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Large deviations for stochastic fluid networks with Weibullian tails

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Abstract
We consider a stochastic fluid network where the external input processes are compound Poisson with heavy-tailed Weibullian jumps. Our results comprise of large deviations estimates for the buffer content process in the vector-valued Skorokhod space which is endowed with the product J_1 topology. To illustrate our framework, we provide explicit results for a tandem queue. At the heart of our proof is a recent sample-path large deviations result, and a novel continuity result for the Skorokhod reflection map in the product J_1 topology.

Keywords Fluid networks · Large deviations · Skorokhod map · Heavy tails

Mathematics Subject Classification 60K25 · 60F10

1 Introduction

The past 25 years have witnessed a significant research activity on queueing systems with heavy tails, but the important case of queueing networks has received less atten-

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tion. Early papers focused on generalized Jackson networks [2], monotone separable networks [1], and max-plus networks [3]. Recent work on tail asymptotics of transient cycle times and waiting times for closed tandem queueing networks can be seen in Kim and Ayhan [16]. In two joint papers with Foss, Masakiyo Miyazawa investigated queue lengths in a queueing network with feedback in Foss and Miyazawa [12] and tandem queueing networks in Foss and Miyazawa [13]. Compared to standard queueing networks tracking movements of discrete customers, fluid networks are somewhat more tractable. In an early paper, it was recognized that tail asymptotics for downstream nodes could be obtained by analyzing the busy period of upstream nodes, under certain assumptions [7]. The case of a tandem fluid queue where the input to the first node is a Lévy process with regularly varying jump sizes has been investigated in Lieshout and Mandjes [17] exploiting a Laplace transform expression which is available in that case.

More recently, multidimensional asymptotics for the time-dependent buffer content vector in a fluid queue fed by compound Poisson processes were investigated in Chen et al. [8]. The framework in Chen et al. [8] allows for the analysis of situations in which a large buffer content may be caused by multiple big jumps in the input process. Such results were established before for multiple server queues and fluid queues fed by on–off sources (see, for example, [10, 11, 21]). The results on fluid networks in Chen et al. [8] were derived assuming regular variation of the jumps in the arrival processes. Work on fluid networks with light-tailed input is surveyed in Miyazawa [18]. The goal of the present paper is to investigate the case where jumps are semi-exponential (e.g. of Weibull type $\exp\{-x^\alpha\}$ with $\alpha \in (0, 1)$). This case is somewhat more difficult to analyze, especially in the case where rare events of interest are caused by multiple big jumps in the input process, as exhibited in the case of the multiple server queue [4].

We focus on a stochastic fluid network comprised of d nodes, with external inputs modeled as compound Poisson processes with semi-exponential increments. We are interested in the event that an arbitrary linear combination of the buffer contents in the network exceeds a large value. We write this functional as a mapping of the input processes using the well-known multidimensional Skorokhod reflection map on the positive orthant (see, e.g. [20]) and apply a sample-path large deviations principle for the superposition of Poisson processes, which has recently been derived in Bazhba et al. [5]. This sample-path large deviation principle has been established for Poisson processes with semi-exponential jumps and holds in the product J_1 topology. To apply the contraction principle (the analogue of the continuous mapping argument in a large deviations context), we need to show that the Skorokhod map has suitable continuity properties. The J_1 product topology is not as strong as the standard J_1 topology on $\mathbb{R}^d$, and it turns out that continuity can only be established for input processes with nonnegative jumps. However, this result, presented in Theorem 2.1, is sufficient for our proof strategy to work.

The contraction principle leads to an expression of the rate function which we analyze in detail. Under some generality, we show that the upper and lower bound of the large deviations bounds matches. We conjecture that each input process contributes to a large fluid level by a finite number of big jumps, and the computation of the rate function can be reduced to a concave optimization problem with a finite number of

decision variables. We illustrate this by reducing the optimization problem to a finite-dimensional problem and then explicitly solving it for the case $d = 2$ in Section 5.

The outline of this paper is as follows: Sect. 2 contains a description of our model, the topological space in which the input processes are defined, and an introduction to the reflection map. In Sects. 3, 4, and 5, we present our main results: upper and lower large deviation bounds for the buffer content process, logarithmic asymptotics for overflow probabilities of the buffer content process over fixed times, and an explicit analysis of the two-node tandem network. Section 6 contains technical proofs. We end this paper with an appendix where we develop several auxiliary large deviations results.

2 Model description and preliminary results

2.1 The model

In this section, we describe our model and we present some preliminary results that are used in our analysis. We consider a single-class open stochastic fluid network with d nodes. We denote the total amount of external work that arrives at station i with $J_i(t) \triangleq \sum_{j=1}^{N_i(t)} J_i^{(j)}$ which is a compound Poisson process with mean γ_i where $\{J_i^{(j)}\}_{j=1,2,\ldots}$ is an iid jump size sequence for each $i = 1, \ldots, d$. If no exogenous input is assigned to node i, then we set $J_i(\cdot) \equiv 0$, and $\gamma_i \triangleq 0$. We define $\mathcal{J}$ as the subset of nodes that have an exogenous input. We assume that $\{J_1(t) : t \geq 0\}, \{J_2(t) : t \geq 0\}, \ldots, \{J_d(t) : t \geq 0\}$'s are independent. For notational convenience, we assume that the Poisson processes $\{N_i(t)\}_{t\geq 0}$ have unit rate for $i \in \mathcal{J}$. The key assumption on the distribution of the jump sizes $J_i^{(1)}$ for $i \in \mathcal{J}$ is that they are semi-exponential:

Assumption 1 For each $i \in \mathcal{J} \subseteq \{1, \ldots, d\}$, $\mathbf{P}\big(J_i^{(1)} \geq x\big) = e^{-c_i L(x) x^{\alpha}}$ where $\alpha \in (0, 1)$, $c_i \in (0, \infty)$, and L is a slowly varying function such that $L(x)/x^{1-\alpha}$ is non-increasing for sufficiently large x.

Recall that L is slowly varying if $L(ax)/L(x) \to 1$ as $x \to \infty$ for each $a > 0$. At each node $i \in \{1, \ldots, d\}$, the fluid is processed and released at a deterministic rate r_i. Fractions of the processed fluid from each node are then routed to other nodes or leave the network. We characterize the stochastic fluid network by a four-tuple $(\boldsymbol{J}, \boldsymbol{r}, Q, \boldsymbol{X}(0))$, where $\boldsymbol{J}(\cdot) = \big(J_1(\cdot), \ldots, J_d(\cdot)\big)$ is the vector of the assigned input processes at each one of the d nodes, respectively. The vector $\boldsymbol{r} \triangleq (r_1, \ldots, r_d)^{\mathsf{T}}$ is the vector of deterministic output rates at the d nodes, $Q \triangleq [q_{ij}]_{i,j\in\{1,\ldots,d\}}$ is a $d \times d$ substochastic routing matrix, and $\boldsymbol{X}(0) \triangleq (X_1(0), \ldots, X_d(0))$ is a nonnegative random vector of initial contents at the d nodes. If the buffer at node i and at time t is non-empty, then there is fluid output from node i at a constant rate r_i. On the other hand, if the buffer of node i is empty at time t, the output rate equals the minimum of the combined (i.e., both external and internal) input rates and the output rate r_i.

We now provide more details on the stochastic dynamics of our network. A proportion q_{ij} of all output from node i is immediately routed to node j, while the remaining proportion $q_i \triangleq 1 - \sum_{j=1}^{k} q_{ij}$ leaves the network. We assume that $q_{ii} \triangleq 0$, and the

routing matrix Q is substochastic, so that $q_{ij} \geq 0$, and $q_i \geq 0$ for all i, j. We also assume that $Q^n \to 0$ as $n \to \infty$ which implies that all input eventually leaves the network. Let $Q^\intercal$ be the transpose matrix of Q. Though we focus on time-dependent behavior, we consider the scenario that the fluid network is stable, ensuring that a high level of fluid is a rare event. Let $\mathcal{Q} = (\mathrm{I} - Q^\intercal)$. We guarantee the stability of the network by posing the following assumption, based on Kella [15]:

Assumption 2 Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)^\intercal$, and assume that $\boldsymbol{r} > \mathcal{Q}^{-1}\boldsymbol{\gamma}$.

Due to our model specifics, the buffer content at station i is processed at a constant rate r_i from the i-th server, and a proportion q_{ij} is routed from the i-th station to the j-th server. To define the buffer content process, we first define the potential content vector $\boldsymbol{X}(t)$

$$\boldsymbol{X}(t) \triangleq \boldsymbol{X}(0) + \boldsymbol{J}(t) - \mathcal{Q}\boldsymbol{r} \cdot t, \quad t \geq 0.$$

Let $\boldsymbol{Z}_i(t)$ denote the buffer content of the i-th station at time t. We can define the buffer content process by the so-called reflection map. We first provide an intuitive description of this map. It is defined in terms of a pair of processes $(\boldsymbol{Z}, \boldsymbol{Y})$ that solve the differential equation

$$d\boldsymbol{Z}(t) = d\boldsymbol{X}(t) + \mathcal{Q}d\boldsymbol{Y}(t), \ t \geq 0. \tag{2.1}$$

Here, $\boldsymbol{Y}(\cdot)$ is non-decreasing and $\boldsymbol{Y}_i(t)$ only increases at times where $\boldsymbol{Z}_i(t) = 0$ for all i and all t. Consequently, as we assume $\boldsymbol{Z}(0) = 0$, the buffer content is

$$\boldsymbol{Z}(t) = \boldsymbol{X}(t) + \mathcal{Q}\boldsymbol{Y}(t), \ t \geq 0. \tag{2.2}$$

We call the map $\boldsymbol{X} \mapsto (\boldsymbol{Y}, \boldsymbol{Z})$ the reflection map. We now provide a more rigorous definition of this map.

2.2 The reflection map with discontinuities

We start with the definition of the reflection map. Fix an arbitrary $T > 0$. Let $\mathbb{D}[0, T]$ denote the Skorokhod space: the space of càdlàg paths over the time horizon $[0, T]$. Note that for our large deviations analyses, we will consider linearly scaled processes in $\mathbb{D}[0, T]$, and hence, this translates considering the time horizon $[0, nT]$ for the original unscaled processes. Denote with $\mathbb{D}^\uparrow[0, T]$ the subspace of the Skorokhod space consisting of non-decreasing functions that are nonnegative at the origin. Note that we use the component-wise partial order on $\mathbb{D}[0, T]$ and $\mathbb{R}^d$. That is, we write $\boldsymbol{x} \triangleq (x_1, \ldots, x_d) \leq \boldsymbol{y} \triangleq (y_1, \ldots, y_d)$ in $\mathbb{R}^d$ if $x_i \leq y_i$ in $\mathbb{R}$ for all $i \in \{1, \ldots, d\}$, and we write $\boldsymbol{\xi} \triangleq (\xi_1, \ldots, \xi_d) \leq \boldsymbol{\zeta} \triangleq (\zeta_1, \ldots, \zeta_d)$ in $\prod_{i=1}^d \mathbb{D}[0, T]$ if $\boldsymbol{\xi}(t) \leq \boldsymbol{\zeta}(t)$ in $\mathbb{R}^d$ for all $t \in [0, T]$.

Definition 2.1 (Definition 14.2.1 of Whitt [20]) For any $\boldsymbol{\xi} \in \prod_{i=1}^{d} \mathbb{D}[0,T]$ and any reflection matrix $\mathcal{Q} = (\mathrm{I} - Q^{\intercal})$, let the feasible regulator set be

$$\Psi(\boldsymbol{\xi}) \triangleq \left\{ \boldsymbol{\zeta} \in \prod_{i=1}^{d} \mathbb{D}^{\uparrow}[0,T] : \boldsymbol{\xi} + \mathcal{Q}\boldsymbol{\zeta} \geq 0 \right\},$$

and let the reflection map be

$$\boldsymbol{R} \triangleq (\psi, \phi) : \prod_{i=1}^{d} \mathbb{D}[0,T] \to \prod_{i=1}^{d} \mathbb{D}[0,T] \times \prod_{i=1}^{d} \mathbb{D}[0,T],$$

with regulator component

$$\psi(\boldsymbol{\xi}) \triangleq \inf\{\Psi(\boldsymbol{\xi})\} = \inf\left\{ \boldsymbol{w} \in \prod_{i=1}^{d} \mathbb{D}[0,T] : \boldsymbol{w} \in \Psi(\boldsymbol{\xi}) \right\},$$

and content component

$$\phi(\boldsymbol{\xi}) \triangleq \boldsymbol{\xi} + \mathcal{Q}\psi(\boldsymbol{\xi}).$$

The infimum in the definition of ψ may not exist in general. However, in Theorem 14.2.1 of Whitt [20], it is proven that the reflection map is properly defined with the component-wise order. That is,

$$\psi_i(\boldsymbol{\xi})(t) = \inf\{\omega_i(t) \in \mathbb{R} : \boldsymbol{\omega} \in \Psi(\xi)\} \text{ for all } i \in \{1, \ldots, d\} \text{ and } t \in [0,T].$$

In addition, the regulator set $\Psi(\boldsymbol{\xi})$ is non-empty and its infimum is attained in $\Psi(\boldsymbol{\xi})$ itself. Now, we state some important results regarding the properties of (ϕ, ψ). The following result gives an explicit representation of the solution of the Skorokhod problem.

Result 2.1 *(Theorem 14.2.1, Theorem 14.2.5 and Theorem 14.2.7 of Whitt* [20]*) If* $Y(\cdot) = \psi(\boldsymbol{X})(\cdot)$ *and* $\boldsymbol{Z}(\cdot) = \phi(\boldsymbol{X})(\cdot)$*, then* $(\boldsymbol{Y}(\cdot), \boldsymbol{Z}(\cdot))$ *solves the Skorokhod problem associated with the equation* (2.1)*. The mappings* ψ *and* ϕ *are Lipschitz continuous maps w.r.t. the uniform metric.*

The next result is a useful property of the Skorokhod map. It allows us to describe the discontinuities of the reflection map under some mild assumptions.

