0}\left|f_T(X_0,\ldots,X_T)\prod_{i=1}^T p(\theta^*,Z_i)\sum_{i=1}^T\frac{p'(\theta^*,Z_i)}{p(\theta^*,Z_i)}\right|^a\\ &\le\underbrace{\left(\mathbf{E}_x^{\theta_0}\prod_{i=1}^T v_\epsilon^{q/\delta}(Z_i)\right)^{a\delta/q}}_{\triangleq\text{ (I)}}\underbrace{\left(\mathbf{E}_x^{\theta_0}\left(\sum_{i=1}^T\omega_\epsilon(Z_i)\right)^{r/\delta}\right)^{a\delta/r}}_{\triangleq\text{ (II)}}\underbrace{\left(\mathbf{E}_x^{\theta_0}|f_T(X_0,\ldots,X_T)|^{s/\delta}\right)^{a\delta/s}}_{\triangleq\text{ (III)}}\end{aligned}$$

uniformly in h where $v_\epsilon(x)=\sup_{|\theta-\theta_0|<\epsilon}|p(\theta,x)|$ and $\omega_\epsilon(x)=\sup_{|\theta-\theta_0|<\epsilon}|p'(\theta,x)/p(\theta,x)|$. For (I), note that $\mathbf{E}^{\theta_0}v_\epsilon^q(Z_1)=\mathbf{E}^{\theta_0}\sup_{|\theta-\theta_0|<\epsilon}p^q(\theta,Z_1)\to 1$ as $\epsilon\to 0$ and $\mathbf{P}(T\ge n)\le\gamma^n\mathbf{E}_x^{\theta_0}\gamma^{-T}$. Therefore, one can choose ϵ small enough, so that $(\mathbf{E}^{\theta_0}v_\epsilon^q(Z_1))^{n/\delta}(\mathbf{P}^{\theta_0}(T\ge n))^{(\delta-1)/\delta}$ decreases at a geometric rate w.r.t. n. For such ϵ,

$$\begin{aligned}\mathbf{E}_x^{\theta_0}\prod_{i=1}^T v_\epsilon^{q/\delta}(Z_i)&=\sum_{n=1}^\infty\mathbf{E}_x^{\theta_0}\left[\prod_{i=1}^n v_\epsilon^{q/\delta}(Z_i);T=n\right]\le\sum_{n=1}^\infty\left(\mathbf{E}_x^{\theta_0}\prod_{i=1}^n v_\epsilon^q(Z_i)\right)^{1/\delta}(\mathbf{P}_x^{\theta_0}(T=n))^{\frac{\delta-1}{\delta}}\\ &\le\sum_{n=1}^\infty(\mathbf{E}_x^{\theta_0}v_\epsilon^q(Z_1))^{n/\delta}(\mathbf{P}_x^{\theta_0}(T\ge n))^{\frac{\delta-1}{\delta}}<\infty.\end{aligned}$$

For (II), from Wald's inequality

$$
\begin{aligned}
\mathbf{E}_x^{\theta_0}\left(\sum_{i=1}^{T}\omega_\epsilon(Z_i)\right)^{r/\delta} &\le \mathbf{E}_x^{\theta_0}\left(\sum_{i=1}^{T}\left(\max_{i=1}^{T}\omega_\epsilon(Z_i)\right)\right)^{r/\delta} \le \mathbf{E}_x\left(T\max_{i=1}^{T}\omega_\epsilon(Z_i)\right)^{r/\delta}\\
&\le \left(\mathbf{E}_x^{\theta_0}T^{\frac{r}{(\delta-1)}}\right)^{\frac{\delta-1}{\delta}}\left(\mathbf{E}_x^{\theta_0}\max_{i=1}^{T}\omega_\epsilon^r(Z_i)\right)^{\frac{1}{\delta}} \le \left(\mathbf{E}_x^{\theta_0}T^{\frac{r}{(\delta-1)}}\right)^{\frac{\delta-1}{\delta}}\left(\mathbf{E}_x^{\theta_0}\sum_{i=1}^{T}\omega_\epsilon^r(Z_i)\right)^{\frac{1}{\delta}}\\
&\le \left(\mathbf{E}_x^{\theta_0}T^{\frac{r}{(\delta-1)}}\right)^{\frac{\delta-1}{\delta}}\left(\mathbf{E}_x^{\theta_0}T\cdot\mathbf{E}\omega_\epsilon^r(Z_1)\right)^{\frac{1}{\delta}} < \infty.
\end{aligned}
$$

The argument for (III) comes from (5.9) in the same way as in Theorem 5.1, and the rest of the argument is identical as well.