Result 2.2 *(Lemma 14.3.3, Corollary 14.3.4 and Corollary 14.3.5 of Whitt* [20]*) Consider* $\boldsymbol{\xi} \in \prod_{i=1}^{d} \mathbb{D}[0,T]$*. Let* $Disc(\psi(\boldsymbol{\xi}))$ *and* $Disc(\phi(\boldsymbol{\xi}))$ *denote the sets of discontinuity points of* $\psi(\boldsymbol{\xi})$ *and* $\phi(\boldsymbol{\xi})$*, respectively. Then, it holds that* $Disc(\psi(\boldsymbol{\xi})) \cup Disc(\phi(\boldsymbol{\xi})) = Disc(\boldsymbol{\xi})$*. In addition, if* $\boldsymbol{\xi}$ *has only positive jumps, then* $\psi(\boldsymbol{\xi})$ *is continuous and*

$$\phi(\boldsymbol{\xi})(t) - \phi(\boldsymbol{\xi})(t-) = \boldsymbol{\xi}(t) - \boldsymbol{\xi}(t-).$$

Result 2.3 *(Theorem 14.2.6 of Whitt [20]) If $\boldsymbol{\xi} \leq \boldsymbol{\zeta}$ in $\prod_{i=1}^{d} \mathbb{D}[0,T]$, $T > 0$, then $\psi(\boldsymbol{\xi}) \geq \psi(\boldsymbol{\zeta})$.*

2.3 Topologies and large deviations

In this section, we introduce our preliminary results on sample-path large deviations for the input and the content process. We begin with setting the notation. For any $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$, let $\|\boldsymbol{\beta}\|_1$ denote the usual ℓ_1-norm: $\|\boldsymbol{\beta}\|_1 = \sum_{i=1}^{d} |\beta_i|$. For $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \prod_{i=1}^{d} \mathbb{D}[0,T]$, let $\|\boldsymbol{\xi}\| \triangleq \sup_{t\in[0,T]} \|\boldsymbol{\xi}(t)\|_1$. For large deviations results, we mainly work with the J_1 topology on $\mathbb{D}[0,T]$, and its product topology on $\prod_{i=1}^{d} \mathbb{D}[0,T]$. Recall that in $\mathbb{D}[0,T]$, J_1 topology $\mathcal{T}_{J_1}$ is the one induced by the J_1 metric d_{J_1}:

$$\begin{aligned} d_{J_1}(\xi,\zeta) &= \inf_{\lambda\in\Lambda[0,T]} \left(\sup_{t\in[0,T]} \left|\xi\circ\lambda(t) - \zeta(t)\right| \right) \vee \left(\sup_{t\in[0,T]} \left|\lambda(t) - e(t)\right| \right) \\ &= \inf_{\lambda\in\Lambda[0,T]} \|\xi\circ\lambda - \zeta\| \vee \|\lambda - e\|, \end{aligned}$$

for $\xi, \zeta \in \mathbb{D}[0,T]$, where $e : [0,T] \to [0,T]$ is the identity map $t \mapsto t$, and $\Lambda[0,T]$ is the set of all increasing homeomorphisms from $[0,T]$ to $[0,T]$. In order to study networks, we need to set a topology in the vector-valued function space. That is, we work in the functional space $(\prod_{i=1}^{d} \mathbb{D}[0,T], \prod_{i=1}^{d} \mathcal{T}_{J_1})$ which is a product space equipped with the product J_1 topology $\prod_{i=1}^{d} \mathcal{T}_{J_1}$, which is induced by the product metric d_p:

$$d_p(\boldsymbol{\xi},\boldsymbol{\zeta}) = \sum_{i=1}^{d} d_{J_1}(\xi_i,\zeta_i)$$

for $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \prod_{i=1}^{d} \mathbb{D}[0,T]$ such that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)$ and $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d)$. Unless specified otherwise, all the topological properties discussed in this paper are w.r.t. the topology generated by d_p.

2.3.1 Some useful continuous functions

The following two lemmas are elementary. Their proofs are provided in Appendix A.

Lemma 2.2 *For $\boldsymbol{\beta} \in \mathbb{R}^d$, let $\Upsilon^{\boldsymbol{\beta}} : \prod_{i=1}^{d} \mathbb{D}[0,T] \to \prod_{i=1}^{d} \mathbb{D}[0,T]$ be such that $\Upsilon^{\boldsymbol{\beta}}(\boldsymbol{\xi})(t) = \boldsymbol{\xi}(t) + \boldsymbol{\beta}\cdot t$. Then,*

(i) *$\Upsilon^{\boldsymbol{\beta}}$ is Lipschitz continuous w.r.t. d_p,*
(ii) *$\Upsilon^{\boldsymbol{\beta}}$ is a homeomorphism.*

Lemma 2.3 *For any $\boldsymbol{b} \in \mathbb{R}^d$, the mapping $\boldsymbol{\xi} \mapsto \boldsymbol{b}^{\intercal}\boldsymbol{\xi}(T)$ from $\prod_{i=1}^{d} \mathbb{D}[0,T]$ to $\mathbb{R}$ is Lipschitz continuous w.r.t. d_p.*

A key step in our approach is to establish the Lipschitz continuity of the regulator and the buffer content maps w.r.t. d_p. This is executed in Proposition 2.1 and Theorem 2.1. Their proofs are provided in Sect. 6. Recall that $\mathbb{D}^{\uparrow}[0,T]$ is the subspace of the Skorokhod space containing non-decreasing paths which are nonnegative at the origin. We say that $\xi \in \mathbb{D}[0,T]$ is a pure jump function if $\xi = \sum_{j=1}^{\infty} x^{(j)} \mathbb{1}_{[u^{(j)},T]}$ for some $x^{(j)}$'s and $u^{(j)}$'s such that $x^{(j)} \in \mathbb{R}$ and $u^{(j)} \in [0,T]$ for each j, and the $u^{(j)}$'s are all distinct. Let $\mathbb{D}^{\uparrow}_{\leqslant\infty}[0,T]$ be the subspace of $\mathbb{D}[0,T]$ consisting of non-decreasing pure jump functions that assume nonnegative values at the origin. Subsequently, let $\mathbb{D}^{\uparrow}_{\leqslant k}[0,T] \triangleq \{\xi \in \mathbb{D}[0,T] : \xi = \sum_{j=1}^{k} x^{(j)} \mathbb{1}_{[u^{(j)},T]},\ x^{(j)} \geq 0,\ u^{(j)} \in [0,T],\ j = 1,\ldots,k\}$ be the subset of $\mathbb{D}^{\uparrow}_{\leqslant\infty}[0,T]$ containing pure jump functions of at most k jumps. In addition, for $\beta \in \mathbb{R}$, let $\mathbb{D}^{\beta}_{\leqslant k}[0,T] \triangleq \{\zeta \in \mathbb{D}[0,T] : \zeta(t) = \xi(t) + \beta \cdot t,\ \xi \in \mathbb{D}^{\uparrow}_{\leqslant k}[0,T]\}$ and $\mathbb{D}^{\beta}_{\leqslant\infty}[0,T] \triangleq \{\zeta \in \mathbb{D}[0,T] : \zeta(t) = \xi(t) + \beta \cdot t,\ \xi \in \mathbb{D}^{\uparrow}_{\leqslant\infty}[0,T]\}$. Let $\mathbb{D}_{\leqslant k}[0,T]$ denote the subspace of $\mathbb{D}[0,T]$ consisting of paths with at most k jumps, i.e. $\mathbb{D}_{\leqslant k}[0,T] = \{\xi \in \mathbb{D}[0,T] : |Disc(\xi)| \leq k\}$. Finally, let $\mathbb{D}^{\beta}[0,T] \triangleq \{\zeta \in \mathbb{D}[0,T] : \zeta(t) = \xi(t) + \beta \cdot t,\ \xi \in \mathbb{D}^{\uparrow}[0,T]\}$.

Proposition 2.1 *Let $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_d) \in \mathbb{R}^d$. The regulator map ψ is Lipschitz continuous w.r.t. d_p on $\prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T]$ with Lipschitz constant at most $d(2d^2(2d+1)K\|\boldsymbol{\beta}\|_1 + Kd \vee 1)$.*

Since $\phi(\boldsymbol{\xi}) = \boldsymbol{\xi} + \mathcal{Q}\psi(\boldsymbol{\xi})$, the following is a corollary of Proposition 2.1.

Theorem 2.1 *Let $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_d) \in \mathbb{R}^d$. The reflection map $\boldsymbol{R} = (\phi,\psi)$ is Lipschitz continuous w.r.t. d_p on $\prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T]$.*

Note that the restriction of the domain to the paths without downward jumps is essential for this type of results to hold. Since the order in which the jumps take place matters for the action of the reflection map, we cannot ensure the continuity of the reflection map without such extra regularity conditions. The main difficulty arises with paths which have jumps with different signs in multiple coordinates appearing almost simultaneously (K. Ramanan, personal communication).

2.3.2 The extended sample-path LDP for the potential buffer content process

We first review the notion of extended LDP. Let $(\mathbb{S},d)$ be a metric space, and $\mathcal{T}$ denote the topology induced by the metric d. Let $\{X_n\}$ be a sequence of $\mathbb{S}$-valued random variables. Let I be a nonnegative lower semi-continuous function on $\mathbb{S}$, and $\{a_n\}$ be a sequence of positive real numbers that tends to infinity as $n \to \infty$.

Definition 2.4 The probability measures of (X_n) satisfy an *extended* LDP in $(\mathbb{S},d)$ with speed a_n and rate function I if

$$-\inf_{x\in A^{\circ}} I(x) \leq \liminf_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq -\lim_{\epsilon\to 0} \inf_{x\in A^{\epsilon}} I(x)$$

for any measurable set A.

Here, we denote $A^\epsilon \triangleq \{\xi \in \mathbb{S} : d(\xi, A) \leq \epsilon\}$ where $d(\xi, A) = \inf_{\zeta \in A} d(\xi, \zeta)$. The notion of an extended LDP has been introduced in Borovkov and Mogulskii [6] and is useful in the setting of semi-exponential random variables, in which a full LDP is provably impossible, as shown in Bazhba et al. [5]. One important implication of extended LDP is an analog of the contraction principle. In the context of the extended LDP, the contraction principle requires Lipschitz continuity as opposed to mere continuity; see Lemma B.3.

The main results of this paper in Sects. 3, 4, and 5 are based on such contraction principles coupled with an extended LDP associated with the probability measures of the input process $\boldsymbol{J}(\cdot)$. Specifically, the time evolution of $\boldsymbol{Z}(\cdot)$ may be written as

$$\boldsymbol{Z}(t) = \boldsymbol{J}(t) - \boldsymbol{\gamma} t + (\boldsymbol{\gamma} - \mathcal{Q}\boldsymbol{r})t + \mathcal{Q}\boldsymbol{Y}(t), \quad t \geq 0.$$

Equivalently, if we consider the scaled and centered input process $\bar{\boldsymbol{J}}_n(\cdot) \triangleq \frac{1}{n}\boldsymbol{J}(n\cdot) - \boldsymbol{\gamma} \cdot e(\cdot)$, scaled potential buffer content process $\boldsymbol{X}_n(\cdot) \triangleq \frac{1}{n}\boldsymbol{X}(n\cdot)$, scaled regulator $\boldsymbol{Y}_n \triangleq \frac{1}{n}\boldsymbol{Y}(n\cdot)$, and scaled buffer content $\boldsymbol{Z}_n \triangleq \frac{1}{n}\boldsymbol{Z}(n\cdot)$, then

$$\boldsymbol{Z}_n(t) = \bar{\boldsymbol{J}}_n(t) + \boldsymbol{\kappa} t + \mathcal{Q}\boldsymbol{Y}_n(t), \quad t \geq 0,$$

where $\boldsymbol{\kappa} \triangleq \boldsymbol{\gamma} - \mathcal{Q}\boldsymbol{r}$. Note that $\boldsymbol{Z}_n = \phi(\boldsymbol{X}_n) = \phi \circ \Upsilon^{\boldsymbol{\kappa}}(\bar{\boldsymbol{J}}_n)$. Therefore, an extended LDP for $\boldsymbol{Z}_n$ can be deduced from that of $\boldsymbol{X}_n$, which, in turn, can be deduced from that of $\bar{\boldsymbol{J}}_n$, if ϕ and $\Upsilon^{\boldsymbol{\kappa}}$ are Lipschitz continuous in J_1 topology. Hence, the Lipschitz continuity of the shifting operator $\Upsilon^{\boldsymbol{\kappa}}$ and the content component map ϕ proved earlier in this section will play pivotal roles in our approach.

Now, we conclude this section with establishing the desired extended LDP for the multidimensional input process $\bar{\boldsymbol{J}}_n$ and the potential buffer content process $\boldsymbol{X}_n$ of the stochastic fluid network. For any $\xi \in \mathbb{D}[0, T]$, let

$$I(\xi) = \sum_{\{t:\xi(t)\neq\xi(t-)\}} (\xi(t) - \xi(t-))^\alpha .$$

The next result is an immediate consequence of Theorem 2.3 and Remark 2.2 in Bazhba et al. [5], combined with Lemma B.1.

Result 2.4 *The probability measures of $\bar{\boldsymbol{J}}_n$ satisfy the extended LDP in $\left(\prod_{i=1}^d \mathbb{D}^{-\gamma_i}[0, T], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$ with speed $L(n)n^\alpha$ and rate function $I^{(d)} : \prod_{i=1}^d \mathbb{D}^{-\gamma_i}[0, T] \to [0, \infty]$, where*

$$I^{(d)}(\boldsymbol{\xi}) = \begin{cases} \sum_{j\in\mathcal{J}} c_j I(\xi_j) & \text{if } \xi_j \in \mathbb{D}^{\uparrow}_{\lneq\infty}[0, T] \text{ for } j \in \mathcal{J} \text{ and } \xi_j \equiv 0 \text{ for } j \notin \mathcal{J}, \\ \infty & \text{otherwise.} \end{cases} \tag{2.3}$$

Next, recall that $\boldsymbol{X}_n = \Upsilon^{\boldsymbol{\kappa}}(\bar{\boldsymbol{J}}_n)$. Due to Lemma 2.2, $\Upsilon^{\boldsymbol{\kappa}}$ is Lipschitz continuous and is a homeomorphism with respect to the product J_1 metric. The following extended large deviation principle for $\boldsymbol{X}_n(\cdot)$ is a direct consequence of Result 2.4 and *ii)* of Lemma B.3.

Result 2.5 *The probability measures of $\boldsymbol{X}_n$ satisfy an extended LDP in $\left(\prod_{i=1}^d \mathbb{D}^{-(\mathcal{Q}\boldsymbol{r})_i}[0,T], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$ with speed $L(n)n^\alpha$ and with rate function*

$$\tilde{I}^{(d)}(\boldsymbol{\xi}) = \begin{cases} \sum_{j\in\mathcal{J}} c_j I(\xi_j) & \text{if } \xi_j \in \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_j}[0,T] \text{ for } j\in\mathcal{J} \\ & \quad \text{and } \xi_j = -(\mathcal{Q}\boldsymbol{r})_j \cdot e \text{ for } j\notin\mathcal{J}, \\ \infty & \text{otherwise.} \end{cases} \tag{2.4}$$

We are now ready to state our first main result in the next section.

3 Large deviations for the buffer content process

In this section, we state large deviation bounds for the scaled buffer content process $\boldsymbol{Z}_n$. We apply an analogue of the contraction principle for extended LDPs (Lemma B.3) to obtain asymptotic estimates for the probability measures of $(\boldsymbol{Z}_n)$:

Theorem 3.1 *The probability measures of $\boldsymbol{Z}_n$ satisfy:*

i) For any set F that is closed in $\left(\prod_{i=1}^d \mathbb{D}[0,T], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$,

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\boldsymbol{Z}_n \in F\right) \leq -\lim_{\epsilon\to 0} \inf_{\boldsymbol{\xi}\in F^\epsilon} I_{\boldsymbol{Z}}(\boldsymbol{\xi}).$$

ii) For set G that is open in $\left(\prod_{i=1}^d \mathbb{D}[0,T], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$,

$$\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\boldsymbol{Z}_n \in G\right) \geq -\inf_{\boldsymbol{\xi}\in G} I_{\boldsymbol{Z}}(\boldsymbol{\xi}),$$

where

$$\begin{aligned} I_{\boldsymbol{Z}}(\zeta) &= \inf\left\{\tilde{I}^{(d)}(\boldsymbol{\xi}) : \boldsymbol{\zeta} = \phi(\boldsymbol{\xi}),\ \boldsymbol{\xi} \in \prod_{i=1}^d \mathbb{D}^{-(\mathcal{Q}\boldsymbol{r})_i}[0,T]\right\} \\ &= \inf\left\{\tilde{I}^{(d)}(\boldsymbol{\xi}) : \boldsymbol{\xi} \in \phi^{-1}(\boldsymbol{\zeta})\right\}. \end{aligned}$$

Note that $I_{\boldsymbol{Z}}$ may not be lower semi-continuous, because $\tilde{I}^{(d)}$ is not a good rate function; see Bazhba et al. [5] for details.

Proof Theorem 2.1 ensures that ϕ is Lipschitz continuous w.r.t. d_p. Therefore, the upper and lower bounds in i) and ii) follow immediately from the extended LDP for $\boldsymbol{X}_n$ (Result 2.5) and the (Lipschitz) contraction principle (Lemma B.3). □

The function $I_{\boldsymbol{Z}}$ is the solution of a constrained minimization problem over step functions, with a concave objective function, and a constraint that depends on the solution of the Skorokhod problem displayed in Theorem 3.1. Though this Skorokhod problem only needs to be evaluated for step functions, this minimization problem is in general not tractable. To get more concrete results, we look at more specific functionals of the buffer content process in subsequent sections.

4 Asymptotics for overflow probabilities

This section examines the probability that the buffer content associated with a subset of nodes in the system exceeds a high level. In particular, we fix $\boldsymbol{b} = (b_1, \ldots, b_d) \in \mathbb{R}_+^d$ and study the probability of linear combination of the buffer content at the end of the time horizon exceeding a threshold given by $\mathbf{P}(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y)$. Note that for the unscaled process Z, this is the probability of congestion at time nT. Let

$$I'(x) \triangleq \inf \left\{ \tilde{I}^{(d)}(\boldsymbol{\xi}) : \boldsymbol{b}^\intercal \phi(\boldsymbol{\xi})(T) = x,\ \boldsymbol{\xi} \in \prod_{i=1}^{d} \mathbb{D}^{-(\mathcal{Q}\boldsymbol{r})_i}[0, T] \right\}$$

Define the set $V_{\geqslant}(y) \triangleq \{\boldsymbol{\xi} \in \prod_{i=1}^d \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_i}[0, T] : \boldsymbol{b}^\intercal \phi(\boldsymbol{\xi})(T) \geq y\}$, and let $V_{\geqslant}^*(y)$ be the optimal value of $\tilde{I}^{(d)}$ over the set $V_{\geqslant}(y)$, i.e. $V_{\geqslant}^*(y) \triangleq \inf_{\boldsymbol{\xi} \in V_{\geqslant}(y)} \tilde{I}^{(d)}(\boldsymbol{\xi})$. Similarly, let $V_{>}(y) \triangleq \{\boldsymbol{\xi} \in \prod_{i=1}^d \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_i}[0, T] : \boldsymbol{b}^\intercal \phi(\boldsymbol{\xi})(T) > y\}$ and set $V_{>}^*(y) \triangleq \inf_{\boldsymbol{\xi} \in V_{>}(y)} \tilde{I}^{(d)}(\boldsymbol{\xi})$. Note that $V_{\geqslant}^*(y)$ and $V_{>}^*(y)$ depend on T, but we suppress the dependence for notational simplicity.

Recall that $\mathcal{J}$ is the set of nodes with exogenous input. Next, let $I^+ \triangleq \{j \in \{1, \ldots, d\} : b_j > 0\}$. The following two lemmas, proven in Sect. 6, ensure the continuity of $V_{\geqslant}^*(\cdot)$.

Lemma 4.1 *Assume that $\mathcal{J} \cap I^+ \neq \emptyset$. The map $x \mapsto V_{\geqslant}^*(x)$ is α-Hölder continuous:*

$$|V_{\geqslant}^*(y) - V_{\geqslant}^*(x)| \leq \left(\max_{i \in I^+} \frac{c_i}{b_i^\alpha} \right) \cdot |y - x|^\alpha.$$

Lemma 4.2 *Assume that $\mathcal{J} \cap I^+ \neq \emptyset$. It holds that $V_{\geqslant}^*(y) = V_{>}^*(y)$.*

We are ready to prove the main result of this section:

Theorem 4.1 *For a fixed $\boldsymbol{b} = (b_1, \ldots, b_d) \in \mathbb{R}_+^d$, assume that $\mathcal{J} \cap I^+ \neq \emptyset$. The overflow probabilities $\mathbf{P}\left(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y\right)$ satisfy the following logarithmic asymptotics:*

$$\lim_{n \to \infty} \frac{1}{L(n) n^\alpha} \log \mathbf{P}(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y) = -V_{\geqslant}^*(y). \tag{4.1}$$

Proof Note first that from Lemma 2.3, (1) $\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T)$ is a Lipschitz (w.r.t. d_p) image of $\boldsymbol{Z}_n$. Note also that $I'(y) = \inf\{I_{\boldsymbol{Z}}(\xi) : \boldsymbol{b}^\intercal \xi(T) = y\}$. Therefore, applying Lemma B.3 *i)* and Theorem 3.1, we get the asymptotic upper and lower bounds for $\frac{1}{L(n)n^\alpha} \log \mathbf{P}(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y)$ as follows:

$$\limsup_{n \to \infty} \frac{1}{L(n) n^\alpha} \log \mathbf{P}(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y) \leq - \lim_{\epsilon \to 0} \inf_{x \in [y-\epsilon, \infty)} I'(x)$$

and

$$\liminf_{n \to \infty} \frac{1}{L(n) n^\alpha} \log \mathbf{P}(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y) \geq - \inf_{x \in (y, \infty)} I'(x).$$

However, from Lemmas (4.1,4.2),

$$-\lim_{\epsilon\to 0}\inf_{x\in[y-\epsilon,\infty)} I'(x) = -\lim_{\epsilon\to 0} V^*_{\geqslant}(y-\epsilon) = -V^*_{\geqslant}(y);$$
$$-\inf_{x\in(y,\infty)} I'(x) = -V^*_{>}(y) = -V^*_{\geqslant}(y).$$

That is, the upper and lower bounds for lim sup and lim inf match, and hence, the limit (4.1) exists and equals $-V^*_{\geqslant}(y)$. □

Note that $V^*_{\geqslant}(y)$ is the solution of an infinite-dimensional optimization problem. We conjecture that in many problem instances, there exists a $k \geq 1$ (that depends on the specific network) such that $\prod_{i=1}^{d} \mathbb{D}_{\leqslant k}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_i}[0,T]$ contains an optimal path that minimizes the rate function $\tilde{I}(\cdot)$ over $V_{\geqslant}(y)$. In such cases, $V^*_{\geqslant}(y)$ can be computed by solving the following optimization problem. For given $\boldsymbol{b} \in \mathbb{R}^d_+$ and $y > 0$, let $P^*_{y,k}$ denote the optimal value of the following optimization problem:

$$\begin{aligned}
\inf\quad & \sum_{i=1}^{d} c_i \sum_{j=1}^{k} \left(x_i^{(j)}\right)^{\alpha} \\
\text{s.t.}\quad & \boldsymbol{b}^{\mathsf{T}}\phi(\boldsymbol{\xi})(T) \geq y; \\
& \xi_i = \textstyle\sum_{j=1}^{k} x_i^{(j)} \mathbb{1}_{[u_i^{(j)},T]} + (\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_1 \cdot e; \\
& x_i^{(j)} \geq 0 \text{ for } i \in \mathcal{J},\ j \in \{1,\ldots,k\}, \quad \text{and} \quad x_i^{(j)} = 0 \text{ for } i \notin \mathcal{J},\ j \in \{1,\ldots,k\}; \\
& u_i^{(j)} \in [0,T] \text{ for } i \in \{1,\ldots,d\},\ j \in \{1,\ldots,k\}.
\end{aligned} \tag{$P_{y,k}$}$$

Then, $P^*_{y,k} = V^*_{\geqslant}(y)$. Note that this means that the large deviations rate is the solution of a $2kd$-dimensional optimization problem: The decision variables are the size $x_i^{(j)}$, and the time $u_i^{(j)}$ of the k jumps ($j \in \{1,\ldots,k\}$) in the d coordinates ($i \in \{1,\ldots,d\}$). This provides a significant reduction in complexity compared to the general setting of Sect. 3. Nevertheless, even the finite-dimensional problem ($P_{y,k}$) is still rather intricate: It is an L^α-norm minimization problem with $\alpha \in (0,1)$. In general, such problems are strongly NP-hard; see Ge et al. [14], for example. In addition, checking whether a solution to ($P_{y,k}$) is feasible requires one to compute the Skorokhod map ϕ for step functions, which is non-trivial. To get more explicit results and gain some physical insights, we consider a two-node tandem network in the next section, where we can reduce the computation of $V^*_{\geqslant}(y)$ down to solving ($P_{y,k}$) with $k = 1$.

5 A two-node example

We consider a two-node tandem network where content from node 1 flows into node 2, and content from node 2 leaves the system, i.e. $q_{12} = 1$, and $q_{ij} = 0$ otherwise. We

assume that each node has an exogenous input process (i.e. $\mathcal{J} = \{1, 2\}$). We consider the problem of identifying the log-asymptotics of the probability of congestion in the second node, i.e., $\mathbf{P}\left(\boldsymbol{b}^{\intercal}\mathbf{Z}_n(T) \geq y\right)$ as $n \to \infty$ where $\boldsymbol{b} = (0, 1)$. That is, our goal is to compute $V^*_{\geqslant}(y)$ in this specific example.

The next lemma enables us to reduce the feasible region of the optimization problem associated with $V^*_{\leqslant}(y)$ from $\mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_1}_{\leqslant\infty}[0, T] \times \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_2}_{\leqslant\infty}[0, T]$ down to $\mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_1}_{\leqslant 1}[0, T] \times \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_2}_{\leqslant 1}[0, T]$. In other words, we can restrict the class of functions to those that have at most one discontinuity in each coordinate.

Lemma 5.1 *Consider the two-node tandem network where $d = 2$ and $\mathcal{Q} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Let $\boldsymbol{\xi} \in \prod_{i=1}^{d} \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_i}_{\leqslant\infty}[0, T]$. Then, there exists a path $\tilde{\boldsymbol{\xi}} \in \prod_{i=1}^{d} \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_i}_{\leqslant 1}[0, T]$ such that*

i) $\tilde{I}^{(d)}(\tilde{\boldsymbol{\xi}}) \leq \tilde{I}^{(d)}(\boldsymbol{\xi})$,
ii) $\phi(\tilde{\boldsymbol{\xi}})(T) \geq \phi(\boldsymbol{\xi})(T)$.