It should be noted that the conditions (5.1), (5.2), (5.3), (5.4) are much stronger conditions than the Lyapunov conditions in Sections 3 and 4. The expectations in Sections 3 and 4 that arise as "solutions of linear systems" have the special structure that allows one to fully leverage the fact that the underlying process is a Markov chain. As a consequence, we can establish Lyapunov criteria that come close to allowing one to obtain minimal conditions for smoothness in the setting of such expectations. On the other hand, for general expectations, it seems to be difficult to fully utilize the Markov structure.

Appendix A. The G/G/1 Queue Example

In this section, we prove the following statement: if $\mathbf{P}^\theta(V_0 > v) = (1+\theta v)^{-r-1}$, then $\alpha(\theta) = \pi(\theta)f_p$ is differentiable for $1 \le p < r$, and the derivative is

$$\alpha'(\theta_0) = \mathbf{E}_{\pi(\theta_0)}^{\theta_0} p'(\theta_0, V_0)\Gamma f_p(W_1). \tag{A.1}$$

It turns out to be handy to have the following bound. The proof of the claim will be provided at the end of this section.

Claim A.1. Let $f_{p;m} \triangleq m \wedge f_p$. There is a constant $d > 0$ and $h_0 > 0$ such that

$$|(P(\theta_0+h)\Gamma f_{p;m} - P(\theta_0)\Gamma f_{p;m})(x)| \le hd(x^p+1), \tag{A.2}$$

for $h < h_0$ and $m \in [0,\infty]$.

First note that (4.6) can be established as in Section 4 with $v_0(x) = x^{p+1}$ and $q(x) = x^p$ for any $p < r$, and hence, $|\Gamma f_p(x)| \le c(x^{p+1}+1)$ for $p < r$ by Glynn and Meyn [4], and f_p is $\pi(\theta)$-integrable for $p < r$ by Glynn and Zeevi [5]. Because $\Gamma f_{p;m}$ is $\pi(\theta_0+h)$-integrable (because it is bounded by an affine function), if we let $\alpha_m(\theta) \triangleq \pi(\theta)f_{p;m}$, then $\alpha_m(\theta_0+h) - \alpha_m(\theta_0) = \pi(\theta_0+h)(P(\theta_0+h) - P(\theta_0))\Gamma f_{p;m}$. Monotone convergence theorem guarantees that $\alpha_m(\theta_0+h) - \alpha_m(\theta_0)$ converges to $\alpha(\theta_0+h) - \alpha(\theta_0)$ as $m \to \infty$; on the other hand, applying monotone convergence and then bounded convergence twice along with Glynn and Zeevi [5] and (A.2), one can check that $\pi(\theta_0+h)(P(\theta_0+h) - P(\theta_0))\Gamma f_{p;m} \to \pi(\theta_0+h)(P(\theta_0+h) - P(\theta_0))\Gamma f_p$. Therefore, (4.4) is valid. Now, set $s(x) = x^p+1$ and put $\mathbb{I}_m(x) = \mathbb{I}(x \ge m)$ and $\mathbb{I}_m^c = \mathbb{I}(x < m)$. Then,

$$
\begin{aligned}
\int_{\mathbb{R}_+}\pi(\theta_0+h,dx)\mathbb{I}_m(x)\frac{P(\theta_0+h) - P(\theta_0)}{h}\Gamma f_p(x) &\le d\int \pi(\theta_0+h,dx)\mathbb{I}_m(x)s(x)\\
&\le d\int \pi(\theta_0+h,dx)\frac{x^{p+\epsilon}+1}{s(x)}s(x)\frac{m^p+1}{m^{p+\epsilon}+1}\\
&= cd\frac{m^p+1}{m^{p+\epsilon}+1}
\end{aligned}
\tag{A.3}
$$

for $0 < \epsilon < r - p$ and some constant $c > 0$. On the other hand, let

$$\int_{\mathbb{R}_+}\pi(\theta_0+h,dx)\mathbb{I}_m^c(x)\frac{P(\theta_0+h) - P(\theta_0)}{h}\Gamma f_p(x) \triangleq \pi(\theta_0+h)s_h^m,$$

then, s_h^m is bounded by $c(m^p+1)$ for some c, and hence $\Gamma s_h^m \le a(m^p+1)(x+1)$ for some $a > 0$.