Lemma 5.1 implies that computing $V^*_{\geqslant}(y)$ is equivalent to solving ($P_{y,k}$) with $k = 1$ in case of the two-node tandem networks. Such computation is the subject of the rest of this section. To keep the presentation concise, we give an outline of the key steps and focus on physical insight.

We first develop an explicit expression for the buffer content at time T for input processes of the form $\xi_i = (\boldsymbol{\gamma} - \mathcal{Q}\boldsymbol{r})_i \cdot e + x_i \mathbb{1}_{[u_i, T]}$, $t \in [0, T]$, $x_i \geq 0$, $u_i \in [0, T]$, $i = 1, 2$. To develop physical intuition is it instructive to write the buffer content process at node 2 as the solution of a one-dimensional reflection mapping, fed by the superposition of ξ_2 and the output process of node 1, which in turn is governed by a one-dimensional reflection mapping as well. To this end, observe that $\psi_1(\boldsymbol{\xi})(t) = -\inf_{s\leq t}\{0 \wedge \xi_1(s)\}$, and $\phi_1(\boldsymbol{\xi})(t) = \xi_1(t) - \inf_{s\leq t}\{0 \wedge \xi_1(s)\}$. Note also that $(\boldsymbol{\xi} + \mathcal{Q}\psi(\boldsymbol{\xi}))_2 = \xi_2 - \psi_1(\boldsymbol{\xi}) + \psi_2(\boldsymbol{\xi})$, and the minimal $\psi_2(\boldsymbol{\xi})$ that regulates this process above zero is $\psi_2(\boldsymbol{\xi})(t) = -\inf_{s\leq t}\{0 \wedge (\xi_2(s) + \inf_{u\leq s}\{0 \wedge \xi_1(u)\})\}$. Consequently, we can write

$$\phi_2(\boldsymbol{\xi})(T) = \xi_2(T) + \inf_{s\leq T}\{0 \wedge \xi_1(s)\} - \inf_{u\leq T}\left\{0 \wedge \left\{\xi_2(u) + \inf_{s\leq u}\{0 \wedge \xi_1(s)\}\right\}\right\}. \quad (5.1)$$

Our goal is to minimize the cost $c_1 x_1^{\alpha} + c_2 x_2^{\alpha}$ subject to the constraint $\phi_2(\boldsymbol{\xi})(T) \geq y$, over $x_1 \geq 0$, $x_2 \geq 0$, $u_1 \in [0, T]$, $u_2 \in [0, T]$. We simplify this problem by identifying convenient choices of u_1 and u_2 which do not lose optimality.

To this end, observe that a jump of size x_2 at time u_2 can instead take place at time $u_2 = T$ without decreasing $\phi_2(\boldsymbol{\xi})(T)$. To determine a convenient choice of u_1, note that a jump of size x_1 in node 1 at time u_1 causes an outflow of rate r_1 from node 1 to node 2 in the interval $[u_1, u_1 + x_1/(r_1 - \gamma_1)]$, and rate γ_1 after time $u_1 + x_1/(r_1 - \gamma_1)$. Therefore, we can take u_1 such that $u_1 + x_1/(r_1 - \gamma_1) = T$, without decreasing $\phi_2(\boldsymbol{\xi})(T)$. This choice is feasible as long as u_1 remains nonnegative, i.e. we require that $x_1/(r_1 - \gamma_1) \leq T$. Observe that choosing $x_1/(r_1 - \gamma_1) > T$ would not be optimal, as it would increase the cost term involving x_1^{α} without increasing $\phi_2(\boldsymbol{\xi})(T)$.

We proceed by solving (5.1) by taking $\xi_1 = (\boldsymbol{\gamma} - \mathcal{Q}\boldsymbol{r})_1 \cdot e + x_1 \mathbb{1}_{[T-x_1/(r_1-\gamma_1),T]}$ and $\xi_2 = (\boldsymbol{\gamma} - \mathcal{Q}\boldsymbol{r})_2 \cdot e + x_2 \mathbb{1}_{[T,T]}$. Straightforward manipulations show that

$$\phi_2(\boldsymbol{\xi})(T) = x_2 + (r_1 + \gamma_2 - r_2)^+ \frac{x_1}{r_1 - \gamma_1}. \tag{5.2}$$

We see that a jump at node 1 has no effect on the buffer content in node 2 if $r_2 \geq r_1 + \gamma_2$, which is intuitively obvious since node 2 is still rate stable when the output of node 1 equals r_1. Therefore, $x_1 = 0$ and $x_2 = y$ are feasible and minimize the rate function. Our first conclusion is that

$$\lim_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y\right) = -c_2 y^\alpha, \quad r_2 \geq r_1 + \gamma_2. \tag{5.3}$$

We now turn to the more interesting case $r_2 < r_1 + \gamma_2$. We do not lose optimality if the constraint on $\phi_2(\boldsymbol{\xi})(T)$ is tight, so we can impose the constraints

$$x_2 + \frac{r_1 + \gamma_2 - r_2}{r_1 - \gamma_1} x_1 = y, \quad x_1 \in [0, (r_1 - \gamma_1)T], \quad x_2 \geq 0. \tag{5.4}$$

From convex optimization theory, see Corollary 32.3.2 in Rockafellar [19], the minimum of the concave objective function $c_1 x_1^\alpha + c_2 x_2^\alpha$ subject to the constraints (5.4) is achieved over the extreme points of (5.4). In our particular situation, this implies that an optimal solution should correspond to one of the following 3 cases: (i) $x_1 = 0$, (ii) $x_2 = 0$, and (iii) $x_1 = (r_1 - \gamma_1)T$. In case (iii), we would have $x_2 = y - (r_1 + \gamma_2 - r_2)T$, which is only feasible if $y \geq (r_1 + \gamma_2 - r_2)T$. Note also that if $y = (r_1 + \gamma_2 - r_2)T$, then (ii) is the case.

Therefore, if $y \leq (r_1 + \gamma_2 - r_2)T$, we can conclude that either case (i) holds with $x_1 = 0, x_2 = y$, and cost $c_2 y^\alpha$, or case (ii) holds with $x_2 = 0$, $x_1 = y\frac{r_1-\gamma_1}{r_1+\gamma_2-r_2}$, and cost $c_1 \left(y\frac{r_1-\gamma_1}{r_1+\gamma_2-r_2}\right)^\alpha$. We conclude that for $y \leq (r_1 + \gamma_2 - r_2)T$,

$$\lim_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y\right) = -\min\left\{c_1 \left(\frac{r_1 - \gamma_1}{r_1 + \gamma_2 - r_2}\right)^\alpha, c_2\right\} y^\alpha. \tag{5.5}$$

We now turn to the case $y > (r_1 + \gamma_2 - r_2)T$. In this case, the time horizon T is small w.r.t. y: The output of node 1 alone is never enough to cause the buffer content of node 2 to reach level y at time T. Thus, case (ii) can be excluded, and we only have to compare case (i) and case (iii). Case (i) has solution $x_2 = y$ with cost $c_2 y^\alpha$. Case (iii) has solution $x_1 = (r_1 - \gamma_1)T$, $x_2 = y - (r_1 + \gamma_2 - r_2)T$, with cost $c_1((r_1 - \gamma_1)T)^\alpha + c_2(y - (r_1 + \gamma_2 - r_2)T)^\alpha$. We conclude that if $y > (r_1 + \gamma_2 - r_2)T$,

$$\begin{aligned} &\lim_{n\to\infty} \tfrac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\boldsymbol{b}^\intercal \boldsymbol{Z}_n(T) \geq y\right) \\ &= -\min\{c_2 y^\alpha, c_1((r_1 - \gamma_1)T)^\alpha + c_2(y - (r_1 + \gamma_2 - r_2)T)^\alpha\}. \end{aligned} \tag{5.6}$$

To give a numerical example, take $y = 2$, $T = 1$, $r_1 = r_2 = 3$, $\gamma_1 = \gamma_2 = 1$. In this case, the inequality $y > (r_1 + \gamma_2 - r_2)T$ holds. To evaluate (5.6), note that the cost of

case (i) equals $c_2 2^\alpha$ and the cost of case (iii) equals $c_1 2^\alpha + c_2$. So we conclude that case (iii) is the most likely way for the event $\{\boldsymbol{b}^\top \boldsymbol{Z}_n(1) \geq 2\}$ to occur if $c_1 \leq c_2(1 - 2^{-\alpha})$, corresponding to a most likely behavior of two big jumps: $x_1 = 2$, occurring at node 1 at time 0, and $x_2 = 1$, occurring at node 2 at time 1.

One may wonder if Lemma 5.1 can be extended to general stochastic fluid networks so that the computation of $V^*_{\geqslant}(y)$ can always be reduced to solving $(P_{y,k})$ with $k = 1$. (This means that their large deviations behaviors are consequences of at most one jump in the external input process to each node.) Unfortunately, this is not the case. We conclude this section with an example for which restricting the number of jumps in each coordinate to at most one is strictly suboptimal.

Consider $\alpha = 1/2$, $T = 2$, $y = 2 + \delta\theta$,

$$\boldsymbol{\gamma} = \begin{pmatrix} \epsilon \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} 4+\epsilon \\ 2+\epsilon \\ 1+\epsilon \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} \delta \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{\gamma} - Q\boldsymbol{r} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix},$$

where $\epsilon = 0.1$, $\delta < 1/4$, $\theta < 1$, and $c_1 = c_2 = 1$. Let $\boldsymbol{\xi}$ be the superposition of the fluid limit $(\boldsymbol{\gamma} - Q\boldsymbol{r}) \cdot e$ of the potential buffer content vector and two jumps of size 4 and θ in the first coordinate at the beginning and at the end of the time horizon, respectively. That is,

$$\boldsymbol{\xi}(t) = \begin{pmatrix} -4t + 4\mathbb{1}_{[0,T]}(t) + \theta\mathbb{1}_{[T,T]}(t) \\ 2t \\ t \end{pmatrix}.$$

Then, $\tilde{I}^{(d)}(\boldsymbol{\xi}) = 2 + \sqrt{\theta}$ and

$$\phi(\boldsymbol{\xi})(T) = \begin{pmatrix} \theta \\ 0 \\ 2 \end{pmatrix}.$$

However, any $\tilde{\boldsymbol{\xi}}$ (in the effective domain of $\tilde{I}^{(d)}$) with only one jump in the first coordinate takes the following form:

$$\tilde{\boldsymbol{\xi}}(t) = \begin{pmatrix} -4t + x\mathbb{1}_{[s,T]}(t) \\ 2t \\ t \end{pmatrix}$$

for some $s \in [0, T]$ and $x \in (0, \infty)$. Note that if $s > 0$, the third coordinate cannot reach 2. Therefore, we see that s has to be zero. Now, we see that for $\phi(\tilde{\boldsymbol{\xi}})(T)$ to be greater than $\phi(\boldsymbol{\xi})(T)$ coordinate-wise as claimed in ii) of Lemma 5.1, x has to be at least $4T + \theta$. However, since $\delta < 1$, this means that $\tilde{I}^{(d)}(\tilde{\boldsymbol{\xi}}) \geq \sqrt{4T + \theta} >$

$\sqrt{4}+\sqrt{\theta} = \tilde{I}^{(d)}(\boldsymbol{\xi})$. That is, no $\tilde{\boldsymbol{\xi}}$ with only one jump in the first coordinate satisfies the conclusion of Lemma 5.1. In fact, this system of tandem queues still turns out to be a counterexample even if we change the statement of Lemma 5.1 so that *ii)* is $\boldsymbol{b}^{\mathsf{T}}\phi(\tilde{\xi})(T) \geq \boldsymbol{b}^{\mathsf{T}}\phi(\xi)(T)$. To see this, note first that if $x < 4(T-s)$, then $\boldsymbol{b}^{\mathsf{T}}\phi(\tilde{\boldsymbol{\xi}})(T) < y$, and hence, we only consider the case $x \geq 4(T-s)$, where

$$\boldsymbol{b}^{\mathsf{T}}\phi(\tilde{\boldsymbol{\xi}})(T) = \delta(x-4(T-s)) + T - s = \delta x + (1-4\delta)(T-s).$$

Note also that since we assume $\delta < 1/4$, this is maximized at $s = 0$. Therefore, for $\boldsymbol{b}^{\mathsf{T}}\phi(\tilde{\boldsymbol{\xi}})(T)$ to be greater than or equal to y, we need x to be greater than or equal to $4T+\theta$. This implies that $\tilde{I}^{(d)}(\tilde{\boldsymbol{\xi}}) \geq \sqrt{4T+\theta} > \sqrt{4}+\sqrt{\theta} = \tilde{I}^{(d)}(\boldsymbol{\xi})$. Therefore, solving $(P_{y,k})$ with $k = 1$ will not give the correct log asymptotics for $\mathbf{P}(\boldsymbol{b}^{\mathsf{T}}\phi(\boldsymbol{X}_n)(T) \geq y)$ in general.

6 Complementary proofs

6.1 Proofs of Lemmas 4.1 and 4.2

Next, we focus on the continuity of $\mathrm{V}^*_{\geqslant}(\cdot)$. Let $\mathbb{D}_+[0,T]$ be the subspace of $\mathbb{D}[0,T]$ that contains paths with only positive discontinuities: $\mathbb{D}_+[0,T] = \{\xi \in \mathbb{D}[0,T] : \xi(t)-\xi(t-) \geq 0,\ \forall t \in [0,T]\}$. Recall that $\mathbb{D}_{\leqslant k}[0,T] = \{\xi \in \mathbb{D}[0,T] : |Disc(\xi)| \leq k\}$.