Therefore, by the same argument as in (4.4),

$$
\begin{aligned}
|\pi(\theta_0+h)s_h^m - \pi(\theta_0)s_h^m| &= |\pi(\theta_0+h)(P(\theta_0+h) - P(\theta_0))(\Gamma s_h^m)|\\
&= \int \pi(\theta_0+h,dx)\mathbf{E}_x\Gamma s_h^m([x+V_0-\chi_1]^+)(p(\theta_0+h,V_0)-1)\\
&\le hd'(m^p+1)
\end{aligned}
\tag{A.4}
$$

for some $d' > 0$. Finally,

$$\pi(\theta_0)s_h^m = \int_{\mathbb{R}_+}\pi(\theta_0,dx)\mathbb{I}_m^c(x)\mathbf{E}_x^{\theta_0}\frac{p(\theta_0+h,V_0)-1}{h}\Gamma f_p(W_1).$$

For each x, $s_h^m(x) \to \mathbb{I}_m^c(x)\mathbf{E}_x^{\theta_0}p'(\theta_0+h,V_0)\Gamma f_p(W_1)$ as $h \to 0$ by bounded convergence. Also, due to the definition of $\mathbb{I}_m^c(x)$ and boundedness of p', s_h^m itself is bounded w.r.t. h and x. Therefore, applying the bounded convergence theorem, we conclude that

$$\pi(\theta_0)s_h^m \to \int_{\mathbb{R}_+} \pi(\theta_0, dx)\mathbb{I}_m^c(x)\mathbf{E}_x^{\theta_0} p'(\theta_0, V_0)\Gamma f_p(W_1), \tag{A.5}$$

as $h \to 0$. Therefore, if we let $h \to 0$ and then let $m \to \infty$, (A.3), (A.4), and (A.5) imply (A.1).

Proof of the Claim. First note that

$$P(\theta_0 + h)\Gamma f_{p;m}(x) - P(\theta_0)\Gamma f_{p;m}(x) = \mathbf{E}^{\theta_0} \sum_{j=1}^{\tau_x(0)-1} [m \wedge (x + S_j)^p - \pi(\theta_0) f_{p;m}](p(\theta_0 + h, V_0) - 1).$$

Set $\sigma_x(0) = \inf\{n \geq 1 : x + S_n - V_0 \leq 0\}$, then obviously $\sigma_x(0) \leq \tau_x(0)$. Considering the Taylor expansion of f_p up to $\lfloor p \rfloor$ th term, $(x + S_j)^p = (x + S_j - V_0)^p + R(x, S_j, V_0)$, where

$$\begin{aligned} R(x, S_j, V_0) &= \sum_{n=1}^{\lfloor p \rfloor - 1} c_n (x + S_j - V_0)^{p-n} V_0^n + c_{\lfloor p \rfloor}(x + S_j - V_0 + V_{0,j}^*)^{p - \lfloor p \rfloor} V_0^{\lfloor p \rfloor} \\ &\leq \sum_{n=1}^{\lfloor p \rfloor} c_n (x + S_j - V_0)^{p-n} V_0^n + c_{\lfloor p \rfloor} V_0^p, \end{aligned}$$

where $c_n = \frac{p(p-1)\cdots(p-n)}{n!}$ and $0 \leq V_{0,j}^* \leq V_0$. Note that because $\sigma_x(0)$ and $(x + S_j - V_0)$ are independent of V_0, and $p(\theta_0 + h, V_0) - 1$ is a mean zero rv, $\mathbf{E}^{\theta_0} \sum_{j=1}^{\sigma_x(0)-1} m \wedge (x + S_j - V_0)^p (p(\theta_0 + h), V_0) - 1) = 0$ and $\mathbf{E}^{\theta_0} \sum_{j=1}^{\sigma_x(0)-1} (\pi(\theta_0) f_{p,m})(p(\theta_0 + h), V_0) - 1) = 0$. Applying the generic inequality

$$(a \wedge b)d - c|d| \leq \{a \wedge (b + c)\}d \leq (a \wedge b)d + c|d|, \quad \forall a, b, c > 0,$$

with $a = m$, $b = (x + S_j - V_0)^p$, $c = R(x, S_j, V_0)$, and $d = p(\theta_0 + h, V_0) - 1$,

$$\begin{aligned} &\left| \mathbf{E}^{\theta_0} \sum_{j=1}^{\sigma_x(0)-1} [m \wedge (x + S_j)^p - \pi(\theta_0) f_{p;m}](p(\theta_0 + h, V_0) - 1) \right| \leq \sum_{j=1}^{\sigma_x(0)-1} R(x + S_j - V_0)|p(\theta_0 + h, V_0) - 1| \\ &\leq \sum_{n=1}^{\lfloor p \rfloor} c_n \mathbf{E}^{\theta_0} V_0^n |p(\theta_0 + h, V_0) - 1| \cdot \mathbf{E}^{\theta_0} \sum_{j=1}^{\sigma_x(0)-1} (x + S_j - V_0)^{p-n} + c_{\lfloor p \rfloor} \mathbf{E}^{\theta_0}(\sigma_x(0) - 1)\mathbf{E}^{\theta_0} V_0^p |p(\theta_0 + h, V_0) - 1|. \end{aligned}$$