Lemma 6.1 *Suppose that* $\boldsymbol{a} = (a_1,\ldots,a_d) \in \mathbb{R}^d_+$, $\boldsymbol{\xi} \in \prod_{i=1}^d \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_i}_{\leqslant\infty}[0,T]$, *and* $\boldsymbol{\zeta} = \boldsymbol{\xi} + \boldsymbol{a}\mathbb{1}_{\{T\}}$. *Then*

(i) $\psi(\boldsymbol{\zeta}) = \psi(\boldsymbol{\xi})$,
(ii) $\phi(\boldsymbol{\zeta})(T) = \phi(\boldsymbol{\xi})(T) + a$, *and*
(iii) $\tilde{I}^{(d)}(\boldsymbol{\zeta}) \leq \tilde{I}^{(d)}(\boldsymbol{\xi}) + \sum_{i=1}^d c_i a_i^{\alpha}$.

Proof For *i)*, from the proof of Theorem 14.2.2 in Whitt [20], we see that for any $\boldsymbol{\omega} \in \prod_{i=1}^k \mathbb{D}[0,T]$ the regulator component $\psi(\boldsymbol{\omega})$ is the limit (w.r.t. $\|\cdot\|$) of $\rho^n_{\boldsymbol{\omega}}(\mathbf{0})$ where $\mathbf{0}$ is the zero function and $\rho^n_{\boldsymbol{\omega}}$ is the n fold composition of $\rho_{\boldsymbol{\omega}} : \prod_{i=1}^d \mathbb{D}^{\uparrow}[0,T] \to \prod_{i=1}^d \mathbb{D}^{\uparrow}[0,T]$ such that $\rho_{\boldsymbol{\omega}}(\boldsymbol{\eta})(t) = 0 \vee \sup_{s\in[0,t]}\{\mathcal{Q}\boldsymbol{\eta}(s) - \boldsymbol{\omega}(s)\}$. Note that $\rho_{\boldsymbol{\omega}}(\boldsymbol{\eta})(t)$ depends only on $\boldsymbol{\eta}(s)$ and $\boldsymbol{\omega}(s)$ for $s \in [0,t]$. Therefore, $\psi(\boldsymbol{\omega})(t)$ depends on $\boldsymbol{\omega}(s)$ for $s \in [0,t]$ only. Therefore, $\psi(\boldsymbol{\zeta})(t) = \psi(\boldsymbol{\xi})(t)$ for $t \in [0, T-\epsilon]$ for any $\epsilon > 0$. The continuity implies that $\psi(\boldsymbol{\zeta})(T) = \psi(\boldsymbol{\xi})(T)$ as well, which concludes the proof of part i).

For *ii)*, observe that $\phi(\boldsymbol{\zeta})(T) = \boldsymbol{\zeta}(T) + \mathcal{Q}\psi(\boldsymbol{\zeta})(T) = \boldsymbol{\xi}(T) + \boldsymbol{a} + \mathcal{Q}\psi(\boldsymbol{\xi})(T) = \phi(\boldsymbol{\xi})(T) + \boldsymbol{a}$.

For *iii)*, we assume that $\xi^{(j)}(t) = -(\mathcal{Q}\boldsymbol{r})_j(t)$ for $j \notin \mathcal{J}$ since if not $\tilde{I}^{(d)}(\boldsymbol{\zeta}) = \tilde{I}^{(d)}(\boldsymbol{\xi}) = \infty$, and the inequality holds trivially. Let $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_d)$, and $\boldsymbol{\xi} =$

$(\xi_1, \ldots, \xi_d)$. Since the function $x \mapsto x^\alpha$, $\alpha \in (0, 1)$, is subadditive,

$$\begin{aligned} I(\zeta_i) &= \sum_{t\in[0,T):\xi_i(t)\neq\xi_i(t-)} (\xi_i(t) - \xi_i(t-))^\alpha + (\xi_i(T) - \xi_i(T-) + a_i)^\alpha \\ &\leq \sum_{t\in[0,T):\xi_i(t)\neq\xi_i(t-)} (\xi_i(t) - \xi_i(t-))^\alpha + (\xi_i(T) - \xi_i(T-))^\alpha + a_i^\alpha \\ &= I(\xi_i) + a_i^\alpha. \end{aligned}$$

Therefore, $\tilde{I}^{(d)}(\boldsymbol{\zeta}) = \sum_{j\in\mathcal{J}} c_j I(\zeta_j) \leq \sum_{j\in\mathcal{J}} c_j I(\xi_j) + \sum_{j\in\mathcal{J}} c_j a_j^\alpha \leq \tilde{I}^{(d)}(\boldsymbol{\xi}) + \sum_{j=1}^d c_j a_j^\alpha$. □

Proof of Lemma 4.1 W.l.o.g., let $y \geq x \geq 0$. Then, $V_{\geqslant}(y) \subseteq V_{\geqslant}(x)$, and hence, $V^*_{\geqslant}(y) \geq V^*_{\geqslant}(x) \geq 0$. For any $\epsilon > 0$, there exists a $\boldsymbol{\zeta} \in V_{\geqslant}(x)$ so that $\tilde{I}^{(d)}(\boldsymbol{\zeta}) < V^*_{\geqslant}(x) + \epsilon$. Next, fix $j \in I^+$ and let $\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{v}\mathbb{1}_{\{T\}}$ where $\boldsymbol{v} = (0, \ldots, \frac{y-x}{b_j}, \ldots, 0)$. Due to *ii)* of Lemma 6.1,

$$\boldsymbol{b}^\intercal \phi(\boldsymbol{\xi})(T) = \boldsymbol{b}^\intercal(\phi(\boldsymbol{\zeta})(T) + \boldsymbol{v}) = \boldsymbol{b}^\intercal \phi(\boldsymbol{\zeta})(T) + b_j \frac{(y-x)}{b_j} \geq x + y - x = y.$$

Hence, $\boldsymbol{\xi} \in V_{\geqslant}(y)$. Due to *iii)* of Lemma 6.1,

$$\tilde{I}^{(d)}(\boldsymbol{\xi}) \leq \tilde{I}^{(d)}(\boldsymbol{\zeta}) + \frac{c_j}{b_j^\alpha} \cdot (y-x)^\alpha \leq \tilde{I}^{(d)}(\boldsymbol{\zeta}) + \Big(\max_{i\in I^+} \frac{c_i}{b_i^\alpha}\Big) \cdot (y-x)^\alpha.$$

We see that

$$\begin{aligned} V^*_{\geqslant}(y) &\leq \tilde{I}^{(d)}(\boldsymbol{\xi}) \\ &\leq \tilde{I}^{(d)}(\boldsymbol{\zeta}) + \max_{1\leq i\leq d: b_i>0} \frac{c_i}{b_i^\alpha}(y-x)^\alpha < V^*_{\geqslant}(x) + \max_{\{1\leq i\leq d: b_i>0\}} \frac{c_i}{b_i^\alpha}(y-x)^\alpha + \epsilon. \end{aligned}$$

This leads to $V^*_{\geqslant}(y) - V^*_{\geqslant}(x) \leq \max_{\{1\leq i\leq d: b_i>0\}} \frac{c_i}{b_i^\alpha}(y-x)^\alpha + \epsilon$. We obtain the desired result by letting ϵ tend to 0. Thus, $|V^*_{\geqslant}(y) - V^*_{\geqslant}(x)| \leq \max_{\{1\leq i\leq d: b_i>0\}} \frac{c_i}{b_i^\alpha} \cdot |y-x|^\alpha$. □

We conclude this section with the proof of Lemma 4.2.

Proof of Lemma 4.2 For any $\epsilon > 0$, we have that $V^*_{\geqslant}(y+\epsilon) \geq V^*_{>}(y)$. Hence, in view of Lemma 4.1,

$$|V^*_{>}(y) - V^*_{\geqslant}(y)| = V^*_{>}(y) - V^*_{\geqslant}(y) \leq V^*_{\geqslant}(y+\epsilon) - V^*_{\geqslant}(y) \leq \Big(\max_{i\in I^+} \frac{c_i}{b_i^\alpha}\Big) \cdot |\epsilon|^\alpha.$$

Now, we let ϵ go to 0 to obtain the desired result. □

6.2 Proof of Lemma 5.1

For any $\eta \in \mathbb{D}[0,T]$, let $\eta^{\downarrow} \in \mathbb{D}[0,T]$ denote the running infimum $\eta^{\downarrow}(t) \triangleq \inf_{s\in[0,t]} 0 \wedge \eta(s)$ for all $t \in [0,T]$. The following simple lemma is useful for proving Lemma 5.1.

Lemma 6.2 *Suppose that $\eta, \omega \in \mathbb{D}[0,T]$ are such that $\eta \geq \omega$ and $\eta(T) = \omega(T)$. Then, $(\eta - \eta^{\downarrow})(T) \leq (\omega - \omega^{\downarrow})(T)$.*

Proof Since $\eta^{\downarrow} \geq \omega^{\downarrow}$, we have $\eta - \eta^{\downarrow} \leq \eta - \omega^{\downarrow}$. Therefore, $(\eta - \eta^{\downarrow})(T) \leq (\eta - \omega^{\downarrow})(T) = (\omega - \omega^{\downarrow})(T)$ □

Now, we prove Lemma 5.1.

Proof of Lemma 5.1 Since we assume that $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_1}_{\leqslant\infty}[0,T] \times \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_2}_{\leqslant\infty}[0,T]$, we can write $\xi_1 = (\gamma_1 - r_1)e + \sum_{j=1}^{\infty} x^{(j)} \mathbb{1}_{[u^{(j)},T]}$ and $\xi_2 = (\gamma_2 + r_1 - r_2)e + \sum_{j=1}^{\infty} y^{(j)} \mathbb{1}_{[v^{(j)},T]}$ for $x^{(j)}, y^{(j)} \geq 0$ and $u^{(j)}, v^{(j)} \in [0,T]$, $j = 1, 2, \ldots$. Consider $\boldsymbol{\xi}' = (\xi_1, \xi_2')$ where $\xi_2' = (\gamma_2 + r_1 - r_2)e + \bar{y}\mathbb{1}_{[T,T]}$ and $\bar{y} = \sum_{j=1}^{\infty} y^{(j)}$. Then, by the subadditivity of $x \mapsto x^{\alpha}$, $\tilde{I}^{(d)}(\boldsymbol{\xi}') \leq \tilde{I}^{(d)}(\boldsymbol{\xi})$.

Note that since

$$\boldsymbol{\xi} + \mathcal{Q}\psi(\boldsymbol{\xi}) = \begin{pmatrix} \xi_1 + \psi_1(\boldsymbol{\xi}) \\ \xi_2 - \psi_1(\boldsymbol{\xi}) + \psi_2(\boldsymbol{\xi}) \end{pmatrix}$$

$$\text{and} \quad \boldsymbol{\xi}' + \mathcal{Q}\psi(\boldsymbol{\xi}') = \begin{pmatrix} \xi_1 + \psi_1(\boldsymbol{\xi}') \\ \xi_2' - \psi_1(\boldsymbol{\xi}') + \psi_2(\boldsymbol{\xi}') \end{pmatrix},$$

we see that $\psi_1(\boldsymbol{\xi}) = \psi_1(\boldsymbol{\xi}') = -\xi_1^{\downarrow}$, and hence, $\phi_1(\boldsymbol{\xi}) = \phi_1(\boldsymbol{\xi}') = \xi_1 - \xi_1^{\downarrow}$. Also, $\phi_2(\boldsymbol{\xi}) = \xi_2 - \psi_1(\boldsymbol{\xi}) - (\xi_2 - \psi_1(\boldsymbol{\xi}))^{\downarrow}$ and $\phi_2(\boldsymbol{\xi}') = \xi_2' - \psi_1(\boldsymbol{\xi}) - (\xi_2' - \psi_1(\boldsymbol{\xi}))^{\downarrow}$. Note that since $\xi_2 - \psi_1(\boldsymbol{\xi}) \geq \xi_2' - \psi_1(\boldsymbol{\xi})$ and $(\xi_2 - \psi_1(\boldsymbol{\xi}))(T) = (\xi_2' - \psi_1(\boldsymbol{\xi}))(T)$, Lemma 6.2 implies that

$$\phi_2(\boldsymbol{\xi})(T) = \xi_2 - \psi_1(\boldsymbol{\xi}) - (\xi_2 - \psi_1(\boldsymbol{\xi}))^{\downarrow} \leq \xi_2 - \psi_1(\boldsymbol{\xi}) - (\xi_2' - \psi_1(\boldsymbol{\xi}))^{\downarrow} = \phi_2(\boldsymbol{\xi}')(T).$$

Therefore, we found $\boldsymbol{\xi}' \in \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_1}_{\leqslant\infty}[0,T] \times \mathbb{D}^{(\boldsymbol{\gamma}-\mathcal{Q}\boldsymbol{r})_2}_{\leqslant 1}[0,T]$ such that $\tilde{I}^{(d)}(\boldsymbol{\xi}') \leq \tilde{I}^{(d)}(\boldsymbol{\xi})$ and $\phi(\boldsymbol{\xi}')(T) \geq \phi(\boldsymbol{\xi})(T)$. Now, let $\boldsymbol{\xi}'' \triangleq (\xi_1', \xi_2')$ where $\xi_1' = (\gamma_1 - r_1)e + \bar{x}\mathbb{1}_{[T - \frac{\bar{x} - \phi_1(\boldsymbol{\xi}')(T)}{r_1 - \gamma_1}, T]}$ and $\bar{x} = \sum_{j=1}^{\infty} x^{(j)}$. Note that $\psi_1(\boldsymbol{\xi}'') \geq \psi_1(\boldsymbol{\xi}')$ and $\psi_1(\boldsymbol{\xi}'')(T) = \psi_1(\boldsymbol{\xi}')(T)$. To see this, let $T' \triangleq T - \frac{\bar{x} - \phi_1(\boldsymbol{\xi}')(T)}{r_1 - \gamma_1}$. Note that

$$T' = T - \frac{\bar{x} - \xi_1(T) + \xi_1^{\downarrow}(T)}{r_1 - \gamma_1} = T - \frac{\bar{x} - (\gamma_1 - r_1)T - \bar{x} + \xi_1^{\downarrow}(T)}{r_1 - \gamma_1} = -\frac{\xi_1^{\downarrow}(T)}{r_1 - \gamma_1}.$$

From the construction of ξ_1', it "attains" its infimum at $T'-$, and hence, $(\xi_1')^{\downarrow}(t) = \xi_1'(T'-) = T'(\gamma_1 - r_1) = \xi_1^{\downarrow}(T)$ for $t \in [T', T]$. Note also that from the forms

of ξ_1 and ξ_1', we clearly have $\xi_1'(t) \le \xi(t)$ for $t \in [0, T']$. Therefore, $(\xi_1')^{\downarrow} \le \xi_1^{\downarrow}$ and $(\xi_1')^{\downarrow}(T) = \xi_1^{\downarrow}(T)$. Since $\psi_1(\boldsymbol{\xi}'') = -(\xi_1')^{\downarrow}$ and $\psi_1(\boldsymbol{\xi}') = -\xi_1^{\downarrow}$, we obtain the relationships between $\psi_1(\boldsymbol{\xi}'')$ and $\psi_1(\boldsymbol{\xi}')$ claimed above. Now , again from Lemma 6.2, we get $\phi_2(\boldsymbol{\xi}'')(T) \ge \phi_2(\boldsymbol{\xi}')(T)$. Note that we constructed $\boldsymbol{\xi}''$ in such a way that $\phi_1(\boldsymbol{\xi}'')(T) = \phi_1(\boldsymbol{\xi}')(T) = \phi_1(\boldsymbol{\xi})(T)$. Note also that $\tilde{I}^{(d)}(\boldsymbol{\xi}'') \le \tilde{I}^{(d)}(\boldsymbol{\xi}')$. We arrive at the conclusion of the lemma by setting $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}''$. □

6.3 Proof of Proposition 2.1 and Theorem 2.1

Recall that $\prod_{i=1}^d \mathbb{D}[0, T]$ is the Skorokhod space equipped with the product J_1 topology and $\mathbb{D}^{\uparrow}[0, T] \triangleq \{\xi \in \mathbb{D}[0, T] : \xi$ is non-decreasing on $[0, T]$ and $\xi(0) \ge 0\}$. $\mathbb{D}^{\uparrow}[0, T]$ is a closed subspace of $\mathbb{D}[0, T]$ w.r.t. the J_1 topology. Hence, $\prod_{i=1}^d \mathbb{D}^{\uparrow}[0, T]$ is a closed subspace of $\prod_{i=1}^d \mathbb{D}[0, T]$ w.r.t. the product J_1 topology. Since $\mathbb{D}^{\beta}[0, T]$ is the image of $\mathbb{D}^{\uparrow}[0, T]$ under the homeomorphism $\Upsilon^{\boldsymbol{\beta}}$, we have that $\prod_{i=1}^d \mathbb{D}^{\beta_i}[0, T]$ is a closed subset of $\prod_{i=1}^d \mathbb{D}[0, T]$.

6.3.1 Some supporting lemmas

Lemma 6.3 *Suppose that* $\lambda, \mu \in \Lambda[0, T]$. *Then,* $\|\lambda \circ \mu - e\| \le \|\lambda - e\| + \|\mu - e\|$.

Proof $\|\lambda \circ \mu - e\| = \|\lambda - \mu^{-1}\| \le \|\lambda - e\| + \|\mu^{-1} - e\| = \|\lambda - e\| + \|e - \mu\| \le 2\delta$. □

We now consider properties of continuous and increasing time deformations w_i, $i = 1, \ldots, d$.

Lemma 6.4 *If* $w_i \in \Lambda[0, T]$ *for each* $i = 1, \ldots, d$, *then* $\hat{w}(s) = \min\{w_1(s), \ldots, w_d(s)\}$ *and* $\check{w}(s) = \max\{w_1(s), \ldots, w_d(s)\}$ *also belong to* $\Lambda[0, T]$.

Proof The min and max of continuous and increasing functions are increasing and continuous. The other properties are easily verified. □

Recall that ψ is Lipschitz continuous w.r.t. $\|\cdot\|$ (Theorem 14.2.5 of Whitt [20]). Let K denote the Lipschitz constant of ψ w.r.t. $\|\cdot\|$, which only depends on Q; in particular, K does not depend on T.

Lemma 6.5 *Let* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$ *and* $\boldsymbol{\zeta} \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0, T]$. *For any* $w \in \Lambda[0, T]$, *it holds that*

$$\|\psi(\boldsymbol{\zeta}) \circ w - \psi(\boldsymbol{\zeta})\| < K\|\boldsymbol{\beta}\|_1 \cdot \|w - e\|.$$

Proof Consider an arbitrary $s \in [0, T]$. If $w(s) \ge s$, since $\psi(\boldsymbol{\zeta})$ is an increasing function, $\psi(\boldsymbol{\zeta})(w(s)) \ge \psi(\boldsymbol{\zeta})(s)$. Moreover, since $\boldsymbol{\zeta} \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0, T]$, $\boldsymbol{\zeta}$ has the following representation: $\boldsymbol{\zeta}(t) = \boldsymbol{\xi}(t) + \boldsymbol{\beta} \cdot t$, where $\boldsymbol{\xi} \in \prod_{i=1}^d \mathbb{D}^{\uparrow}[0, T]$. Consequently, for $t > u$, $\boldsymbol{\zeta}(t) - \boldsymbol{\zeta}(u) = \boldsymbol{\xi}(t) - \boldsymbol{\xi}(u) + \boldsymbol{\beta} \cdot (t - u) \ge \boldsymbol{\beta} \cdot (t - u)$. This implies that

$$\boldsymbol{\zeta}(w(s)) = \boldsymbol{\zeta}((w(s) - s) + s) \ge \boldsymbol{\zeta}(s) + \boldsymbol{\beta} \cdot (w(s) - s). \tag{6.1}$$

Next, consider the path $\tilde{\boldsymbol{\zeta}}_1$ where

$$\tilde{\boldsymbol{\zeta}}_1(t) = \begin{cases} \boldsymbol{\zeta}(t), & t \in [0, s], \\ \boldsymbol{\zeta}(s) + \boldsymbol{\beta} \cdot (t - s), & t \in [s, w(s)]. \end{cases}$$

Since $\tilde{\boldsymbol{\zeta}}_1 \leq \boldsymbol{\zeta}$ over $[0, w(s)]$, Result 2.3 gives that $\psi(\tilde{\boldsymbol{\zeta}}_1)(w(s)) \geq \psi(\boldsymbol{\zeta})(w(s))$. Furthermore, let

$$\tilde{\boldsymbol{\zeta}}_2(t) = \begin{cases} \boldsymbol{\zeta}(t), & t \in [0, s], \\ \boldsymbol{\zeta}(s), & t \in [s, w(s)]. \end{cases}$$

Then, we have that $\psi(\tilde{\boldsymbol{\zeta}}_2)(w(s)) = \psi(\boldsymbol{\zeta})(s)$. Therefore,

$$\begin{aligned} 0 &\leq \psi(\boldsymbol{\zeta})(w(s)) - \psi(\boldsymbol{\zeta})(s) \\ &\leq \psi(\tilde{\boldsymbol{\zeta}}_1)(w(s)) - \psi(\tilde{\boldsymbol{\zeta}}_2)(w(s)) \leq K \sup_{t \in [0, w(s)]} \|\tilde{\boldsymbol{\zeta}}_1(t) - \tilde{\boldsymbol{\zeta}}_2(t)\|_1 \\ &\leq K \|\boldsymbol{\beta}\|_1 \cdot |w(s) - s| \leq K \|\boldsymbol{\beta}\|_1 \cdot \|w - e\|. \end{aligned} \tag{6.2}$$

Next, we consider the case $w(s) \leq s$. Since $\psi(\boldsymbol{\zeta})$ is an increasing function, $\psi(\boldsymbol{\zeta})(s) \geq \psi(\boldsymbol{\zeta})(w(s))$. Furthermore, since $\boldsymbol{\zeta} \in \prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0, T]$, we have that

$$\boldsymbol{\zeta}(s) = \boldsymbol{\zeta}((s - w(s)) + w(s)) \geq \boldsymbol{\zeta}(w(s)) + \boldsymbol{\beta}(s - w(s)). \tag{6.3}$$

Next, consider the path $\tilde{\boldsymbol{\zeta}}_1$, where

$$\tilde{\boldsymbol{\zeta}}_1(t) = \begin{cases} \boldsymbol{\zeta}(t), & t \in [0, w(s)], \\ \boldsymbol{\zeta}(s) + \boldsymbol{\beta}(t - w(s)), & t \in [w(s), s]. \end{cases}$$

Since $\tilde{\boldsymbol{\zeta}}_1 \leq \boldsymbol{\zeta}$ over $[0, s]$, Result 2.3 gives that $\psi(\tilde{\boldsymbol{\zeta}}_1)(s) \geq \psi(\boldsymbol{\zeta})(s)$. On the other hand, let

$$\tilde{\boldsymbol{\zeta}}_2(t) = \begin{cases} \boldsymbol{\zeta}(t), & t \in [0, w(s)], \\ \boldsymbol{\zeta}(s), & t \in [w(s), s]. \end{cases}$$

We then have that $\psi(\tilde{\boldsymbol{\zeta}}_2)(s) = \psi(\boldsymbol{\zeta})(w(s))$. Therefore,

$$\begin{aligned} 0 \leq \psi(\boldsymbol{\zeta})(s) - \psi(\boldsymbol{\zeta})(w(s)) &\leq \psi(\tilde{\boldsymbol{\zeta}}_1)(s) - \psi(\tilde{\boldsymbol{\zeta}}_2)(s) \leq K \sup_{t \in [0, s]} \|\tilde{\boldsymbol{\zeta}}_1(t) - \tilde{\boldsymbol{\zeta}}_2(t)\|_1 \\ &\leq K \|\boldsymbol{\beta}\|_1 \cdot |w(s) - s| \leq K \|\boldsymbol{\beta}\|_1 \cdot \|w - e\|. \end{aligned} \tag{6.4}$$

From (6.2) and (6.4), we get (regardless of the value of $w(\cdot)$ at s)

$$\|\psi(\boldsymbol{\zeta})(w(s)) - \psi(\boldsymbol{\zeta})(s)\|_1 \leq dK \|\boldsymbol{\beta}\|_1 \cdot \|w - e\| = K \|\boldsymbol{\beta}\|_1 \cdot \|w - e\|.$$

Taking the supremum over $s \in [0, T]$, we arrive at the conclusion of the lemma. □

Note that if $\boldsymbol{\beta} = \mathbf{0}$ and $\zeta \in \prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T]$, then ζ belongs to $\prod_{i=1}^{d} \mathbb{D}^{\uparrow}[0,T]$ and is nonnegative at the origin. This implies $\psi(\zeta) = 0$, and the upper bound in Lemma 6.5 holds trivially. Next, we state two more lemmas which are needed in our proof for the Lipschitz continuity of the regulator map in $\prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T]$. Let $\iota \in \mathbb{D}[0,T]$ be $\iota(t) \equiv 1$, and $\boldsymbol{\iota} = (\iota, \ldots, \iota) \in \prod_{i=1}^{d} \mathbb{D}[0,T]$.

Lemma 6.6 *Consider $\boldsymbol{w} = (w_1, \ldots, w_d)$, each component of which is a time deformation in $\Lambda[0,T]$. Recall $\hat{w}$ and $\check{w}$ in Lemma 6.4. That is, $\check{w}(t) = \max\{w_1(t), \ldots, w_d(t)\}$, and $\hat{w}(t) = \min\{w_1(t), \ldots, w_d(t)\}$. Define the vector-valued functions $\hat{\boldsymbol{w}}$, $\check{\boldsymbol{w}}$, and $\boldsymbol{e}$ from $[0,T]$ to $[0,T]^d$ as $\hat{\boldsymbol{w}} \triangleq (\hat{w}, \ldots, \hat{w})$, $\check{\boldsymbol{w}} \triangleq (\check{w}, \ldots, \check{w})$, and $\boldsymbol{e} \triangleq (e, \ldots, e)$. For any $\xi \in \prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T]$,*

i) $\psi(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) \le \psi(\boldsymbol{\xi}) \circ \hat{w} + (d+1)K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota}$, *and*
ii) $\psi(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) + (d+1)K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota} \ge \psi(\boldsymbol{\xi}) \circ \check{w}$.