Note that for $s \leq p$,

$$0 \leq \mathbf{E}^{\theta_0} \sum_{j=1}^{\sigma_x(0)-1} (x + S_j - V_0)^s \leq \mathbf{E}^{\theta_0} \sum_{j=1}^{\tau_x(0)-1} (x + S_j)^s = \Gamma f_s + \pi(\theta_0) f_s \mathbf{E}^{\theta_0} \tau_x(0) \leq c_{s+1}(x^{s+1} + 1). \tag{A.6}$$

Therefore, $\mathbf{E}^{\theta_0} \sum_{j=1}^{\sigma_x(0)-1} \left[m \wedge \left(x + S_j\right)^p - \pi(\theta_0) f_{p;m}\right] \left(p(\theta_0 + h, V_0) - 1\right) = O(hx^p)$. On the other hand,

$$\begin{aligned} \left| \mathbf{E} \sum_{j=\sigma_x(0)}^{\tau_x(0)-1} (x + S_j)^p \right| &\leq \left| \mathbf{E} \sum_{j=0}^{\tau_V(0)-1} (V + S_j)^p \right| \leq \left| \mathbf{E} \sum_{j=0}^{\tau_V(0)-1} (V + S_j)^p - \pi(\theta_0) f_p \right| + \left| \mathbf{E} \sum_{j=0}^{\tau_V(0)-1} \pi(\theta_0) f_p \right|, \\ &\leq |\mathbf{E}\Gamma f_p(V)| + |\pi(\theta_0) f_p \mathbf{E}\gamma(V)| < \infty \end{aligned}$$

where $\gamma(x) \triangleq \mathbf{E}\tau_x(0)$. Likewise, $\mathbf{E} \sum_{j=\sigma_x(0)}^{\tau_x(0)-1} \pi(\theta_0) f_p$ can be bounded by a constant, and the conclusion of the claim follows. □

References

[1] Dunford N, Schwartz JT, Bade WG, Bartle RG (1971) *Linear Operators* (Wiley-Interscience, New York).
[2] Glasserman P (1993) Regenerative derivatives of regenerative sequences. *Adv. Appl. Probab.* 25(1):116–139.
[3] Glynn PW, L'Ecuyer P (1995) Likelihood ratio gradient estimation for stochastic recursions. *Adv. Appl. Probab.* 27(4):1019–1053.
[4] Glynn PW, Meyn SP (1996) A Liapunov bound for solutions of the Poisson equation. Ethier SN, Feng J, Stockbridge RH, eds. *Ann. Probab.* 24(2):916–931.
[5] Glynn PW, Zeevi A (2008) Bounding stationary expectations of Markov processes. *Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz* (Institute of Mathematical Statistics), 195–214.
[6] Heidergott B, Hordijk A (2003) Taylor series expansions for stationary Markov chains. *Adv. Appl. Probab.* 35(4):1046–1070.
[7] Heidergott, B, Hordijk A (2009) Measure-valued differentiation for the cycle cost performance in the G/G/1 queue in the presence of heavy-tailed distributions. *Markov Processes Related Fields* 15:225–253.
[8] Heidergott B, Vázquez-Abad F (2006) Measure-valued differentiation for random horizon problems. *Markov Processes Related Fields* 12(3):509–536.
[9] Heidergott B, Hordijk A, Weisshaupt H (2006) Measure-valued differentiation for stationary Markov chains. *Math. Oper. Res.* 31(1):154–172.
[10] Ibragimov IA, Has'minskii RZ (1981) *Statistical Estimation: Asymptotic Theory*, vol. 16 (Springer Science & Business Media, New York).

[11] Kiefer J, Wolfowitz J (1956) On the characteristics of the general queueing process, with applications to random walk. *Ann. Math. Statist.* 27(1):147–161.
[12] Kifer Y (1986) *Ergodic Theory of Random Transformations.* Progress in Probability, vol. 10 (Birkhäuser, Boston).
[13] Leahu H, Heidergott B, Hordijk A (2013) Perturbation analysis of waiting times in the G/G/1 queue. *Discrete Event Dyn. Syst.* 23(3):277–305.
[14] Lehmann EL, Casella G (2006) *Theory of Point Estimation* (Springer Science & Business Media, New York).
[15] Meyn S, Tweedie R (2009) *Markov Chains and Stochastic Stability* (Cambridge University Press, Cambridge).
[16] Pflug GC (1992) Gradient estimates for the performance of Markov chains and discrete event processes. *Ann. Oper. Res.* 39(1):173–194.
[17] Rosenblatt, M. (1985) *Stationary Sequences and Random Fields* (Birkhäuser, Boston, Basel, Stuttgart).
[18] Serfling RJ (1980) *Approximation Theorems of Mathematical Statistics* (Wiley, New York).
[19] Vázquez-Abad FJ, Kushner HJ (1992) Estimation of the derivative of a stationary measure with respect to a control parameter. *J. Appl. Probab.* 29(2):343–352.