Proof We start with *i)*. Since $\boldsymbol{\xi} \in \prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T]$ and $\hat{w}(s) \le w_i(s)$, we have that for each $i = 1, \ldots, d$,

$$\xi_i(w_i(s)) \ge \xi_i(\hat{w}(s)) - \|\boldsymbol{\beta}\|_\infty (w_i(s) - \hat{w}(s)), \ s \in [0,T].$$

Note also that since $\left|\hat{w}(t) - e(t)\right| = \left|w_j(t) - e(t)\right|$ for some j,

$$\begin{aligned}
\|\boldsymbol{e} - \hat{\boldsymbol{w}}\| &= \sup_{0\in[0,T]} \sum_{i=1}^{d} \left|\hat{w}(t) - e(t)\right| \\
&\le \sup_{0\in[0,T]} \sum_{i=1}^{d} \sum_{j=1}^{d} \left|w_j(t) - e(t)\right| = d \sup_{0\in[0,T]} \sum_{j=1}^{d} \left|w_j(t) - e(t)\right| = d\|\boldsymbol{w} - \boldsymbol{e}\|.
\end{aligned}$$

Similarly, $\|\check{\boldsymbol{w}} - \boldsymbol{e}\| \le d\|\boldsymbol{w} - \boldsymbol{e}\|$. Therefore, due to Result 2.3 and the Lipschitz continuity of ψ w.r.t. $\|\cdot\|$,

$$\begin{aligned}
&\psi\big(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d\big) \\
&\le \psi\big(\xi_1 \circ \hat{w} - \|\boldsymbol{\beta}\|_\infty (w_1 - \hat{w}), \ldots, \xi_d \circ \hat{w} - \|\boldsymbol{\beta}\|_\infty (w_d - \hat{w})\big) \\
&= \psi\big(\boldsymbol{\xi} \circ \hat{w} - \|\boldsymbol{\beta}\|_\infty (\boldsymbol{w} - \hat{\boldsymbol{w}})\big) \\
&\le \psi(\boldsymbol{\xi} \circ \hat{w}) + K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \hat{\boldsymbol{w}}\| \cdot \boldsymbol{\iota} \\
&\le \psi(\boldsymbol{\xi}) \circ \hat{w} + K\|\boldsymbol{\beta}\|_\infty \cdot (\|\boldsymbol{w} - \boldsymbol{e}\| + \|\boldsymbol{e} - \hat{\boldsymbol{w}}\|) \cdot \boldsymbol{\iota} \\
&\le \psi(\boldsymbol{\xi}) \circ \hat{w} + (d+1)K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota}.
\end{aligned}$$

For *ii)*, observe that $\xi_i(\check{w}(s)) \ge \xi_i(w_i(s)) - \|\boldsymbol{\beta}\|_\infty (\check{w}(s) - w_i(s))$ for each $i = 1, \ldots, d$ and $s \in [0,T]$, since $\boldsymbol{\xi} \in \prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T]$, and $\check{w}(s) \ge w_i(s)$ for each $i = 1, \ldots, d$.

Therefore, due to Result 2.3 and the Lipschitz continuity of ψ w.r.t. $\|\cdot\|$,

$$\begin{aligned}
\psi(\boldsymbol{\xi}) \circ \check{w} &= \psi\big(\xi_1 \circ \check{w}, \ldots, \xi_d \circ \check{w}\big) \\
&\le \psi\big(\xi_1 \circ w_1 - \|\boldsymbol{\beta}\|_\infty(\check{w} - w_1), \ldots, \xi_d \circ w_d - \|\boldsymbol{\beta}\|_\infty(\check{w} - w_d)\big) \\
&= \psi\big((\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) - \|\boldsymbol{\beta}\|_\infty(\check{\boldsymbol{w}} - \boldsymbol{w})\big) \\
&\le \psi(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) + K\|\boldsymbol{\beta}\|_\infty \cdot \|\check{\boldsymbol{w}} - \boldsymbol{w}\| \cdot \boldsymbol{\iota} \\
&\le \psi(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) + K\|\boldsymbol{\beta}\|_\infty \cdot (\|\check{\boldsymbol{w}} - \boldsymbol{e}\| + \|\boldsymbol{e} - \boldsymbol{w}\|) \cdot \boldsymbol{\iota} \\
&= \psi(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) + (d+1)K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota}.
\end{aligned}$$

□

Lemma 6.7 *For any* $\boldsymbol{\xi} \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0,T]$ *and* $\boldsymbol{w} = (w_1, \ldots, w_d) \in \prod_{i=1}^d \Lambda[0,T]$,

$$\|\psi(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) - \psi(\boldsymbol{\xi})\| \le d(2d+1)K\|\boldsymbol{\beta}\|_1 \cdot \|\boldsymbol{w} - \boldsymbol{e}\|.$$

Proof Due to Lemmas 6.5, 6.6, and $\|\hat{w} - e\| \le \|\boldsymbol{w} - \boldsymbol{e}\|$,

$$\begin{aligned}
&\psi(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) - \psi(\boldsymbol{\xi}) \\
&\le \psi(\boldsymbol{\xi}) \circ \hat{w} - \psi(\boldsymbol{\xi}) + (d+1)K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota} \\
&\le dK\|\boldsymbol{\beta}\|_1 \cdot \|\hat{w} - e\| \cdot \boldsymbol{\iota} + (d+1)K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota} \\
&\le (2d+1)K\|\boldsymbol{\beta}\|_1 \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\psi(\boldsymbol{\xi}) - \psi(\xi_1 \circ w_1, \ldots, \xi_d \circ w_d) \\
&\le \psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\xi})(\check{w}) + (d+1)K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota} \\
&\le dK\|\boldsymbol{\beta}\|_1 \cdot \|\check{w} - e\| \cdot \boldsymbol{\iota} + (d+1)K\|\boldsymbol{\beta}\|_\infty \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota} \\
&\le (2d+1)K\|\boldsymbol{\beta}\|_1 \cdot \|\boldsymbol{w} - \boldsymbol{e}\| \cdot \boldsymbol{\iota}.
\end{aligned}$$

From these, the conclusion of the lemma follows. □

6.3.2 Lipschitz continuity of the reflection map

Now, we are ready to conclude Sect. 6.3 with the proofs of Proposition 2.1 and Theorem 2.1, which are the Lipschitz continuity of the regulator map and the buffer content component map, respectively, in the product J_1 topology. We start with the Lipschitz continuity of the regulator map ψ.

Proof of Proposition 2.1 Given $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0,T]$, consider an arbitrary δ such that $d_p(\boldsymbol{\xi}, \boldsymbol{\zeta}) < \delta$. Then, there exists $\lambda_i \in \Lambda[0,T]$ such that $\|\xi_i \circ \lambda_i - \zeta_i\| \vee \|\lambda_i - e\| < \delta$

for each $i = 1, \ldots, d$. Notice that

$$\begin{aligned} d_p(\psi(\boldsymbol{\xi}), \psi(\boldsymbol{\zeta})) &\le \sum_{i=1}^{d} \inf_{w_i \in \Lambda[0,T]} \left\| \psi_i(\boldsymbol{\xi}) \circ w_i - \psi_i(\boldsymbol{\zeta}) \right\| \vee \left\| w_i - e \right\| \\ &\le \sum_{i=1}^{d} \left\| \psi_i(\boldsymbol{\xi}) \circ \lambda_i - \psi_i(\boldsymbol{\zeta}) \right\| \vee \left\| \lambda_i - e \right\| \\ &\le \sum_{i=1}^{d} \left\| \psi_i(\boldsymbol{\xi}) \circ \lambda_i - \psi_i(\xi_1 \circ \lambda_1, \ldots, \xi_d \circ \lambda_d) \right\| \vee \left\| \lambda_i - e \right\| \\ &\quad + \sum_{i=1}^{d} \left\| \psi_i(\xi_1 \circ \lambda_1, \ldots, \xi_d \circ \lambda_d) - \psi_i(\zeta_1, \ldots, \zeta_d) \right\| \vee \left\| \lambda_i - e \right\| \end{aligned}$$

Note that from Lemma 6.7,

$$\begin{aligned} &\left\| \psi_i(\boldsymbol{\xi}) \circ \lambda_i - \psi_i(\xi_1 \circ \lambda_1, \ldots, \xi_d \circ \lambda_d) \right\| \\ &\quad = \left\| \psi_i(\boldsymbol{\xi}) - \psi_i(\xi_1 \circ \lambda_1, \ldots, \xi_d \circ \lambda_d) \circ (\lambda_i)^{-1} \right\| \\ &\quad = \left\| \psi_i(\boldsymbol{\xi}) - \psi_i(\xi_1 \circ \lambda_1 \circ (\lambda_i)^{-1}, \ldots, \xi_d \circ \lambda_d \circ (\lambda_i)^{-1}) \right\| \\ &\quad \le d(2d+1)K \|\boldsymbol{\beta}\|_1 \cdot \left\| \boldsymbol{e} - \left(\lambda_1 \circ (\lambda_i)^{-1}, \ldots, \lambda_d \circ (\lambda_i)^{-1} \right) \right\| \\ &\quad \le d(2d+1)K \|\boldsymbol{\beta}\|_1 \cdot \sum_{j=1}^{d} \left\| e - \lambda_j \circ (\lambda_i)^{-1} \right\| \\ &\quad \le d(2d+1)K \|\boldsymbol{\beta}\|_1 \cdot \sum_{j=1}^{d} \left(\left\| e - \lambda_j \right\| + \left\| e - (\lambda_i)^{-1} \right\| \right) \\ &\quad = d(2d+1)K \|\boldsymbol{\beta}\|_1 \cdot \sum_{j=1}^{d} \left(\left\| e - \lambda_j \right\| + \left\| e - \lambda_i \right\| \right) \\ &\quad \le 2d^2(2d+1)K \|\boldsymbol{\beta}\|_1 \cdot \delta, \end{aligned}$$

where the third inequality is due to Lemma 6.3. On the other hand,

$$\begin{aligned} &\left\| \psi_i(\xi_1 \circ \lambda_1, \ldots, \xi_d \circ \lambda_d) - \psi_i(\zeta_1, \ldots, \zeta_d) \right\| \\ &\quad \le \left\| \psi(\xi_1 \circ \lambda_1, \ldots, \xi_d \circ \lambda_d) - \psi(\zeta_1, \ldots, \zeta_d) \right\| \\ &\quad \le K \left\| (\xi_1 \circ \lambda_1, \ldots, \xi_d \circ \lambda_d) - (\zeta_1, \ldots, \zeta_d) \right\| \\ &\quad = K \sum_{i=1}^{d} \left\| \xi_i \circ \lambda_i - \zeta_i \right\| \le K d \delta. \end{aligned}$$

Therefore,

$$d_p(\psi(\boldsymbol{\xi}), \psi(\boldsymbol{\zeta})) \le d(2d^2(2d+1)K\|\boldsymbol{\beta}\|_1 + Kd \vee 1) \cdot \delta. \tag{6.5}$$

Letting $\delta \downarrow d_p(\boldsymbol{\xi}, \boldsymbol{\zeta})$, we obtain Lipschitz continuity of ψ w.r.t. d_p. □

Proof of Theorem 2.1 The Lipschitz continuity of the regulator map has been proven in Proposition 2.1. We only need to verify the Lipschitz continuity of the buffer content component map ϕ. Let δ be such that $d_p(\boldsymbol{\xi}, \boldsymbol{\zeta}) < \delta$. Then, there exists $\lambda_i \in \Lambda[0,T]$ such that $\|\xi_i \circ \lambda_i - \zeta_i\| \vee \|\lambda_i - e\| \le \delta$ for each $i = 1, \dots, d$. Note that $\phi_i(\boldsymbol{\xi}) = \xi_i + \psi_i(\boldsymbol{\xi}) - \sum_{j\in\{1,\dots,d\}\setminus\{i\}} q_{ji}\psi_j(\boldsymbol{\xi})$. Hence,

$$\begin{aligned}
&d_{J_1}(\phi_i(\boldsymbol{\xi}), \phi_i(\boldsymbol{\zeta})) \\
&\quad = d_{J_1}\big(\xi_i + \psi_i(\boldsymbol{\xi}) - \textstyle\sum_{j\in\{1,\dots,d\}\setminus\{i\}} q_{ji}\psi_j(\boldsymbol{\xi}),\ \zeta_i + \psi_i(\boldsymbol{\zeta}) - \sum_{j\in\{1,\dots,d\}\setminus\{i\}} q_{ji}\psi_j(\boldsymbol{\xi})\big) \\
&\quad \le \Big\|\xi_i \circ \lambda_i + \psi_i(\boldsymbol{\xi}) \circ \lambda_i - \textstyle\sum_{j\in\{1,\dots,d\}\setminus\{i\}} q_{ji}\psi_j(\boldsymbol{\xi}) \circ \lambda_i \\
&\qquad - \zeta_i - \psi_i(\boldsymbol{\zeta}) + \sum_{j\in\{1,\dots,d\}\setminus\{i\}} q_{ji}\psi_j(\boldsymbol{\xi})\Big\| \vee \big\|\lambda_i - e\big\| \\
&\quad \le \big\|\xi_i \circ \lambda_i - \zeta_i\big\| \vee \delta + \big\|\psi_i(\boldsymbol{\xi}) \circ \lambda_i - \psi_i(\boldsymbol{\zeta})\big\| \vee \delta \\
&\qquad + \sum_{j\in\{1,\dots,d\}\setminus\{i\}} \|\psi_j(\boldsymbol{\xi}) \circ \lambda_i - \psi_j(\boldsymbol{\xi})\| \vee \delta
\end{aligned}$$

Note that $\big\|\xi_i \circ \lambda_i - \zeta_i\big\| \le \delta$ and $\|\psi_j(\boldsymbol{\xi}) \circ \lambda_i - \psi_j(\boldsymbol{\xi})\|$ can be bounded by $2d^2(2d+1)K\|\boldsymbol{\beta}\|_1 \cdot \delta$ the say way as in the proof of Proposition 2.1. Since $d_p(\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\zeta})) \le \sum_{i=1}^d d_{J_1}(\phi_i(\boldsymbol{\xi}), \phi_i(\boldsymbol{\zeta}))$, we have that ϕ is Lipschitz continuous in $\prod_{i=1}^d \mathbb{D}^{\beta_i}[0,T]$ by letting $\delta \downarrow d_p(\boldsymbol{\xi}, \boldsymbol{\zeta})$. □

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Appendix A Continuity of some useful functions

In this appendix, we include the proofs of continuity of some functions in the product J_1 topology. Recall the function $\Upsilon^{\boldsymbol{\beta}} : \prod_{i=1}^d \mathbb{D}[0,T] \to \prod_{i=1}^d \mathbb{D}[0,T]$ where $\Upsilon^{\boldsymbol{\beta}}(\boldsymbol{\xi})(t) = \boldsymbol{\xi}(t) + \boldsymbol{\beta} \cdot t$ for $t \in [0,T]$.

Proof of Lemma 2.2 For *i)*, suppose that $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are given. For each $i \in \{1, \dots, d\}$, let λ_i be a homeomorphism such that $\|\xi_i - \zeta_i \circ \lambda_i\| \vee \|\lambda_i - e\| < 2 \cdot d_{J_1}(\xi_i, \zeta_i)$. Then,

$$\begin{aligned}
d_{J_1}\left(\Upsilon_i^{\boldsymbol{\beta}}(\boldsymbol{\xi}), \Upsilon_i^{\boldsymbol{\beta}}(\boldsymbol{\zeta})\right) &\le \|\Upsilon_i^{\boldsymbol{\beta}}(\boldsymbol{\xi}) - \Upsilon_i^{\boldsymbol{\beta}}(\boldsymbol{\zeta}) \circ \lambda_i\| \vee \|\lambda_i - e\| &\text{(A.1)}\\
&= \|\xi_i - \zeta_i \circ \lambda_i - \beta_i(\lambda_i - e)\| \vee \|\lambda_i - e\| \\
&\le \|\xi_i - \zeta_i \circ \lambda_i\| \vee \|\lambda_i - e\| + \|\beta_i(\lambda_i - e)\| \vee \|\lambda_i - e\| \\
&\le 2(1 + 1 \vee |\beta_i|) \cdot d_{J_1}(\xi_i, \zeta_i). &\text{(A.2)}
\end{aligned}$$

Consequently,

$$d_p(\Upsilon^\beta(\boldsymbol{\zeta}), \Upsilon^\beta(\boldsymbol{\xi})) = \sum_{i=1}^{d} d_{J_1}(\Upsilon_i^{\boldsymbol{\beta}}(\boldsymbol{\zeta}), \Upsilon_i^{\boldsymbol{\beta}}(\boldsymbol{\xi})) \le \sum_{i=1}^{d} 2(1 + 1 \vee |\beta_i|) \cdot d_{J_1}(\xi_i, \zeta_i)$$
$$\le 2(1 + 1 \vee \|\boldsymbol{\beta}\|_1) \cdot d_p(\boldsymbol{\xi}, \boldsymbol{\zeta}).$$

For *ii)*, note that $(\Upsilon^{\boldsymbol{\beta}})^{-1}(\boldsymbol{\zeta}) = \boldsymbol{\zeta} - \boldsymbol{\beta} \cdot e = \Upsilon^{-\boldsymbol{\beta}}(\boldsymbol{\zeta})$, and hence, $\Upsilon^{\boldsymbol{\beta}}$ is injective and surjective. From this, the continuity of $(\Upsilon^{\boldsymbol{\beta}})^{-1}$ is also an immediate result of *i)*. □

Finally, we prove that the projection map is Lipschitz continuous in the product J_1 topology.

Proof of Lemma 2.3. Let $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \prod_{i=1}^{d} \mathbb{D}[0, T]$ be given. Note first that

$$|\xi_i(T) - \zeta_i(T)| = |\xi_i(T) - \zeta_i(\lambda(T))| \le \|\xi_i - \zeta_i \circ \lambda\|$$

for any $\lambda \in \Lambda[0, T]$ since $\lambda(T) = T$. Taking infimum over all $\lambda \in \Lambda[0, T]$, we see that $|\xi_i(T) - \zeta_i(T)| \le d_{J_1}(\xi_i, \zeta_i)$. Therefore,

$$|\boldsymbol{b}^\mathsf{T}\boldsymbol{\xi}(T) - \boldsymbol{b}^\mathsf{T}\boldsymbol{\zeta}(T)|$$
$$\le \sum_{i=1}^{d} |b_i| \cdot |\xi_i(T) - \zeta_i(T)| \le \sum_{i=1}^{d} |b_i| \cdot d_{J_1}(\xi_i, \zeta_i) \le \|\boldsymbol{b}\|_1 \cdot d_p(\boldsymbol{\xi}, \boldsymbol{\zeta}).$$

□

Appendix B Some useful tools on large deviations

In this appendix, we include results that facilitate the use of the extended LDP. Given that the probability measures of (X_n) satisfy the extended LDP in a metric space $(\mathcal{X}, d)$, our results include the derivation of the extended LDP in closed subspaces of $\mathcal{X}$, and a variation of the contraction principle for Lipschitz continuous maps. Let $\mathcal{D}_I \triangleq \{x \in X : I(x) < \infty\}$ denote the effective domain of I.

Lemma B.1 *Let E be a closed subset of $\mathcal{X}$, and let X_n be such that $\mathbf{P}(X_n \in E) = 1$ for all $n \ge 1$. Suppose that E is equipped with the topology induced by $\mathcal{X}$. Then, if the probability measures of (X_n) satisfy the extended LDP in $(\mathcal{X}, d)$ with speed a_n, and with rate function I so that $\mathcal{D}_I \subseteq E$, then the same extended LDP holds in E.*

Proof Suppose that an extended LDP holds in $\mathcal{X}$. For the upper bound, let F be a closed subset of E so that $F = F' \cap E$ for some F' that is a closed subset of $\mathcal{X}$. Then, F is a closed subset of $\mathcal{X}$. Hence, $\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}(X_n \in F) \le -\inf_{x\in F^\epsilon} I(x) = -\inf_{x\in F^\epsilon \cap E} I(x)$. Next, for the lower bound, let G be an open subset of E. That is,

$G = G' \cap E$, where G' is an open subset of $\mathcal{X}$. Then,

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(X_n \in G\right)\\ &= \liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(X_n \in G'\right) \geq -\inf_{x\in G'} I(x) = -\inf_{x\in G} I(x).\end{aligned}$$

The level sets $\Psi_I(\alpha) \subseteq \mathcal{X}$ are closed, so I restricted to E remains lower semi-continuous. □

We continue with a useful lemma on pre-images of Lipschitz continuous maps on metric spaces.

Lemma B.2 *Let $(\mathbb{S}, \sigma)$ and $(\mathbb{X}, d)$ be metric spaces. Suppose that $\Phi : (\mathbb{X}, d) \to (\mathbb{S}, \sigma)$ is a Lipschitz continuous mapping with Lipschitz constant $\|\Phi\|_{\text{Lip}}$. Then, for any set $F \subset \mathbb{S}$, it holds that*

$$\left(\Phi^{-1}(F)\right)^{\epsilon} \subseteq \Phi^{-1}\left(F^{\epsilon\cdot\|\Phi\|_{\text{Lip}}}\right).$$

Proof Let $\zeta \in \left(\Phi^{-1}(F)\right)^{\epsilon}$. For each n, there exists ξ_n such that $\xi_n \in \Phi^{-1}(F)$ and $d(\zeta, \xi_n) \leq \epsilon + 1/n$. Note that $\sigma(\Phi(\zeta), F) \leq \sigma(\Phi(\zeta), \Phi(\xi_n)) \leq \|\Phi\|_{\text{Lip}} \cdot d(\zeta, \xi_n) \leq \|\Phi\|_{\text{Lip}} \cdot (\epsilon + 1/n)$. Taking $n \to \infty$, we have that $\sigma(\Phi(\zeta), F) \leq \epsilon \cdot \|\Phi\|_{\text{Lip}}$. That is, $\Phi(\zeta) \in F^{\|\Phi\|_{\text{Lip}}\epsilon}$, or $\zeta \in \Phi^{-1}\left(F^{\|\Phi\|_{\text{Lip}}\epsilon}\right)$. Since ζ was chosen arbitrarily from $\left(\Phi^{-1}(F)\right)^{\epsilon}$, we arrive at the desired inclusion. □

The following lemma is a version of the contraction principle adapted to the setting of extended LDP's.

Lemma B.3 *Let $(\mathbb{X}, d)$ and $(\mathbb{S}, \sigma)$ be metric spaces. Suppose that the sequence of probability measures of $(\boldsymbol{X}_n)$ satisfies the lower and upper bounds of extended LDP in $(\mathbb{X}, d)$ with speed a_n and a function I (that is not necessarily a rate function). Moreover, let $\Phi : (\mathbb{X}, d) \to (\mathbb{S}, \sigma)$ be a Lipschitz continuous mapping and set $I'(y) \triangleq \inf_{\Phi(x)=y} I(x)$. Then,*

(i) *$\Phi(\boldsymbol{X}_n)$ satisfies the following lower and upper bounds: for any open set $G \subseteq \mathbb{S}$,*

$$\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\Phi(\boldsymbol{X}_n) \in G\right) \geq -\inf_{x\in G} I'(x),$$

and for any closed set $F \subseteq \mathbb{S}$,

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\Phi(\boldsymbol{X}_n) \in F\right) \leq -\lim_{\epsilon\to 0} \inf_{x\in F^{\epsilon}} I'(x).$$

(ii) *Suppose, in addition, that I is a rate function and Φ is a homeomorphism. Then, I' is a rate function, and $\Phi(\boldsymbol{X}_n)$ satisfies the extended LDP in $(\mathbb{S}, \sigma)$ with speed a_n and rate function I'.*

(iii) *If I' is a good rate function—i.e., $\Psi_{I'}(M) \triangleq \{y \in \mathbb{S} : I'(y) \le M\}$ is compact for each $M \in [0, \infty)$—then $\Phi(\boldsymbol{X}_n)$ satisfies the LDP in $(\mathbb{S}, \sigma)$ with speed a_n and good rate function I'.*

Proof (i) For the upper bound, let F be a closed subset of $(\mathbb{S}, \sigma)$. Thanks to Lemma B.2, for any $\epsilon > 0$, we have that $\left(\Phi^{-1}(F)\right)^{\epsilon} \subseteq \Phi^{-1}\left(F^{\epsilon \cdot \|\Phi\|_{\mathrm{Lip}}}\right)$. Hence,

$$-\inf_{x \in \left(\Phi^{-1}(F)\right)^{\epsilon}} I(x) \le -\inf_{x \in \Phi^{-1}\left(F^{\epsilon \cdot \|\Phi\|_{\mathrm{Lip}}}\right)} I(x). \tag{B.1}$$

Furthermore, by the upper bound of the extended LDP of $\boldsymbol{X}_n$, for any $\delta > 0$ there exists an $n(\delta)$ such that for any $n \ge n(\delta)$,

$$\begin{aligned}
\mathbf{P}(\Phi(\boldsymbol{X}_n) \in F) &= \mathbf{P}(\boldsymbol{X}_n \in \Phi^{-1}(F)) \\
&\le \exp\left(a_n\left(-\inf_{x \in \left(\Phi^{-1}(F)\right)^{\epsilon}} I(x) + \delta\right)\right) \\
&\le \exp\left(a_n\left(-\inf_{x \in \Phi^{-1}\left(F^{\epsilon \cdot \|\Phi\|_{\mathrm{Lip}}}\right)} I(x) + \delta\right)\right),
\end{aligned} \tag{B.2}$$

for any $n \ge n(\delta)$ and $\epsilon > 0$. Therefore,

$$\limsup_{n \to \infty} \frac{1}{a_n} \log \mathbf{P}\left(\Phi(\boldsymbol{X}_n) \in F\right) \le -\inf_{x \in \Phi^{-1}\left(F^{\epsilon \cdot \|\Phi\|_{\mathrm{Lip}}}\right)} I(x) + \delta = -\inf_{y \in F^{\epsilon \cdot \|\Phi\|_{\mathrm{Lip}}}} I'(y) + \delta.$$

Letting $\delta \to 0$ and then $\epsilon \to 0$, we arrive at the desired large deviation upper bound.

For the lower bound, consider an open set G. Since $\Phi^{-1}(G)$ is open,

$$\begin{aligned}
&\liminf_{n \to \infty} \frac{1}{a_n} \log \mathbf{P}\left(\Phi(\boldsymbol{X}_n) \in G\right) \\
&= \liminf_{n \to \infty} \frac{1}{a_n} \log \mathbf{P}\left(\boldsymbol{X}_n \in \Phi^{-1}(G)\right) \ge -\inf_{y \in \Phi^{-1}(G)} I(y) = -\inf_{x \in G} I'(x).
\end{aligned}$$

(ii) Since the upper and lower bounds for the extended large deviation principle have been proved in i), we only have to prove that I' is lower semi-continuous. To see this, note first that $I'(y) = I(\Phi^{-1}(y))$, and hence, for any $M > 0$,

$$\begin{aligned}
\{y \in \mathbb{S} : I'(y) \le M\} &= \{y \in \mathbb{S} : I(\Phi^{-1}(y)) \le M\} \\
&= \{\Phi(x) : I(x) \le M\} = \Phi(\Psi_I(M)).
\end{aligned}$$

Since Φ is a homeomorphism, the r.h.s. is closed. Hence, $\Phi(\boldsymbol{X}_n)$ satisfies the extended LDP.

(iii) From the standard argument—see, for example, the proof of Theorem 4.2.1 of Dembo and Zeitouni [9]—I' is a good rate function. From Lemma 4.1.6 of Dembo

and Zeitouni [9], we obtain $\lim_{\epsilon\to 0}\inf_{y\in F^{\epsilon\|\Phi\|_{\mathrm{Lip}}}} I'(y) = \inf_{y\in F} I'(y)$. Consequently,

$$\limsup_{n\to\infty}\frac{\log \mathbf{P}\left(\boldsymbol{S}_n\in F\right)}{a_n}\le -\lim_{\epsilon\to 0}\inf_{y\in F^{\epsilon\|\Phi\|_{\mathrm{Lip}}}} I'(y) = -\inf_{y\in F} I'(y).$$

□

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